## 19. **Structure of semisimple Lie algebras, I**

19.1. **Semisimple elements.** Let $x$ be an element of a Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbf{k}$. Let $\mathfrak{g}_\lambda \subset \mathfrak{g}$ be the generalized eigenspace of $\mathrm{ad}x$ with eigenvalue $\lambda$. Then $\mathfrak{g} = \oplus_\lambda \mathfrak{g}_\lambda$.

**Lemma 19.1.** *We have* $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.

*Proof.* Let $y \in \mathfrak{g}_\lambda, z \in \mathfrak{g}_\mu$. We have

$$(\mathrm{ad}x - \lambda - \mu)^N([y,z]) =$$

$$\sum_{p+q+r+s=N} (-1)^{r+s} \frac{N!}{p!q!r!s!} \lambda^r \mu^s [(\mathrm{ad}x)^p(y), (\mathrm{ad}x)^q(z)] =$$

$$\sum_{k+\ell=N} \frac{N!}{k!\ell!} [(\mathrm{ad}x - \lambda)^k(y), (\mathrm{ad}x - \mu)^\ell(z)].$$

Thus if $(\mathrm{ad}x - \lambda)^n(y) = 0$ and $(\mathrm{ad}x - \mu)^m(z) = 0$ then

$$(\mathrm{ad}x - \lambda - \mu)^{m+n}([y,z]) = 0,$$

so $[y,z] \in \mathfrak{g}_{\lambda+\mu}$. □

**Definition 19.2.** An element $x$ of a Lie algebra $\mathfrak{g}$ is called **semisimple** if the operator $\mathrm{ad}x$ is semisimple and **nilpotent** if this operator is nilpotent.

It is clear that any element which is both semisimple and nilpotent is central, so for a semisimple Lie algebra it must be zero. Note also that for $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{k})$ this coincides with the usual definition.

**Proposition 19.3.** *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero. Then every element $x \in \mathfrak{g}$ has a unique decomposition as $x = x_s + x_n$, where $x_s$ is semisimple, $x_n$ is nilpotent and $[x_s, x_n] = 0$. Moreover, if $y \in \mathfrak{g}$ and $[x,y] = 0$ then $[x_s, y] = [x_n, y] = 0$.*

*Proof.* Recall that $\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ via the adjoint representation. So we can consider the Jordan decomposition $x = x_s + x_n$, with $x_s, x_n \in \mathfrak{gl}(\mathfrak{g})$. We have $x_s(y) = \lambda y$ for $y \in \mathfrak{g}_\lambda$. Thus $y \mapsto x_s(y)$ is a derivation of $\mathfrak{g}$ by Lemma 19.1. But by Proposition 17.9 every derivation of $\mathfrak{g}$ is inner, which implies that $x_s \in \mathfrak{g}$, hence $x_n \in \mathfrak{g}$. It is clear that $x_s$ is semisimple, $x_n$ is nilpotent, and $[x_s, x_n] = 0$. Also if $[x,y] = 0$ then $\mathrm{ad}y$ preserves $\mathfrak{g}_\lambda$ for all $\lambda$, hence $[x_s, y] = 0$ as linear operators on $\mathfrak{g}$ and thus as elements of $\mathfrak{g}$. This also implies that the decomposition is unique since if $x = x_s' + x_n'$ then $[x_s, x_s'] = [x_n, x_n'] = 0$, so $x_s - x_s' = x_n' - x_n$ is both semisimple and nilpotent, hence zero. □

**Corollary 19.4.** *Any semisimple Lie algebra $\mathfrak{g} \neq 0$ over a field of characteristic zero contains nonzero semisimple elements.*

*Proof.* Otherwise, by Proposition 19.3, every element $x \in \mathfrak{g}$ is nilpotent, which by Engel's theorem would imply that $\mathfrak{g}$ is nilpotent, hence solvable, hence zero. $\square$

19.2. **Toral subalgebras.** From now on we assume that $\mathrm{char}(\mathbf{k}) = 0$ unless specified otherwise.

**Definition 19.5.** An abelian Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a **toral subalgebra** if it consists of semisimple elements.

**Proposition 19.6.** *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a toral subalgebra, and $B$ a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ (e.g., the Killing form).*

*(i) We have a decomposition $\mathfrak{g} = \oplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the subspace of $x \in \mathfrak{g}$ such that for $h \in \mathfrak{h}$ we have $[h, x] = \alpha(h)x$, and $\mathfrak{g}_0 \supset \mathfrak{h}$.*

*(ii) We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.*

*(iii) If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal under $B$.*

*(iv) $B$ restricts to a nondegenerate pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbf{k}$.*

*Proof.* (i) is just the joint eigenspace decomposition for $\mathfrak{h}$ acting in $\mathfrak{g}$. (ii) is a very easy special case of Lemma 19.1. (iii) and (iv) follow from the fact that $B$ is nondegenerate and invariant. $\square$

**Corollary 19.7.** *(i) The Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is reductive.*

*(ii) if $x \in \mathfrak{g}_0$ then $x_s, x_n \in \mathfrak{g}_0$.*

*Proof.* (i) This follows from Proposition 16.14 and the fact that the form $(x, y) \mapsto \mathrm{Tr}|_{\mathfrak{g}}(xy)$ on $\mathfrak{g}_0$ is nondegenerate (Proposition 19.6(iv) for the Killing form of $\mathfrak{g}$).

(ii) We have $[h, x] = 0$ for $h \in \mathfrak{h}$, so $[h, x_s] = 0$, hence $x_s \in \mathfrak{g}_0$. $\square$

19.3. **Cartan subalgebras.**

**Definition 19.8.** A **Cartan subalgebra** of a semisimple Lie algebra $\mathfrak{g}$ is a toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g}_0 = \mathfrak{h}$.

**Example 19.9.** Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{k})$. Then the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of diagonal matrices is a Cartan subalgebra.

It is clear that any Cartan subalgebra is a maximal toral subalgebra of $\mathfrak{g}$. The following theorem, stating the converse, shows that Cartan subalgebras exist.

**Theorem 19.10.** *Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra.*

*Proof.* Let $x \in \mathfrak{g}_0$, then by Corollary 19.7(ii) $x_s \in \mathfrak{g}_0$, so $x_s \in \mathfrak{h}$ by maximality of $\mathfrak{h}$. Thus $\mathrm{ad}x|_{\mathfrak{g}_0} = \mathrm{ad}x_n|_{\mathfrak{g}_0}$ is nilpotent. So by Engel's theorem $\mathfrak{g}_0$ is nilpotent. But it is also reductive, hence abelian.

Now let us show that every $x \in \mathfrak{g}_0$ which is nilpotent in $\mathfrak{g}$ must be zero. Indeed, in this case, for any $y \in \mathfrak{g}_0$, the operator $\mathrm{ad}x \cdot \mathrm{ad}y : \mathfrak{g} \to \mathfrak{g}$ is nilpotent (as $[x, y] = 0$), so $\mathrm{Tr}|_{\mathfrak{g}}(\mathrm{ad}x \cdot \mathrm{ad}y) = 0$. But this form is nondegenerate on $\mathfrak{g}_0$, which implies that $x = 0$.

Thus for any $x \in \mathfrak{g}_0$, $x_n = 0$, so $x = x_s$ is semisimple. Hence $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{h}$ is a Cartan subalgebra. □

We will show in Theorem 20.10 that all Cartan subalgebras of $\mathfrak{g}$ are conjugate under $\mathrm{Aut}(\mathfrak{g})$, in particular they all have the same dimension, which is called the **rank** of $\mathfrak{g}$.

### 19.4. Root decomposition.

**Proposition 19.11.** *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and $B$ a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ (e.g., the Killing form).*

*(i) We have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the subspace of $x \in \mathfrak{g}$ such that for $h \in \mathfrak{h}$ we have $[h, x] = \alpha(h)x$, and $R$ is the (finite) set of $\alpha \in \mathfrak{h}^*$, $\alpha \neq 0$, such that $\mathfrak{g}_\alpha \neq 0$.*

*(ii) We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.*

*(iii) If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal under $B$.*

*(iv) $B$ restricts to a nondegenerate pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbf{k}$.*

*Proof.* This immediately follows from Theorem 19.6. □

**Definition 19.12.** The set $R$ is called the **root system** of $\mathfrak{g}$ and its elements are called **roots**.

**Proposition 19.13.** *Let $\mathfrak{g}_1, ..., \mathfrak{g}_n$ be simple Lie algebras and let $\mathfrak{g} = \oplus_i \mathfrak{g}_i$.*

*(i) Let $\mathfrak{h}_i \subset \mathfrak{g}_i$ be Cartan subalgebras of $\mathfrak{g}_i$ and $R_i \subset \mathfrak{h}_i^*$ the corresponding root systems of $\mathfrak{g}_i$. Then $\mathfrak{h} = \oplus_i \mathfrak{h}_i$ is a Cartan subalgebra in $\mathfrak{g}$ and the corresponding root system $R$ is the disjoint union of $R_i$.*

*(ii) Each Cartan subalgebra in $\mathfrak{g}$ has the form $\mathfrak{h} = \oplus_i \mathfrak{h}_i$ where $\mathfrak{h}_i \subset \mathfrak{g}_i$ is a Cartan subalgebra in $\mathfrak{g}_i$.*

*Proof.* (i) is obvious. To prove (ii), given a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, let $\mathfrak{h}_i$ be the projections of $\mathfrak{h}$ to $\mathfrak{g}_i$. It is easy to see that $\mathfrak{h}_i \subset \mathfrak{g}_i$ are Cartan subalgebras. Also $\mathfrak{h} \subset \oplus_i \mathfrak{h}_i$ and the latter is toral, which implies that $\mathfrak{h} = \oplus_i \mathfrak{h}_i$ since $\mathfrak{h}$ is a Cartan subalgebra. □

**Example 19.14.** Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{k})$. Then the subspace of diagonal matrices $\mathfrak{h}$ is a Cartan subalgebra (cf. Example 19.9), and it can be naturally identified with the space of vectors $\mathbf{x} = (x_1, ..., x_n)$ such that $\sum_i x_i = 0$. Let $\mathbf{e}_i$ be the linear functionals on this space given by $\mathbf{e}_i(\mathbf{x}) = x_i$. We have $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbf{k} E_{ij}$ and $[\mathbf{x}, E_{ij}] = (x_i - x_j) E_{ij}$. Thus the root system $R$ consists of vectors $\mathbf{e}_i - \mathbf{e}_j \in \mathfrak{h}^*$ for $i \neq j$ (so there are $n(n-1)$ roots).

Now let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Let $(,)$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, for example the Killing form. Since the restriction of $(,)$ to $\mathfrak{h}$ is nondegenerate, it defines an isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ given by $h \mapsto (h, ?)$. The inverse of this isomorphism will be denoted by $\alpha \mapsto H_\alpha$. We also have the inverse form on $\mathfrak{h}^*$ which we also will denote by $(,)$; it is given by $(\alpha, \beta) := \alpha(H_\beta) = (H_\alpha, H_\beta)$.

**Lemma 19.15.** *For any $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ we have*

$$[e, f] = (e, f) H_\alpha.$$

*Proof.* We have $[e, f] \in \mathfrak{h}$ so it is enough to show that the inner product of both sides with any $h \in \mathfrak{h}$ is the same. We have

$$([e, f], h) = (e, [f, h]) = \alpha(h)(e, f) = ((e, f) H_\alpha, h),$$

as desired. □

**Lemma 19.16.** *(i) If $\alpha$ is a root then $(\alpha, \alpha) \neq 0$.*

*(ii) Let $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ be such that $(e, f) = \frac{2}{(\alpha,\alpha)}$, and let $h_\alpha := \frac{2H_\alpha}{(\alpha,\alpha)}$. Then $e, f, h_\alpha$ satisfy the commutation relations of the Lie algebra $\mathfrak{sl}_2$.*

*(iii) $h_\alpha$ is independent on the choice of $(,)$.*

*Proof.* (i) Pick $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ with $(e, f) \neq 0$. Let $h := [e, f] = (e, f) H_\alpha$ (by Lemma 19.15) and consider the Lie algebra $\mathfrak{a}$ generated by $e, f, h$. Then we see that

$$[h, e] = \alpha(h) e = (\alpha, \alpha)(e, f) e, \ [h, f] = -\alpha(h) f = (\alpha, \alpha)(e, f) f.$$

Thus if $(\alpha, \alpha) = 0$ then $\mathfrak{a}$ is a solvable Lie algebra. By Lie's theorem, we can choose a basis in $\mathfrak{g}$ such that operators $\mathrm{ad} e$, $\mathrm{ad} f$, $\mathrm{ad} h$ are upper triangular. Since $h = [e, f]$, $\mathrm{ad} h$ will be strictly upper-triangular and thus nilpotent. But since $h \in \mathfrak{h}$, it is also semisimple. Thus, $\mathrm{ad} h = 0$, so $h = 0$ as $\mathfrak{g}$ is semisimple. On the other hand, $h = (e, f) H_\alpha \neq 0$. This contradiction proves the first part of the theorem.

(ii) This follows immediately from the formulas in the proof of (i).

(iii) It's enough to check the statement for a simple Lie algebra, and in this case this is easy since $(,)$ is unique up to scaling. □

The Lie subalgebra of $\mathfrak{g}$ spanned by $e, f, h_\alpha$, which we've shown to be isomorphic to $\mathfrak{sl}_2(\mathbf{k})$, will be denoted by $\mathfrak{sl}_2(\mathbf{k})_\alpha$ (we will see that $\mathfrak{g}_\alpha$ are 1-dimensional so it is independent on the choices).

**Proposition 19.17.** *Let* $\mathfrak{a}_\alpha = \mathbf{k}H_\alpha \oplus \bigoplus_{k\neq 0} \mathfrak{g}_{k\alpha} \subset \mathfrak{g}$. *Then* $\mathfrak{a}_\alpha$ *is a Lie subalgebra of* $\mathfrak{g}$.

*Proof.* This follows from the fact that for $e \in \mathfrak{g}_{k\alpha}, f \in \mathfrak{g}_{-k\alpha}$ we have $[e, f] = (e, f)H_{k\alpha} = k(e, f)H_\alpha$. □

**Corollary 19.18.** *(i) The space* $\mathfrak{g}_\alpha$ *is 1-dimensional for each root* $\alpha$ *of* $\mathfrak{g}$.

*(ii) If* $\alpha$ *is a root of* $\mathfrak{g}$ *and* $k \geq 2$ *is an integer then* $k\alpha$ *is not a root of* $\mathfrak{g}$.

*Proof.* For a root $\alpha$ the Lie algebra $\mathfrak{a}_\alpha$ contains $\mathfrak{sl}_2(\mathbf{k})_\alpha$, so it is a finite dimensional representation of this Lie algebra. Also the kernel of $h_\alpha$ on this representation is spanned by $h_\alpha$, hence 1-dimensional, and eigenvalues of $h_\alpha$ are even integers since $\alpha(h_\alpha) = 2$. Thus by the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4), this representation is irreducible, i.e., eigenspaces of $h_\alpha$ (which are $\mathfrak{g}_{k\alpha}$ and $\mathbf{k}H_\alpha$) are 1-dimensional. Therefore the map $[e, ?] : \mathfrak{g}_\alpha \to \mathfrak{g}_{2\alpha}$ is zero (as $\mathfrak{g}_\alpha$ is spanned by $e$). So again by representation theory of $\mathfrak{sl}_2$ we have $\mathfrak{g}_{k\alpha} = 0$ for $|k| \geq 2$. □

**Theorem 19.19.** *Let* $\mathfrak{g}$ *be a semisimple Lie algebra with Cartan subalgebra* $\mathfrak{h}$ *and root decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$. *Let* $(,)$ *be a nondegenerate symmetric invariant bilinear form on* $\mathfrak{g}$.

*(i)* $R$ *spans* $\mathfrak{h}^*$ *as a vector space, and elements* $h_\alpha$, $\alpha \in R$ *span* $\mathfrak{h}$ *as a vector space.*

*(ii) For any two roots* $\alpha, \beta$, *the number* $a_{\alpha,\beta} := \beta(h_\alpha) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ *is an integer.*

*(iii) For* $\alpha \in R$, *define the* **reflection operator** $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ *by*

$$s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha = \lambda - 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)}\alpha.$$

*Then for any roots* $\alpha$, $\beta$, $s_\alpha(\beta)$ *is also a root.*

*(iv) For roots* $\alpha, \beta \neq \pm\alpha$, *the subspace* $V_{\alpha,\beta} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}_{\beta+k\alpha} \subset \mathfrak{g}$ *is an is an irreducible representation of* $\mathfrak{sl}_2(\mathbf{k})_\alpha$.

*Proof.* (i) Suppose $h \in \mathfrak{h}$ is such that $\alpha(h) = 0$ for all roots $\alpha$. Then $\mathrm{ad}h = 0$, hence $h = 0$ as $\mathfrak{g}$ is semisimple. This implies both statements.

(ii) $a_{\alpha,\beta}$ is the eigenvalue of $h_\alpha$ on $e_\beta$, hence an integer by the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4).

(iii) Let $x \in \mathfrak{g}_\beta$ be nonzero. If $\beta(h_\alpha) \geq 0$ then let $y = f_\alpha^{\beta(h_\alpha)} x$. If $\beta(h_\alpha) \leq 0$ then let $y = e_\alpha^{-\beta(h_\alpha)} x$. Then by representation theory of $\mathfrak{sl}_2$, $y \neq 0$. We also have $[h, y] = s_\alpha(\beta)(h)y$. This implies the statement.

(iv) It is clear that $V_{\alpha,\beta}$ is a representation. Also all $h_\alpha$-eigenspaces in $V_{\alpha,\beta}$ are 1-dimensional, and the eigenvalues are either all odd or all even. This implies that it is irreducible. □

**Corollary 19.20.** *Let $\mathfrak{h}_\mathbb{R}$ be the $\mathbb{R}$-span of $h_\alpha$. Then $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$ and the restriction of the Killing form to $\mathfrak{h}_\mathbb{R}$ is real-valued and positive definite.*

*Proof.* It follows from the previous theorem that the eigenvalues of $\mathrm{ad}h$, $h \in \mathfrak{h}_\mathbb{R}$, are real. So $\mathfrak{h}_\mathbb{R} \cap i\mathfrak{h}_\mathbb{R} = 0$, which implies the first statement. Now, $K(h, h) = \sum_i \lambda_i^2$ where $\lambda_i$ are the eigenvalues of $\mathrm{ad}h$ (which are not all zero if $h \neq 0$). Thus $K(h, h) > 0$ if $h \neq 0$. □

18.745 Lie Groups and Lie Algebras I
Fall 2020