22. **Properties of the Weyl group**

22.1. **Weyl chambers.** Suppose we have two polarizations of a root system R defined by $t, t' \in E$, and Π, Π' are the corresponding systems of simple roots. Are Π, Π' equivalent in a suitable sense? The answer turns out to be yes. To show this, we will need the notion of a Weyl chamber.

Note that the polarization defined by t depends only on the signs of (t, α), so does not change when t is continuously deformed without crossing the hyperplanes $(t, \alpha) = 0$. This motivates the following definition:

Definition 22.1. A **Weyl chamber** is a connected component of the complement of the root hyperplanes L_α given by the equations $(\alpha, x) = 0$ in E $(\alpha \in R)$.

Thus a Weyl chamber is defined by a system of strict homogeneous linear inequalities $\pm(\alpha, x) = 0$, $\alpha \in R$. More precisely, the set of solutions of such a system is either empty of a Weyl chamber.

Thus the polarization defined by t depends only on the Weyl chamber containing t.

The following lemma is geometrically obvious.

Lemma 22.2. *(i) The closure $\overline{C}$ of a Weyl chamber C is a convex cone.*

(ii) The boundary of $\overline{C}$ is a union of codimension 1 faces F_i which are convex cones inside one of the root hyperplanes defined inside it by a system of non-strict homogeneous linear inequalities.

The root hyperplanes containing the faces F_i are called the **walls** of C.

We have seen above that every Weyl chamber defines a polarization of R. Conversely, every polarization defines the corresponding **positive Weyl chamber** C_+ defined by the conditions $(\alpha, x) > 0$ for $\alpha \in R_+$ (this set is nonempty since it contains t, hence is a Weyl chamber). Thus C_+ is the set of vectors of the form $\sum_{i=1}^r c_i\omega_i$ with $c_i > 0$. So C_+ has r faces $L_{\alpha_1} \cap C_+, ..., L_{\alpha_r} \cap C_+$.

Lemma 22.3. *These assignments are mutually inverse bijections between polarizations of R and Weyl chambers.*

Exercise 22.4. Prove Lemma 22.3.

Since the Weyl group W permutes the roots, it acts on the set of Weyl chambers.

Theorem 22.5. *W acts transitively on the set of Weyl chambers.*

Proof. Let us say that Weyl chambers C, C' are **adjacent** if they share a common face $F \subset L_\alpha$. In this case it is easy to see that $s_\alpha(C) = C'$. Now given any Weyl chambers C, C', pick generic $t \in C, t' \in C'$ and connect them with a straight segment. This will define a sequence of Weyl chambers visited by this segment: $C_0 = C, C_1, ..., C_m = C'$, and C_i, C_{i+1} are adjacent for each i. So C_i, C_{i+1} lie in the same W-orbit. Hence so do C, C'. □

Corollary 22.6. *Every Weyl chamber has r walls.*

Proof. This follows since it is true for the positive Weyl chamber and by Theorem 22.5 the Weyl group acts transitively on the Weyl chambers. □

Corollary 22.7. *Any two polarizations of R are related by the action of an element $w \in W$. Thus if Π, Π' are systems of simple roots corresponding to two polarizations then there is $w \in W$ such that $w(\Pi) = \Pi'$.*

22.2. **Simple reflections.** Given a polarization of R and the corresponding system of simple roots $\Pi = \{\alpha_1, ..., \alpha_r\}$, the **simple reflections** are the reflections s_{α_i}, denoted by s_i.

Lemma 22.8. *For every Weyl chamber C there exist $i_1, ..., i_m$ such that $C = s_{i_1}...s_{i_m}(C_+)$.*

Proof. Pick $t \in C, t_+ \in C_+$ generically and connect them with a straight segment as before. Let m be the number of chamber walls crossed by this segment. The proof is by induction in m (with obvious base). Let C' be the chamber entered by our segment from C and L_α the wall separating C, C', so that $C = s_\alpha(C')$. By the induction assumption $C' = u(C_+)$, where $u = s_{i_1}...s_{i_{m-1}}$. So $L_\alpha = u(L_{\alpha_j})$ for some j. Thus $s_\alpha = us_ju^{-1}$. Hence $C = s_\alpha(C') = s_\alpha u(C_+) = us_j(C_+)$, so we get the result with $i_m = j$. □

Corollary 22.9. *(i) The simple reflections s_i generate W;*
(ii) $W(\Pi) = R$.

Proof. (i) This follows since for any root α, the hyperplane L_α is a wall of some Weyl chamber, so s_α is a product of s_i.
(ii) Follows from (i). □

Thus R can be reconstructed from Π as $W(\Pi)$, where W is the subgroup of $O(E)$ generated by s_i.

Example 22.10. For root system A_{n-1} part (i) says that any element of S_n is a product of transpositions of neighbors.

22.3. **Length of an element of the Weyl group.** Let us say that a root hyperplane L_α **separates** two Weyl chambers C, C' if they lie on different sides of L_α.

Definition 22.11. The **length** $\ell(w)$ of $w \in W$ is the number of root hyperplanes separating C_+ and $w(C_+)$.

We have $t \in C_+, w(t) \in w(C_+)$, so $\ell(w)$ is the number of roots α such that $(t, \alpha) > 0$ but $(w(t), \alpha) = (t, w^{-1}\alpha) < 0$. Note that if α is a root satisfying this condition then $\beta = -w^{-1}\alpha$ satisfies the conditions $(t, \beta) > 0$, $(t, w\beta) < 0$. Thus $\ell(w) = \ell(w^{-1})$ and $\ell(w)$ is the number of positive roots which are mapped by w to negative roots. Note also that the notion of length depends on the polarization of R (as it refers to the positive chamber C_+ defined using the polarization).

Example 22.12. Let s_i be a simple reflection. Then $s_i(C_+)$ is adjacent to C_+, with the only separating hyperplane being L_{α_i}. Thus $\ell(s_i) = 1$. It follows that the only positive root mapped by s_i to a negative root is α_i (namely, $s_i(\alpha_i) = -\alpha_i$), and thus s_i permutes $R_+ \setminus \{\alpha_i\}$.

Proposition 22.13. *Let* $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$. *Then* $(\rho, \alpha_i^\vee) = 1$ *for all* i. *Thus* $\rho = \sum_{i=1}^r \omega_i$.

Proof. We have $\rho = \frac{1}{2}\alpha_i + \frac{1}{2}\sum_{\alpha \in R_+, \alpha \neq \alpha_i} \alpha$. Since s_i permutes $R_+ \setminus \{\alpha_i\}$, we get $s_i\rho = \rho - \alpha_i$. But for any λ, $s_i\lambda = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$. This implies the statement. $\square$

The weight ρ plays an important role in representation theory of semisimple Lie algebras. For instance, it occurs in the Weyl character formula for these representations which we will soon derive.

Theorem 22.14. *Let* $w = s_{i_1}...s_{i_l}$ *be a representation of* $w \in W$ *as a product of simple reflections that has minimal possible length. Then* $l = \ell(w)$.

Proof. As before, define a chain of Weyl chambers $C_k = s_{i_1}...s_{i_k}(C_+)$, so that $C_0 = C_+$ and $C_k = w(C_+)$. We have seen that C_k and C_{k-1} are adjacent. So there is a zigzag path from C_+ to $w(C_+)$ that intersects at most l root hyperplanes (namely, the segment from C_{k-1} to C_k intersects only one hyperplane). Thus $\ell(w) \le l$. On the other hand, pick generic points in C_+ and $w(C_+)$ and connect them with a straight segment. This segment intersects every separating root hyperplane exactly once and does not intersect other root hyperplanes, so produces an expression of w as a product of $\ell(w)$ simple reflections. This implies the statement. $\square$

An expression $w = s_{i_1}...s_{i_l}$ is called **reduced** if $l = \ell(w)$.

Proposition 22.15. *The Weyl group W acts simply transitively on Weyl chambers.*

Proof. By Theorem 22.5 the action is transitive, so we just have to show that if $w(C_+) = C_+$ then $w = 1$. But in this case $\ell(w) = 0$, so w has to be a product of zero simple reflections, i.e., indeed $w = 1$. □

Thus we see that $\overline{C}_+$ is a *fundamental domain* of the action of W on E.

Moreover, we have

Proposition 22.16. $E/W = \overline{C}_+$, *i.e., every W-orbit on E has a unique representative in $\overline{C}_+$.*

Proof. Suppose $\lambda, \mu \in \overline{C}_+$ and $\lambda = w\mu$, where $w \in W$ is shortest possible. Assume the contrary, that $w \neq 1$. Pick a reduced decomposition $w = s_{i_l}...s_{i_1}$. Let γ be the positive root which is mapped to a negative root by w but not by $s_{i_l}w$, i.e., $\gamma = s_{i_1}...s_{i_{l-1}}\alpha_{i_l}$. Then $0 \le (\mu, \gamma) = (\lambda, w\gamma) \le 0$. so $(\mu, \gamma) = 0$. Thus

$$\lambda = w\mu = s_{i_l}...s_{i_1}\mu = s_{i_{l-1}}...s_{i_1}s_\gamma\mu = s_{i_{l-1}}...s_{i_1}\mu$$

which is a contradiction since w was the shortest possible. □

Corollary 22.17. *Let $C_- = -C_+$ be the* **negative Weyl chamber**. *Then there exists a unique $w_0 \in W$ such that $w_0(C_+) = C_-$. We have $\ell(w_0) = |R_+|$ and for any $w \neq w_0$, $\ell(w) < \ell(w_0)$. Also $w_0^2 = 1$.*

Exercise 22.18. Prove Corollary 22.17.

The element w_0 is therefore called the **longest element** of W.

Example 22.19. For the root system A_{n-1} the element w_0 is the order reversing involution: $w_0(1, 2, ..., n) = (n, ..., 2, 1)$.