22. Properties of the Weyl group

22.1. Weyl chambers. Suppose we have two polarizations of a root system \( R \) defined by \( t, t' \in E \), and \( \Pi, \Pi' \) are the corresponding systems of simple roots. Are \( \Pi, \Pi' \) equivalent in a suitable sense? The answer turns out to be yes. To show this, we will need the notion of a Weyl chamber.

Note that the polarization defined by \( t \) depends only on the signs of \( (t, \alpha) \), so does not change when \( t \) is continuously deformed without crossing the hyperplanes \( (t, \alpha) = 0 \). This motivates the following definition:

**Definition 22.1.** A Weyl chamber is a connected component of the complement of the root hyperplanes \( L_\alpha \) given by the equations \( (\alpha, x) = 0 \) in \( E \) \( (\alpha \in R) \).

Thus a Weyl chamber is defined by a system of strict homogeneous linear inequalities \( \pm(\alpha, x) = 0, \alpha \in R \). More precisely, the set of solutions of such a system is either empty or a Weyl chamber.

Thus the polarization defined by \( t \) depends only on the Weyl chamber containing \( t \).

The following lemma is geometrically obvious.

**Lemma 22.2.** (i) The closure \( \overline{C} \) of a Weyl chamber \( C \) is a convex cone.

(ii) The boundary of \( \overline{C} \) is a union of codimension 1 faces \( F_i \) which are convex cones inside one of the root hyperplanes defined inside it by a system of non-strict homogeneous linear inequalities.

The root hyperplanes containing the faces \( F_i \) are called the walls of \( C \).

We have seen above that every Weyl chamber defines a polarization of \( R \). Conversely, every polarization defines the corresponding positive Weyl chamber \( C_+ \) defined by the conditions \( (\alpha, x) > 0 \) for \( \alpha \in R_+ \) (this set is nonempty since it contains \( t \), hence is a Weyl chamber). Thus \( C_+ \) is the set of vectors of the form \( \sum_{i=1}^r c_i \omega_i \) with \( c_i > 0 \). So \( C_+ \) has \( r \) faces \( L_{\alpha_1} \cap C_+, \ldots, L_{\alpha_r} \cap C_+ \).

**Lemma 22.3.** These assignments are mutually inverse bijections between polarizations of \( R \) and Weyl chambers.

**Exercise 22.4.** Prove Lemma 22.3.

Since the Weyl group \( W \) permutes the roots, it acts on the set of Weyl chambers.

**Theorem 22.5.** \( W \) acts transitively on the set of Weyl chambers.
Proof. Let us say that Weyl chambers $C, C'$ are adjacent if they share a common face $F \subset L_\alpha$. In this case it is easy to see that $s_\alpha(C) = C'$. Now given any Weyl chambers $C, C'$, pick generic $t \in C, t' \in C'$ and connect them with a straight segment. This will define a sequence of Weyl chambers visited by this segment: $C_0 = C, C_1, ..., C_m = C'$, and $C_i, C_{i+1}$ are adjacent for each $i$. So $C_i, C_{i+1}$ lie in the same $W$-orbit. Hence so do $C, C'$.

\[ \square \]

Corollary 22.6. Every Weyl chamber has $r$ walls.

Proof. This follows since it is true for the positive Weyl chamber and by Theorem 22.5 the Weyl group acts transitively on the Weyl chambers.

\[ \square \]

Corollary 22.7. Any two polarizations of $R$ are related by the action of an element $w \in W$. Thus if $\Pi, \Pi'$ are systems of simple roots corresponding to two polarizations then there is $w \in W$ such that $w(\Pi) = \Pi'$.

22.2. Simple reflections. Given a polarization of $R$ and the corresponding system of simple roots $\Pi = \{\alpha_1, ..., \alpha_r\}$, the simple reflections are the reflections $s_\alpha$, denoted by $s_i$.

Lemma 22.8. For every Weyl chamber $C$ there exist $i_1, ..., i_m$ such that $C = s_{i_1}...s_{i_m}(C_+)$. 

Proof. Pick $t \in C, t_+ \in C_+$ generically and connect them with a straight segment as before. Let $m$ be the number of chamber walls crossed by this segment. The proof is by induction in $m$ (with obvious base). Let $C'$ be the chamber entered by our segment from $C$ and $L_\alpha$ the wall separating $C, C'$, so that $C = s_\alpha(C')$. By the induction assumption $C' = u(C_+)$, where $u = s_{i_1}...s_{i_{m-1}}$. So $L_\alpha = u(L_{\alpha_j})$ for some $j$. Thus $s_\alpha = us_ju^{-1}$. Hence $C = s_\alpha(C') = s_\alpha u(C_+) = us_j(C_+)$, so we get the result with $i_m = j$.

\[ \square \]

Corollary 22.9. (i) The simple reflections $s_i$ generate $W$; 
(ii) $W(\Pi) = R$.

Proof. (i) This follows since for any root $\alpha$, the hyperplane $L_\alpha$ is a wall of some Weyl chamber, so $s_\alpha$ is a product of $s_i$. 
(ii) Follows from (i).

Thus $R$ can be reconstructed from $\Pi$ as $W(\Pi)$, where $W$ is the subgroup of $O(E)$ generated by $s_i$.

Example 22.10. For root system $A_{n-1}$ part (i) says that any element of $S_n$ is a product of transpositions of neighbors.
22.3. **Length of an element of the Weyl group.** Let us say that a root hyperplane $L_\alpha$ **separates** two Weyl chambers $C, C'$ if they lie on different sides of $L_\alpha$.

**Definition 22.11.** The **length** $\ell(w)$ of $w \in W$ is the number of root hyperplanes separating $C_+$ and $w(C_+)$. We have $t \in C_+, w(t) \in w(C_+)$, so $\ell(w)$ is the number of roots $\alpha$ such that $(t, \alpha) > 0$ but $(w(t), \alpha) = (t, w^{-1}\alpha) < 0$. Note that if $\alpha$ is a root satisfying this condition then $\beta = -w^{-1}\alpha$ satisfies the conditions $(t, \beta) > 0$, $(t, w\beta) < 0$. Thus $\ell(w) = \ell(w^{-1})$ and $\ell(w)$ is the number of positive roots which are mapped by $w$ to negative roots. Note also that the notion of length depends on the polarization of $R$ (as it refers to the positive chamber $C_+$ defined using the polarization).

**Example 22.12.** Let $s_i$ be a simple reflection. Then $s_i(C_+) = C_+$ is adjacent to $C_+$, with the only separating hyperplane being $L_{\alpha_i}$. Thus $\ell(s_i) = 1$. It follows that the only positive root mapped by $s_i$ to a negative root is $\alpha_i$ (namely, $s_i(\alpha_i) = -\alpha_i$), and thus $s_i$ permutes $R_+ \setminus \{\alpha_i\}$.

**Proposition 22.13.** Let $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$. Then $(\rho, \alpha_i^\vee) = 1$ for all $i$. Thus $\rho = \sum_{i=1}^r \omega_i$.

**Proof.** We have $\rho = \frac{1}{2} \alpha_i + \frac{1}{2} \sum_{\alpha \in R_+, \alpha \neq \alpha_i} \alpha$. Since $s_i$ permutes $R_+ \setminus \{\alpha_i\}$, we get $s_i \rho = \rho - \alpha_i$. But for any $\lambda$, $s_i \lambda = \lambda - (\lambda, \alpha_i^\vee) \alpha_i$. This implies the statement. □

The weight $\rho$ plays an important role in representation theory of semisimple Lie algebras. For instance, it occurs in the Weyl character formula for these representations which we will soon derive.

**Theorem 22.14.** Let $w = s_{i_1}...s_{i_l}$ be a representation of $w \in W$ as a product of simple reflections that has minimal possible length. Then $l = \ell(w)$.

**Proof.** As before, define a chain of Weyl chambers $C_k = s_{i_1}...s_{i_k}(C_+)$, so that $C_0 = C_+$ and $C_k = w(C_+)$. We have seen that $C_k$ and $C_{k-1}$ are adjacent. So there is a zigzag path from $C_+$ to $w(C_+)$ that intersects at most $l$ root hyperplanes (namely, the segment from $C_{k-1}$ to $C_k$ intersects only one hyperplane). Thus $\ell(w) \leq l$. On the other hand, pick generic points in $C_+$ and $w(C_+)$ and connect them with a straight segment. This segment intersects every separating root hyperplane exactly once and does not intersect other root hyperplanes, so produces an expression of $w$ as a product of $\ell(w)$ simple reflections. This implies the statement. □

An expression $w = s_{i_1}...s_{i_l}$ is called **reduced** if $l = \ell(w)$. 

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Proposition 22.15. The Weyl group $W$ acts simply transitively on Weyl chambers.

Proof. By Theorem 22.5 the action is transitive, so we just have to show that if $w(C_+) = C_+$ then $w = 1$. But in this case $\ell(w) = 0$, so $w$ has to be a product of zero simple reflections, i.e., indeed $w = 1$. \(\square\)

Thus we see that $C_+$ is a fundamental domain of the action of $W$ on $E$.

Moreover, we have

Proposition 22.16. $E/W = C_+$, i.e., every $W$-orbit on $E$ has a unique representative in $C_+$.

Proof. Suppose $\lambda, \mu \in C_+$ and $\lambda = w\mu$, where $w \in W$ is shortest possible. Assume the contrary, that $w \neq 1$. Pick a reduced decomposition $w = s_i \ldots s_{i_t}$. Let $\gamma$ be the positive root which is mapped to a negative root by $w$ but not by $s_i w$, i.e., $\gamma = s_i \ldots s_{i_t-1} \alpha_i$. Then

$$0 \leq (\mu, \gamma) = (\lambda, w\gamma) \leq 0.$$  

so $(\mu, \gamma) = 0$. Thus

$$\lambda = w\mu = s_i \ldots s_i \mu = s_i \ldots s_{i_t-1} \ldots s_i \alpha_i \mu = s_i \ldots s_{i_t-1} \ldots s_i \mu$$

which is a contradiction since $w$ was the shortest possible. \(\square\)

Corollary 22.17. Let $C_- = -C_+$ be the negative Weyl chamber. Then there exists a unique $w_0 \in W$ such that $w_0(C_+) = C_-$. We have $\ell(w_0) = |R_+|$ and for any $w \neq w_0$, $\ell(w) < \ell(w_0)$. Also $w_0^2 = 1$.

Exercise 22.18. Prove Corollary 22.17.

The element $w_0$ is therefore called the longest element of $W$.

Example 22.19. For the root system $A_{n-1}$ the element $w_0$ is the order reversing involution: $w_0(1, 2, \ldots, n) = (n, \ldots, 2, 1)$. 

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