20. **Structure of semisimple Lie algebras, II**

20.1. **Strongly regular (regular semisimple) elements.** In this section we will discuss another way of constructing Cartan subalgebras. First consider an example.

Example 20.1. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $x \in \mathfrak{g}$ be a diagonal matrix with distinct eigenvalues. Then the centralizer $\mathfrak{h} = C(x)$ is the space of all diagonal matrices of trace 0, which is a Cartan subalgebra. Thus the same applies to any diagonalizable matrix with distinct eigenvalues, i.e., a generic matrix (one for which the discriminant of the characteristic polynomial is nonzero).

So we may hope that if we take a generic element x in a semisimple Lie algebra then its centralizer is a Cartan subalgebra. But for that we have to define what we mean by generic.

Definition 20.2. The **nullity** $n(x)$ of an element $x \in \mathfrak{g}$ is the multiplicity of the eigenvalue 0 for the operator $\mathrm{ad}x$ (i.e., the dimension of the generalized 0-eigenspace). The **rank** $\mathrm{rank}(\mathfrak{g})$ of $\mathfrak{g}$ is the minimal value of $n(x)$. An element x is **strongly regular** if $n(x) = \mathrm{rank}(\mathfrak{g})$.

Example 20.3. It is easy to check that for $\mathfrak{g} = \mathfrak{sl}_n$, x is strongly regular if and only if its eigenvalues are distinct.

We will need the following auxiliary lemma.

Lemma 20.4. *Let $P(z_1, ..., z_n)$ be a nonzero complex polynomial, and $U \subset \mathbb{C}^n$ be the set of points $(z_1, ..., z_n) \in \mathbb{C}^n$ such that $P(z_1, ..., z_n) \neq 0$. Then U is path-connected, dense and open.*

Proof. It is clear that U is open, since it is the preimage of the open set $\mathbb{C}^\times \subset \mathbb{C}$ under a continuous map. It is also dense, as its complement, the hypersurface $P = 0$, cannot contain a ball. Finally, to see that it is path-connected, take $\mathbf{x}, \mathbf{y} \in U$, and consider the polynomial $Q(t) := P((1 - t)\mathbf{x} + t\mathbf{y})$. It has only finitely many zeros, hence the entire complex line $\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y}$ except finitely many points is contained in U. Clearly, $\mathbf{x}$ and $\mathbf{y}$ can be connected by a path inside this line avoiding this finite set of points. $\square$

Lemma 20.5. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then the set $\mathfrak{g}^{\mathrm{sr}}$ of strongly regular elements is connected, dense and open in $\mathfrak{g}$.*

Proof. Consider the characteristic polynomial $P_x(t)$ of $\mathrm{ad}x$. We have

$$P_x(t) = t^{\mathrm{rank}(\mathfrak{g})}(t^m + a_{m-1}(x)t^{m-1} + ... + a_0(x)),$$

where $m = \dim \mathfrak{g} - \mathrm{rank}\mathfrak{g}$ and a_i are some polynomials of x, with $a_0 \neq 0$. Then x is strongly regular if and only if $a_0(x) \neq 0$. This implies the statement by Lemma 20.4. $\square$

Proposition 20.6. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Then*
(i) $\dim \mathfrak{h} = \mathrm{rank}(\mathfrak{g})$*; and*
(ii) the set $\mathfrak{h}^{\mathrm{reg}} := \mathfrak{h} \cap \mathfrak{g}^{\mathrm{sr}}$ *coincides with the set*

$$V := \{h \in \mathfrak{h} : \alpha(h) \neq 0 \ \forall \alpha \in R\}.$$

In particular, $\mathfrak{h}^{\mathrm{reg}}$ *is open and dense in* $\mathfrak{h}$.

Proof. (i) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ (we know it exists, e.g. we can take G to be the connected component of the identity in $\mathrm{Aut}(\mathfrak{g})$).

Lemma 20.7. *Let $\phi : G \times V \to \mathfrak{g}$ be the map defined by $\phi(g, x) := \mathrm{Ad}g \cdot x$. Then the set $U := \mathrm{Im}\phi \subset \mathfrak{g}$ is open.*

Proof. Let us compute the differential $\phi_* : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g}$ at the point $(1, x)$ for $x \in \mathfrak{h}$. We obtain

$$\phi_*(y, h) = [y, x] + h.$$

The kernel of this map is identified with the set of $y \in \mathfrak{g}$ such that $[y, x] \in \mathfrak{h}$. But then $K([y, x], z) = K(y, [x, z]) = 0$ for all $z \in \mathfrak{h}$, so $[y, x] = 0$. Thus $\mathrm{Ker}\phi_* = C(x)$.

Now let $x \in V$. Then $C(x) = \mathfrak{h}$. Thus ϕ_* is surjective by dimension count, hence ϕ is a submersion at $(1, x)$. This means that $U := \mathrm{Im}\phi$ contains x together with its neighborhood in $\mathfrak{g}$. Hence the same holds for $\mathrm{Ad}g \cdot x$, which implies that U is open. $\square$

Since $\mathfrak{g}^{\mathrm{sr}}$ is open and dense and U is open by Lemma 20.7 and nonempty, we see that $U \cap \mathfrak{g}^{\mathrm{sr}} \neq \emptyset$. But

$$n(\mathrm{Ad}g \cdot x) = n(x) = \dim C(x) = \dim \mathfrak{h}.$$

for $x \in V$. This implies that $\mathrm{rank}\mathfrak{g} = \dim \mathfrak{h}$, which yields (i).

(ii) It is clear that for $x \in \mathfrak{h}$, we have

$$n(x) = \dim \mathrm{Ker}(\mathrm{ad}x) = \dim \mathfrak{h} + \#\{\alpha \in R : \alpha(x) = 0\}.$$

This implies the statement. $\square$

20.2. **Conjugacy of Cartan subalgebras.**

Theorem 20.8. *(i) Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $x \in \mathfrak{g}$ be a strongly regular semisimple element (which exists by Proposition 20.6). Then the centralizer $C(x)$ of x in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.*

(ii) Any Cartan subalgebra of $\mathfrak{g}$ is of this form.

Proof. Consider the eigenspace decomposition of $\mathrm{ad}x$: $\mathfrak{g} = \oplus_\lambda \mathfrak{g}_\lambda$. Since $\mathbb{C}x$ is a toral subalgebra, the Lie algebra $\mathfrak{g}_0 = C(x)$ is reductive, with $\dim(\mathfrak{g}_0) = \mathrm{rank}\mathfrak{g}$.

We claim that $\mathfrak{g}_0$ is also nilpotent. By Engel's theorem, to establish this, it suffices to show that the restriction of $\mathrm{ad}y$ to $\mathfrak{g}_0$ is nilpotent for $y \in \mathfrak{g}_0$. But $\mathrm{ad}(x+ty) = \mathrm{ad}x + t\mathrm{ad}y$ is invertible on $\mathfrak{g}/\mathfrak{g}_0$ for small t, since it is so for $t = 0$ and the set of invertible matrices is open. Thus $\mathrm{ad}(x+ty)$ must be nilpotent on $\mathfrak{g}_0$, as the multiplicity of the eigenvalue 0 for this operator must be (at least) $\mathrm{rank}\mathfrak{g} = \dim \mathfrak{g}_0$. But $\mathrm{ad}(x+ty) = t\mathrm{ad}y$ on $\mathfrak{g}_0$, which implies that $\mathrm{ad}y$ is nilpotent on $\mathfrak{g}_0$, as desired.

Thus $\mathfrak{g}_0$ is abelian. Moreover, for $y, z \in \mathfrak{g}_0$ the operator $\mathrm{ad}y_n \cdot \mathrm{ad}z$ is nilpotent on $\mathfrak{g}$ (as the product of two commuting operators one of which is nilpotent), so $K_\mathfrak{g}(y_n, z) = 0$, which implies that $y_n = 0$, as $K_\mathfrak{g}$ restricts to a nondegenerate form on $\mathfrak{g}_0$ and z is arbitrary. It follows that any $y \in \mathfrak{g}_0$ is semisimple, so $\mathfrak{g}_0$ is a toral subalgebra. Moreover, it is maximal since any element commuting with x is in $\mathfrak{g}_0$. Thus $\mathfrak{g}_0$ is a Cartan subalgebra.

(ii) Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. By Proposition 20.6 it contains a strongly regular element x, which is automatically semisimple. Then $\mathfrak{h} = C(x)$. □

Corollary 20.9. *(i) Any strongly regular element $x \in \mathfrak{g}$ is semisimple.*

(ii) Such x is contained in a unique Cartan subalgebra, namely $\mathfrak{h}_x = C(x)$.

Proof. (i) It is clear that if x is strongly regular then so is x_s. Since $x \in C(x_s)$ and as shown above $C(x_s)$ is a Cartan subalgebra, it follows that x is semisimple.

(ii) Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra containing x. Then $\mathfrak{h} \supset \mathfrak{h}_x$, thus by dimension count $\mathfrak{h} = \mathfrak{h}_x$. □

We note that there is also a useful notion of a **regular element**, which is an $x \in \mathfrak{g}$ for which the **ordinary** (rather than generalized) 0-eigenspace of $\mathrm{ad}x$ (i.e., the centralizer $C(x)$ of x) has dimension $\mathrm{rank}\mathfrak{g}$. Such elements don't have to be semisimple, e.g. the nilpotent Jordan block in $\mathfrak{sl}_n$ is regular. It follows from Proposition 20.9(i) that an element is strongly regular if and only if it is both regular and semisimple. For this reason, from now on we will follow standard terminology and call strongly regular elements **regular semisimple**.

Theorem 20.10. *Any two Cartan subalgebras of a complex semisimple Lie algebra $\mathfrak{g}$ are conjugate. I.e., if $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$ are two Cartan subalgebras and G a connected Lie group with Lie algebra $\mathfrak{g}$ then there exists an element $g \in G$ such that $\mathrm{Ad} g \cdot \mathfrak{h}_1 = \mathfrak{h}_2$.*

Proof. By Corollary 20.9(ii), every element $x \in \mathfrak{g}^{\mathrm{sr}}$ is contained in a unique Cartan subalgebra $\mathfrak{h}_x$. Introduce an equivalence relation on $\mathfrak{g}^{\mathrm{sr}}$ by setting $x \sim y$ if $\mathfrak{h}_x$ is conjugate to $\mathfrak{h}_y$. It is clear that if $x, y \in \mathfrak{h}$ are regular elements in a Cartan subalgebra $\mathfrak{h}$ then $\mathfrak{h}_x = \mathfrak{h}_y = \mathfrak{h}$, so for any $g \in G$, $\mathrm{Ad} g \cdot x \sim y$, and any element equivalent to y has this form. So by Lemma 20.7 the equivalence class U_y of y is open. However, by Lemma 20.5, $\mathfrak{g}^{\mathrm{sr}}$ is connected. Thus there is only one equivalence class. Hence any two Cartan subalgebras of the form $\mathfrak{h}_x$ for regular x are conjugate. This implies the result, since by Theorem 20.8 any Cartan subalgebra is of the form $\mathfrak{h}_x$. □

Remark 20.11. The same results and proofs apply over any algebraically closed field $\mathbf{k}$ of characteristic zero if we use the Zariski topology instead of the usual topology of $\mathbb{C}^n$ when working with the notions of a connected, open and dense set.

20.3. **Root systems of classical Lie algebras.**

Example 20.12. Let $\mathfrak{g}$ be the symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbf{k})$. Thus $\mathfrak{g}$ consists of square matrices A of size $2n$ such that

$$AJ + JA^T = 0$$

where $J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$, with blocks being of size n. So we get $A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$, where b, c are symmetric. A Cartan subalgebra $\mathfrak{h}$ is then spanned by matrices A such that $a = \mathrm{diag}(x_1, ..., x_n)$ and $b = c = 0$. So $\mathfrak{h} \cong \mathbf{k}^n$. In this case we have roots coming from the a-part, which are simply the roots $\mathbf{e}_i - \mathbf{e}_j$ of $\mathfrak{gl}_n \subset \mathfrak{sp}_{2n}$ (defined by the condition that $b = c = 0$) and also the roots coming form the b-part, which are $\mathbf{e}_i + \mathbf{e}_j$ (including $i = j$, when we get $2\mathbf{e}_i$), and the c-part, which gives the negatives of these roots, $-\mathbf{e}_i - \mathbf{e}_j$, including $-2\mathbf{e}_i$.

This is the **root system of type C_n.**

Example 20.13. Let $\mathfrak{g}$ be the orthogonal Lie algebra $\mathfrak{so}_{2n}(\mathbf{k})$, preserving the quadratic form $Q = x_1 x_{n+1} + ... + x_n x_{2n}$. Then the story is almost the same. The Lie algebra $\mathfrak{g}$ consists of square matrices A of size $2n$ such that

$$AJ + JA^T = 0$$

where $J = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$, with blocks being of size n. So we get $A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$, where b, c are now skew-symmetric. A Cartan subalgebra $\mathfrak{h}$ is again spanned by matrices A such that $a = \mathrm{diag}(x_1, ..., x_n)$ and $b = c = 0$. So $\mathfrak{h} \cong \mathbf{k}^n$. In this case we again have roots coming from the a-part, which are simply the roots $\mathbf{e}_i - \mathbf{e}_j$ of $\mathfrak{gl}_n \subset \mathfrak{so}_{2n}$ (defined by the condition that $b = c = 0$) and also the roots coming form the b-part, which are $\mathbf{e}_i + \mathbf{e}_j$ (but now excluding $i = j$, so only for $i \neq j$), and the c-part, which gives the negatives of these roots, $-\mathbf{e}_i - \mathbf{e}_j$, $i \neq j$.

This is the **root system of type D_n.**

Example 20.14. Let $\mathfrak{g}$ be the orthogonal Lie algebra $\mathfrak{so}_{2n+1}(\mathbf{k})$, preserving the quadratic form $Q = x_0^2 + x_1 x_{n+1} + ... + x_n x_{2n}$. Then the Lie algebra $\mathfrak{g}$ consists of square matrices A of size $2n + 1$ such that

$$AJ + JA^T = 0$$

where

$$J = \begin{pmatrix} \mathbf{1}_1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_n \\ 0 & \mathbf{1}_n & 0 \end{pmatrix},$$

So we get

$$A = \begin{pmatrix} 0 & u & -u \\ w & a & b \\ -w & c & -a^T \end{pmatrix},$$

where b, c are skew-symmetric. A Cartan subalgebra $\mathfrak{h}$ is spanned by matrices A such that $a = \mathrm{diag}(x_1, ..., x_n)$ and $b = c = 0$, $u = w = 0$. So $\mathfrak{h} \cong \mathbf{k}^n$. In this case we again have roots coming from the a-part, which are simply the roots $\mathbf{e}_i - \mathbf{e}_j$ of $\mathfrak{gl}_n \subset \mathfrak{so}_{2n+1}$ (defined by the condition that $b = c = 0$, $u = w = 0$) and also the roots coming form the b-part, which are $\mathbf{e}_i + \mathbf{e}_j$, $i \neq j$, and the c-part, which gives the negatives of these roots, $-\mathbf{e}_i - \mathbf{e}_j$, $i \neq j$. But we also have the roots coming from the w-part, which are $\mathbf{e}_i$, and from the u part, which are $-\mathbf{e}_i$.

This is the **root system of type B_n.**

18.745 Lie Groups and Lie Algebras I
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