## 21. **Root systems**

21.1. **Abstract root systems.** Let $E \cong \mathbb{R}^r$ be a Euclidean space with a positive definite inner product.

**Definition 21.1.** An **abstract root system** is a finite set $R \subset E \setminus 0$ satisfying the following axioms:

(R1) $R$ spans $E$;

(R2) For all $\alpha, \beta \in R$ the number $n_{\alpha\beta} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer;

(R3) If $\beta \in R$ then $s_\alpha(\beta) := \beta - n_{\alpha\beta}\alpha \in R$.

Elements of $R$ are called **roots**. The number $r = \dim E$ is called the **rank** of $R$.

In particular, taking $\beta = \alpha$ in R3 yields that $R$ is centrally symmetric, i.e., $R = -R$. Also note that $s_\alpha$ is the reflection with respect to the hyperplane $(\alpha, x) = 0$, so R3 just says that $R$ is invariant under such reflections.

Note also that if $R \subset E$ is a root system, $\bar{E} \subset E$ a subspace, and $R' = R \cap E'$ then $R'$ is also a root system inside $E' = \mathrm{Span}(R') \subset \bar{E}$.

For a root $\alpha$ the corresponding **coroot** $\alpha^\vee \in E^*$ is defined by the formula $\alpha^\vee(x) = \frac{2(\alpha,x)}{(\alpha,\alpha)}$. Thus $\alpha^\vee(\alpha) = 2$, $n_{\alpha\beta} = \alpha^\vee(\beta)$ and $s_\alpha(\beta) = \beta - \alpha^\vee(\beta)\alpha$.

**Definition 21.2.** A root system $R$ is **reduced** if for $\alpha, c\alpha \in R$, we have $c = \pm 1$.

**Proposition 21.3.** *If $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra then the corresponding set of roots $R$ is a reduced root system, and $\alpha^\vee = h_\alpha$.*

*Proof.* This follows immediately from Theorem 19.19. $\square$

**Example 21.4.** 1. The root system of $\mathfrak{sl}_n$ is called $A_{n-1}$. In this case, as we have seen in Example 19.14, the roots are $\mathbf{e}_i - \mathbf{e}_j$, and $s_{\mathbf{e}_i - \mathbf{e}_j} = (ij)$, the transposition of the $i$-th and $j$-th coordinates.

2. The subset $\{1, 2, -1, -2\}$ of $\mathbb{R}$ is a root system which is not reduced.

**Definition 21.5.** Let $R_1 \subset E_1, R_2 \subset E_2$ be root systems. An **isomorphism of root systems** $\phi : R_1 \to R_2$ is an isomorphism $\phi : E_1 \to E_2$ which maps $R_1$ to $R_2$ and preserves the numbers $n_{\alpha\beta}$.

So an isomorphism does not have to preserve the inner product, e.g. it may rescale it.

21.2. **The Weyl group.**

**Definition 21.6.** The **Weyl group** of a root system $R$ is the group of automorphisms of $E$ generated by $s_\alpha$.

**Proposition 21.7.** *$W$ is a finite subgroup of $O(E)$ which preserves $R$.*

*Proof.* Since $s_\alpha$ are orthogonal reflections, $W \subset O(E)$. By R3, $s_\alpha$ preserves $R$. By R1 an element of $W$ is determined by its action on $R$, hence $W$ is finite. □

**Example 21.8.** For the root system $A_{n-1}$, $W = S_n$, the symmetric group. Note that for $n \geq 3$, the automorphism $x \mapsto -x$ of $R$ is not in $W$, so $W$ is, in general, a proper subgroup of $\mathrm{Aut}(R)$.

21.3. **Root systems of rank 2.** If $\alpha, \beta$ are linearly independent roots in $R$ and $E' \subset E$ is spanned by $\alpha, \beta$ then $R' = R \cap E'$ is a root system in $E'$ of rank 2. So to classify reduced root systems, it is important to classify reduced root systems of rank 2 first.

**Theorem 21.9.** *Let $R$ be a reduced root system and $\alpha, \beta \in R$ be two linearly independent roots with $|\alpha| \geq |\beta|$. Let $\phi$ be the angle between $\alpha$ and $\beta$. Then we have one of the following possibilities:*
*(1) $\phi = \pi/2$, $n_{\alpha\beta} = n_{\beta\alpha} = 0$;*
*(2a) $\phi = 2\pi/3$, $|\alpha|^2 = |\beta|^2$, $n_{\alpha\beta} = n_{\beta\alpha} = -1$;*
*(2b) $\phi = \pi/3$, $|\alpha|^2 = |\beta|^2$, $n_{\alpha\beta} = n_{\beta\alpha} = 1$;*
*(3a) $\phi = 3\pi/4$, $|\alpha|^2 = 2|\beta|^2$, $n_{\alpha\beta} = -1$, $n_{\beta\alpha} = -2$;*
*(3b) $\phi = \pi/4$, $|\alpha|^2 = 2|\beta|^2$, $n_{\alpha\beta} = 1$, $n_{\beta\alpha} = 2$;*
*(4a) $\phi = 5\pi/6$, $|\alpha|^2 = 3|\beta|^2$, $n_{\alpha\beta} = -1$, $n_{\beta\alpha} = -3$;*
*(4b) $\phi = \pi/6$, $|\alpha|^2 = 3|\beta|^2$, $n_{\alpha\beta} = 1$, $n_{\beta\alpha} = 3$.*

*Proof.* We have $(\alpha, \beta) = 2|\alpha| \cdot |\beta| \cos\phi$, so $n_{\alpha\beta} = 2\frac{|\beta|}{|\alpha|}\cos\phi$. Thus $n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2\phi$. Hence this number can only take values $0, 1, 2, 3$ (as it is an integer by R2) and $\frac{n_{\alpha\beta}}{n_{\beta\alpha}} = \frac{|\alpha|^2}{|\beta|^2}$ if $n_{\alpha\beta} \neq 0$. The rest is obtained by analysis of each case. □

In fact, all these possibilities are realized. Namely, we have root systems $A_1 \times A_1$, $A_2$, $B_2 = C_2$ (the root system of the Lie algebras $\mathfrak{sp}_4$ and $\mathfrak{so}_5$, which are in fact isomorphic, consisting of the vertices and midpoints of edges of a square), and $G_2$, generated by $\alpha, \beta$ with $(\alpha, \alpha) = 6$, $(\beta, \beta) = 2$, $(\alpha, \beta) = -3$, and roots being $\pm\alpha, \pm\beta$, $\pm(\alpha+\beta)$, $\pm(\alpha + 2\beta)$, $\pm(\alpha + 3\beta)$, $\pm(2\alpha + 3\beta)$.

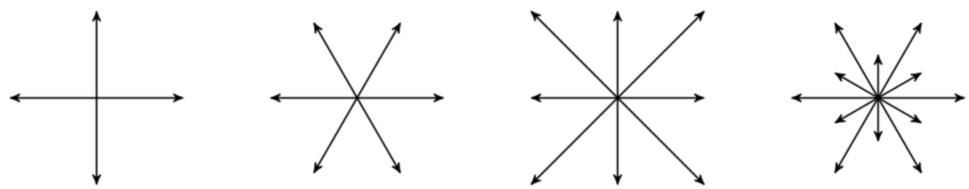
**Theorem 21.10.** *Any reduced rank 2 root system $R$ is of the form $A_1 \times A_1$, $A_2$, $B_2$ or $G_2$.*

*Proof.* Pick independent roots $\alpha, \beta \in R$ such that the angle $\phi$ is as large as possible. Then $\phi \geq \pi/2$ (otherwise can replace $\alpha$ with $-\alpha$), so we are in one of the cases $1, 2a, 3a, 4a$. Now the statement follows by inspection of each case, giving $A_1 \times A_1$, $A_2$, $B_2$ and $G_2$ respectively. $\square$

**Corollary 21.11.** *If $\alpha, \beta \in R$ are independent roots with $(\alpha, \beta) < 0$ then $\alpha + \beta \in R$.*

*Proof.* This is easy to see from the classification of rank 2 root systems. $\square$

The root systems of rank 2 are shown in the following picture.



$A_1 \times A_1$ $\quad$ $A_2$ $\quad$ $B_2 \cong C_2$ $\quad$ $G_2$

21.4. **Positive and simple roots.** Let $R$ be a reduced root system and $t \in E^*$ be such that $t(\alpha) \neq 0$ for any $\alpha \in R$. We say that a root is **positive** (with respect to $t$) if $t(\alpha) > 0$ and **negative** if $t(\alpha) < 0$. The set of positive roots is denoted by $R_+$ and of negative ones by $R_-$, so $R_+ = -R_-$ and $R = R_+ \cup R_-$ (disjoint union). This decomposition is called a **polarization** of $R$; it depends on the choice of $t$.

**Example 21.12.** Let $R$ be of type $A_{n-1}$. Then for $t = (t_1, ..., t_n)$ we have $t(\alpha) \neq 0$ for all $\alpha$ iff $t_i \neq t_j$ for any $i$, $j$. E.g. suppose $t_1 > t_2 > ... > t_n$, then we have $\mathbf{e}_i - \mathbf{e}_j \in R_+$ iff $i < j$. We see that polarizations are in bijection with permutations in $S_n$, i.e., with elements of the Weyl group, which acts simply transitively on them. We will see that this is, in fact, the case for any reduced root system.

**Definition 21.13.** A root $\alpha \in R_+$ is **simple** if it is not a sum of two other positive roots.

**Lemma 21.14.** *Every positive root is a sum of simple roots.*

*Proof.* If $\alpha$ is not simple then $\alpha = \beta + \gamma$ where $\beta, \gamma \in R_+$. We have $t(\alpha) = t(\beta) + t(\gamma)$, so $t(\beta), t(\gamma) < t(\alpha)$. If $\beta$ or $\gamma$ is not simple, we can continue this process, and it will terminate since $t$ has finitely many values on $R$. $\square$

**Lemma 21.15.** *If $\alpha, \beta \in R_+$ are simple roots then $(\alpha, \beta) \leq 0$.*

*Proof.* Assume $(\alpha, \beta) > 0$. Then $(-\alpha, \beta) < 0$ so by Lemma 21.11 $\gamma := \beta - \alpha$ is a root. If $\gamma$ is positive then $\beta = \alpha + \gamma$ is not simple. If $\gamma$ is negative then $-\gamma$ is positive so $\alpha = \beta + (-\gamma)$ is not simple. $\square$

**Theorem 21.16.** *The set $\Pi \subset R_+$ of simple roots is a basis of $E$.*

*Proof.* We will use the following linear algebra lemma:

**Lemma 21.17.** *Let $v_i$ be vectors in a Euclidean space $E$ such that $(v_i, v_j) \le 0$ when $i \ne j$ and $t(v_i) > 0$ for some $t \in E^*$. Then $v_i$ are linearly independent.*

*Proof.* Suppose we have a nontrivial relation

$$\sum_{i \in I} c_i v_i = \sum_{i \in J} c_i v_i$$

where $I, J$ are disjoint and $c_i > 0$ (clearly, every nontrivial relation can be written in this form). Evaluating $t$ on this relation, we deduce that both sides are nonzero. Now let us compute the square of the left hand side:

$$0 < |\sum_{i \in I} c_i v_i|^2 = (\sum_{i \in I} c_i v_i, \sum_{j \in J} c_j v_j) \le 0.$$

This is a contradiction. $\square$

Now the result follows from Lemma 21.15 and Lemma 21.17. $\square$

Thus the set $\Pi$ of simple roots has $r$ elements: $\Pi = (\alpha_1, ..., \alpha_r)$.

**Example 21.18.** Let us describe simple roots for classical root systems. Suppose the polarization is given by $t = (t_1, ..., t_n)$ with decreasing coordinates. Then:

1. For type $A_{n-1}$, i.e., $\mathfrak{g} = \mathfrak{sl}_n$, the simple roots are $\alpha_i := \mathbf{e}_i - \mathbf{e}_{i+1}$, $1 \le i \le n-1$.

2. For type $C_n$, i.e., $\mathfrak{g} = \mathfrak{sp}_{2n}$, the simple roots are

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ...,\ \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n,\ \alpha_n = 2\mathbf{e}_n.$$

3. For type $B_n$, i.e., $\mathfrak{g} = \mathfrak{so}_{2n+1}$, we have the same story as for $C_n$ except $\alpha_n = \mathbf{e}_n$ rather than $2\mathbf{e}_n$. Thus the simple roots are

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ...,\ \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n,\ \alpha_n = \mathbf{e}_n.$$

4. For type $D_n$, i.e., $\mathfrak{g} = \mathfrak{so}_{2n}$, the simple roots are

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ...,\ \alpha_{n-2} = \mathbf{e}_{n-2} - \mathbf{e}_{n-1},\ \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n,\ \alpha_n = \mathbf{e}_{n-1} + \mathbf{e}_n.$$

We thus obtain

**Corollary 21.19.** *Any root $\alpha \in R$ can be uniquely written as $\alpha = \sum_{i=1}^{r} n_i\alpha_i$, where $n_i \in \mathbb{Z}$. If $\alpha$ is positive then $n_i \geq 0$ for all $i$ and if $\alpha$ is negative then $n_i \leq 0$ for all $i$.*

For a positive root $\alpha$, its **height** $h(\alpha)$ is the number $\sum n_i$. So simple roots are the roots of height 1, and the height of $\mathbf{e}_i - \mathbf{e}_j$ in $R = A_{n-1}$ is $j - i$.

21.5. **Dual root system.** For a root system $R$, the set $R^\vee \subset E^*$ of $\alpha^\vee$ for all $\alpha \in R$ is also a root system, such that $(R^\vee)^\vee = R$. It is called the **dual root system** to $R$. For example, $B_n$ is dual to $C_n$, while $A_{n-1}$, $D_n$ and $G_2$ are selfdual.

Moreover, it is easy to see that any polarization of $R$ gives rise to a polarization of $R^\vee$ (using the image $t^\vee$ of $t$ under the isomorphism $E \to E^*$ induced by the inner product), and the corresponding system $\Pi^\vee$ of simple roots consists of $\alpha_i^\vee$ for $\alpha_i \in \Pi$.

21.6. **Root and weight lattices.** Recall that a **lattice** in a real vector space $E$ is a subgroup $Q \subset E$ generated by a basis of $E$. Of course, every lattice is conjugate to $\mathbb{Z}^n \subset \mathbb{R}^n$ by an element of $GL_n(\mathbb{R})$. Also recall that for a lattice $Q \subset E$ the **dual lattice** $Q^* \subset E^*$ is the set of $f \in E^*$ such that $f(v) \in \mathbb{Z}$ for all $v \in Q$. If $Q$ is generated by a basis $\mathbf{e}_i$ of $E$ then $Q^*$ is generated by the dual basis $\mathbf{e}_i^*$.

In particular, for a root system $R$ we can define the **root lattice** $Q \subset E$, which is generated by the simple roots $\alpha_i$ with respect to some polarization of $R$. Since $Q$ is also generated by all roots in $R$, it is independent on the choice of the polarization. Similarly, we can define the **coroot lattice** $Q^\vee \subset E^*$ generated by $\alpha^\vee, \alpha \in R$, which is just the root lattice of $R^\vee$.

Also we define the **weight lattice** $P \subset E$ to be the dual lattice to $Q^\vee$: $P = (Q^\vee)^*$, and the **coweight lattice** $P^\vee \subset E^*$ to be the dual lattice to $Q$: $P^\vee = Q^*$, so $P^\vee$ is the weight lattice of $R^\vee$. Thus

$$P = \{\lambda \in E : (\lambda, \alpha^\vee) \in \mathbb{Z}\, \forall \alpha \in R\},\ P^\vee = \{\lambda \in E^* : (\lambda, \alpha) \in \mathbb{Z}\, \forall \alpha \in R\}.$$

Since for $\alpha, \beta \in R$ we have $(\alpha^\vee, \beta) = n_{\alpha\beta} \in \mathbb{Z}$, we have $Q \subset P$, $Q^\vee \subset P^\vee$.

Given a system of simple roots $\Pi = \{\alpha_1, ..., \alpha_r\}$, we define **fundamental coweights** $\omega_i^\vee$ to be the dual basis to $\alpha_i$ and **fundamental weights** $\omega_i$ to be the dual basis to $\alpha_i^\vee$: $(\omega_i, \alpha_j^\vee) = (\omega_i^\vee, \alpha_j) = \delta_{ij}$. Thus $P$ is generated by $\omega_i$ and $P^\vee$ by $\omega_i^\vee$.

**Example 21.20.** Let $R$ be of type $A_1$. Then $(\alpha, \alpha^\vee) = 2$ for the unique positive root $\alpha$, so $\omega = \frac{1}{2}\alpha$, thus $P/Q = \mathbb{Z}/2$. More generally, if $R$ is of type $A_{n-1}$ and we identify $Q \cong Q^\vee, P \cong P^\vee$, then $P$ becomes the set of

$\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ such that $\sum_i \lambda_i = 0$ and $\lambda_i - \lambda_j = \mathbb{Z}$. So we have a homomorphism $\phi : P \to \mathbb{R}/\mathbb{Z}$ given by $\phi(\lambda) = \lambda_i \bmod \mathbb{Z}$ (for any $i$). Since $\sum_i \lambda_i = 0$, we have $\phi : P \to \mathbb{Z}/n$, and $\mathrm{Ker}\phi = Q$ (integer vectors with sum zero). Also it is easy to see that $\phi$ is surjective (we may take $\lambda_i = \frac{k}{n}$ for $i \neq n$ and $\lambda_n = \frac{k}{n} - k$, then $\phi(\lambda) = \frac{k}{n}$). Thus $P/Q \cong \mathbb{Z}/n$.