18. **Extensions of representations, Whitehead's theorem, compete reducibility**

18.1. **Extensions.** Let $\mathfrak{g}$ be a Lie algebra and U, W be representations of $\mathfrak{g}$. We would like to classify all representations V which fit into a short exact sequence

$$0 \to U \to V \to W \to 0, \tag{18.1}$$

i.e., $U \subset V$ is a subrepresentation such that the surjection $p : V \to W$ has kernel U and thus defines an isomorphism $V/U \cong W$. In other words, V is endowed with a 2-step filtration with $F_0V = U$ and $F_1V = V$ such that $F_1V/F_0V = W$, so $\mathrm{gr}(V) = U \oplus W$. To do so, pick a splitting of this sequence as a sequence of vector spaces, i.e. an injection $i : W \to V$ (not a homomorphism of representations, in general) such that $p \circ i = \mathrm{Id}_W$. This defines a linear isomorphism $\widetilde{i} : U \oplus W \to V$ given by $(u, w) \mapsto u + i(w)$, which allows us to rewrite the action of $\mathfrak{g}$ on V as an action on $U \oplus W$. Since $\widetilde{i}$ is not in general a morphism of representations, this action is given by

$$\rho(x)(u, w) = (xu + a(x)w, xw)$$

where $a : \mathfrak{g} \to \mathrm{Hom}_{\mathbf{k}}(W, U)$ is a linear map, and $\widetilde{i}$ is a morphism of representations iff $a = 0$.

What are the conditions on a to give rise to a representation? We compute:

$$\rho([x, y])(u, w) = ([x, y]u + a([x, y])w, [x, y]w),$$

$$[\rho(x), \rho(y)](u, w) = ([x, y]u + ([x, a(y)] + [a(x), y])w, [x, y]w).$$

Thus the condition to give a representation is the Leibniz rule

$$a([x, y]) = [x, a(y)] + [a(x), y] = [x, a(y)] - [y, a(x)].$$

In general, if E is a representation of $\mathfrak{g}$ then a linear function $a : \mathfrak{g} \to E$ such that

$$a([x, y]) = x \circ a(y) - y \circ a(x)$$

is called a $\mathbf{1-}$**cocycle** of $\mathfrak{g}$ with values in E. The space of 1-cocycles is denoted by $Z^1(\mathfrak{g}, E)$.

Example 18.1. We have $Z^1(\mathfrak{g}, \mathbf{k}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ and $Z^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{Der}\mathfrak{g}$.

Thus we see that in our setting $a : \mathfrak{g} \to \mathrm{Hom}_{\mathbf{k}}(W, U)$ defines a representation if and only if $a \in Z^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U))$. Denote the representation V attached to such a by V_a. Then we have a natural short exact sequence

$$0 \to U \to V_a \to W \to 0.$$

It may, however, happen that some $a \neq 0$ defines a trivial extension $V \cong U \oplus W$, i.e., $V_a \cong V_0$, and more generally $V_a \cong V_b$ for $a \neq b$. Let us determine when this happens. More precisely, let us look for isomorphisms $f : V_a \to V_b$ preserving the structure of the short exact sequences, i.e., such that $\mathrm{gr}(f) = \mathrm{Id}$. Then

$$f(u, w) = (u + Aw, w)$$

where $A : W \to U$ is a linear map. Then we have

$$xf(u, w) = x(u + Aw, w) = (xu + xAw + b(x)w, xw)$$

and

$$fx(u, w) = f(xu + a(x)w, xw) = (xu + a(x)w + Axw, xw),$$

so we get that $xf = fx$ iff

$$[x, A] = a(x) - b(x).$$

In particular, setting $b = 0$, we see that V is a trivial extension if and only if $a(x) = [x, A]$ for some A.

More generally, if E is a $\mathfrak{g}$-module, the linear function $a : \mathfrak{g} \to E$ given by $a(x) = xv$ for some $v \in E$ is called the **1-coboundary** of v, and one writes $a = dv$. The space of 1-coboundaries is denoted by $B^1(\mathfrak{g}, E)$; it is easy to see that it is a subspace of $Z^1(\mathfrak{g}, E)$, i.e., a 1-coboundary is always a 1-cocycle. Thus in our setting $f : V_a \to V_b$ is an isomorphism of representations iff

$$a - b = dA,$$

i.e., there is an isomorphism $f : V_a \cong V_b$ with $\mathrm{gr}(f) = \mathrm{Id}$ if and only if $a = b$ in the quotient space

$$\mathrm{Ext}^1(W, U) := Z^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U))/B^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U)).$$

The notation is justified by the fact that this space parametrizes extensions of W by U. More precisely, every short exact sequence (18.1) gives rise to a class $[V] \in \mathrm{Ext}^1(W, U)$, and the extension defined by this sequence is trivial iff $[V] = 0$.

More generally, for a $\mathfrak{g}$-module E the space

$$H^1(\mathfrak{g}, E) := Z^1(\mathfrak{g}, E)/B^1(\mathfrak{g}, E)$$

is called the **first cohomology** of $\mathfrak{g}$ with coefficients in E. Thus,

$$\mathrm{Ext}^1(W, U) = H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U)).$$

Lemma 18.2. *A short exact sequence* $0 \to U \to V \to W \to 0$ *gives rise to an exact sequence*

$$H^1(\mathfrak{g}, U) \to H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, W).$$

Exercise 18.3. Prove Lemma 18.2.

18.2. **Whitehead's theorem.** We have shown in Corollary 17.6 and Proposition 17.9 that for a semisimple $\mathfrak{g}$ over a field of characteristic zero, $H^1(\mathfrak{g}, \mathbf{k}) = (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^* = 0$, and $H^1(\mathfrak{g},\mathfrak{g}) = \mathrm{Der}\mathfrak{g}/\mathfrak{g} = 0$. In fact, these are special cases of a more general theorem.

Theorem 18.4. *(Whitehead) If $\mathfrak{g}$ is semisimple in characteristic zero then for every finite dimensional representation V of $\mathfrak{g}$, $H^1(\mathfrak{g}, V) = 0$.*

18.3. **Proof of Theorem 18.4.** We will use the following lemma, which holds over any field.

Lemma 18.5. *Let E be a representation of a Lie algebra $\mathfrak{g}$ and $C \in U(\mathfrak{g})$ be a central element which acts by 0 on the trivial representation of $\mathfrak{g}$ and by some scalar $\lambda \neq 0$ on E. Then $H^1(\mathfrak{g}, E) = 0$.*

Proof. We have seen that $H^1(\mathfrak{g}, E) = \mathrm{Ext}^1(\mathbf{k}, E)$, so our job is to show that any extension

$$0 \to E \to V \to \mathbf{k} \to 0$$

splits. Let $p : V \to \mathbf{k}$ be the projection. We claim that there exists a unique vector $v \in V$ such that $p(v) = 1$ and $Cv = 0$. Indeed, pick some $w \in V$ with $p(w) = 1$. Then $Cw \in E$, so set $v = w - \lambda^{-1}Cw$. Since $C^2w = \lambda Cw$, we have $Cv = 0$. Also if v' is another such vector then $v - v' \in E$ so $C(v - v') = \lambda(v - v') = 0$, hence $v = v'$.

Thus $\mathbf{k}v \subset V$ is a $\mathfrak{g}$-invariant complement to E (as C is central), which implies the statement. ☐

It remains to construct a central element of $U(\mathfrak{g})$ for a semisimple Lie algebra $\mathfrak{g}$ to which we can apply Lemma 18.5. This can be done as follows. Let a_i be a basis of $\mathfrak{g}$ and a^i the dual basis under an invariant inner product on $\mathfrak{g}$ (for example, the Killing form). Define the **(quadratic) Casimir element**

$$C := \sum_i a_i a^i.$$

It is easy to show that C is independent on the choice of the basis (although it depends on the choice of the inner product). Also C is central: for $y \in \mathfrak{g}$,

$$[y, C] = \sum_i ([y, a_i]a^i + a_i[y, a^i]) = 0$$

since

$$\sum_i ([y, a_i] \otimes a^i + a_i \otimes [y, a^i]) = 0.$$

Finally, note that for $\mathfrak{g} = \mathfrak{sl}_2$, C is proportional to the Casimir element $2fe + \frac{h^2}{2} + h = ef + fe + \frac{h^2}{2}$ considered previously, as the basis $f, e, \frac{h}{\sqrt{2}}$ is dual to the basis $e, f, \frac{h}{\sqrt{2}}$ under an invariant inner product of $\mathfrak{g}$.

The key lemma used in the proof of Theorem 18.4 is the following.

Lemma 18.6. *Let $\mathfrak{g}$ be semisimple in characteristic zero and V be a nontrivial finite dimensional irreducible $\mathfrak{g}$-module. Then there is a central element $C \in U(\mathfrak{g})$ such that $C|_{\mathbf{k}} = 0$ and $C|_V \neq 0$.*

Proof. Consider the invariant symmetric bilinear form on $\mathfrak{g}$

$$B_V(x, y) = \mathrm{Tr}|_V(xy).$$

We claim that $B_V \neq 0$. Indeed, let $\bar{\mathfrak{g}} \subset \mathfrak{gl}(V)$ be the image of $\mathfrak{g}$. By Lemma 17.1, if $B_V = 0$ then $\bar{\mathfrak{g}}$ is solvable, so, being the quotient of a semisimple Lie algebra $\mathfrak{g}$, it must be zero, hence V is trivial, a contradiction.

Let $I = \mathrm{Ker}(B_V)$. Then $I \subset \mathfrak{g}$ is an ideal, so by Proposition 17.7, $\mathfrak{g} = I \oplus \mathfrak{g}'$ for some semisimple Lie algebra $\mathfrak{g}'$, and B_V is nondegenerate on $\mathfrak{g}'$. Let C be the Casimir element of $U(\mathfrak{g}')$ corresponding to the inner product B_V. Then $\mathrm{Tr}_V(C) = \sum_i B_V(a_i, a^i) = \dim \mathfrak{g}'$, so $C|_V = \frac{\dim \mathfrak{g}'}{\dim V} \neq 0$. Also it is clear that $C|_{\mathbf{k}} = 0$, so the lemma follows. □

Corollary 18.7. *For any irreducible finite dimensional representation V of a semisimple Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ of characteristic zero, we have $H^1(\mathfrak{g}, V) = 0$.*

Proof. If V is nontrivial, this follows from Lemmas 18.5 and 18.6. On the other hand, if $V = \mathbf{k}$ then $H^1(\mathfrak{g}, V) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* = 0$. □

Now we can prove Theorem 18.4. By Lemma 18.2, it suffices to prove the theorem for irreducible V, which is guaranteed by Corollary 18.7.

Corollary 18.8. *A reductive Lie algebra $\mathfrak{g}$ in characteristic zero is uniquely a direct sum of a semisimple and abelian Lie algebra.*

Proof. Consider the adjoint representation of $\mathfrak{g}$. It is a representation of $\mathfrak{g}' = \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, which fits into a short exact sequence

$$0 \to \mathfrak{z}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}' \to 0.$$

By complete reducibility, this sequence splits, i.e. we have a decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$ as a direct sum of ideals, and it is clearly unique. □

18.4. **Complete reducibility of representations of semisimple Lie algebras.**

Theorem 18.9. *Every finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ over a field of characteristic zero is completely reducible, i.e., isomorphic to a direct sum of irreducible representations.*

Proof. Theorem 18.4 implies that for any finite dimensional representations W, U of $\mathfrak{g}$ one has $\mathrm{Ext}^1(W, U) = 0$. Thus any short exact sequence

$$0 \to U \to V \to W \to 0$$

splits, which implies the statement. $\square$