23. **Dynkin diagrams**

23.1. **Cartan matrices and Dynkin diagrams.** Our goal now is to classify reduced root systems, which is a key step in the classification of semisimple Lie algebras. We have shown that classifying root systems is equivalent to classifying sets Π of simple roots. So we need to classify such sets Π. Before doing so, note that we have a nice notion of **direct product** of root systems.

Namely, let $R_1 \subset E_1$ and $R_2 \subset E_2$ be two root systems. Let $E = E_1 \oplus E_2$ (orthogonal decomposition) and $R = R_1 \sqcup R_2$ (with $R_1 \perp R_2$). If $t_1 \in E_1, t_2 \in E_2$ define polarizations of R_1, R_2 with systems of simple roots Π_1, Π_2 then $t = t_1 + t_2$ defines a polarization of R with $\Pi = \Pi_1 \sqcup \Pi_2$ (with $\Pi_1 \perp \Pi_2$ and $\Pi_i = \Pi \cap R_i$).

Definition 23.1. A root system R is **irreducible** if it cannot be written (nontrivially) in this way.

Lemma 23.2. *If R is a root system with system of simple roots $\Pi = \Pi_1 \sqcup \Pi_2$ with $\Pi_1 \perp \Pi_2$ then $R = R_1 \sqcup R_2$ where R_i is the root system generated by Π_i.*

Proof. If $\alpha \in \Pi_1, \beta \in \Pi_2$ then $s_\alpha(\beta) = \beta$, $s_\beta(\alpha) = \alpha$ and s_α and s_β commute. So if W_i is the group generated by $s_\alpha, \alpha \in \Pi_i$ then $W = W_1 \times W_2$, with W_1 acting trivially on Π_2 and W_2 on Π_1. Thus

$$R = W(\Pi) = W(\Pi_1 \sqcup \Pi_2) = W_1(\Pi_1) \sqcup W_2(\Pi_2) = R_1 \sqcup R_2.$$

□

Proposition 23.3. *Any root system is uniquely a union of irreducible ones.*

Proof. The decomposition is given by the maximal decomposition of Π into mutually orthogonal systems of simple roots. □

Thus it suffices to classify irreducible root systems.

As noted above, a root system is determined by pairwise inner products of positive roots. However, it is more convenient to encode them by the **Cartan matrix** A defined by

$$a_{ij} = n_{\alpha_j \alpha_i} = (\alpha_i^\vee, \alpha_j).$$

The following properties of the Cartan matrix follow immediately from Lemma 21.15, Theorem 21.9 and Theorem 21.16:

Proposition 23.4. *(i) $a_{ii} = 2$;*

(ii) a_{ij} is a nonpositive integer;

(iii) for any $i \neq j$, $a_{ij} a_{ij} = 4\cos^2\phi \in \{0, 1, 2, 3\}$, where ϕ is the angle between α_i and α_j;

(iv) Let $d_i = |\alpha_i|^2$. Then the matrix $d_i a_{ij}$ is symmetric and positive definite.

We will see later that conversely, any such matrix defines a root system.

Example 23.5. 1. Type A_{n-1}: $a_{ii} = 2, a_{i,i+1} = a_{i+1,i} = -1$, $a_{ij} = 0$ otherwise.

2. Type B_n: $a_{ii} = 2$, $a_{i,i+1} = a_{i+1,i} = -1$ except that $a_{n,n-1} = -2$.

3. Type C_n: transposed to B_n.

4. Type D_n: same as B_n but $a_{n-1,n-2} = a_{n,n-2} = a_{n-2,n} = a_{n-2,n-1} = -1$, $a_{n,n-1} = a_{n-1,n} = 0$.

5. Type G_2: $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

It is convenient to encode such matrices by **Dynkin diagrams:**

- Indices i are vertices;
- Vertices i and j are connected by $a_{ij}a_{ji}$ lines;
- If $a_{ij} \neq a_{ji}$, i.e., $|\alpha_i|^2 \neq |\alpha_j|^2$, then the arrow on the lines goes from long root to short root ("less than" sign).

It is clear that such diagram completely determines the Cartan matrix (if we fix the labeling of vertices), and vice versa. Also it is clear that the root system is irreducible if and only if its Dynkin diagram is connected.

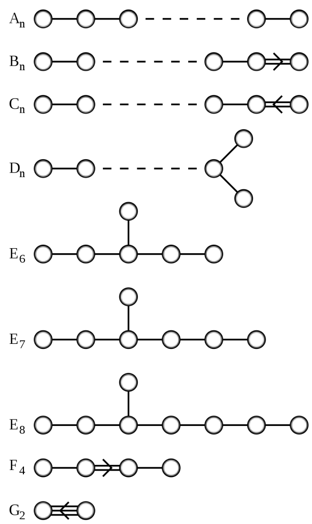
Proposition 23.6. *The Cartan matrix determines the root system uniquely.*

Proof. We may assume the Dynkin diagram is connected. The Cartan matrix determines, for any pair of simple roots, the angle between them (which is right or obtuse) and the ratio of their lengths if they are not orthogonal. By the classification of rank 2 root systems, this determines the inner product on simple roots, up to scaling, which implies the statement. □

23.2. **Classification of Dynkin diagrams.** The following theorem gives a complete classification of irreducible root systems.

Theorem 23.7. *(i) Connected Dynkin diagrams are classified by the list given i in the picture below, i.e., they are A_n, B_n, C_n, D_n, G_2 which we have already met, along with four more: F_4, E_6, E_7, E_8.*

(ii) Every matrix satisfying the conditions of Proposition 23.4 is a Cartan matrix of some root system.



The proof of Theorem 23.7 is rather long but direct. It consists of several steps. The first step is construction of the remaining root systems F_4, E_6, E_7, E_8.

23.3. **The root system F_4.**

Definition 23.8. The root system F_4 is the union of the root system $B_4 \subset \mathbb{R}^4$ with the vectors

$$\left(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\right) = \sum_{i=1}^{4}(\pm\tfrac{1}{2}\mathbf{e}_i),$$

for all choices of signs.

Thus besides the roots of B_4, which are $\pm\mathbf{e}_i \pm \mathbf{e}_j$ (24 of them, squared length 2) and $\pm\mathbf{e}_i$ (8 of them, squared length 1), we have the 16 new roots $\sum_{i=1}^{4}(\pm\frac{1}{2}\mathbf{e}_i)$ (squared length 1); this gives a total of 48.

Exercise 23.9. Check that this is an irreducible root system.

To give a polarization of the F_4 root system, pick $t = (t_1, t_2, t_3, t_4)$ with $t_1 \gg t_2 \gg t_3 \gg t_4$.

Exercise 23.10. Check that for this polarization, the simple positive roots are, $\alpha_1 = \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4)$, $\alpha_2 = \mathbf{e}_4$, $\alpha_3 = \mathbf{e}_3 - \mathbf{e}_4$, $\alpha_4 = \mathbf{e}_2 - \mathbf{e}_3$. Thus $\alpha_1^\vee = \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4$, $\alpha_2^\vee = 2\mathbf{e}_4$, $\alpha_3^\vee = \mathbf{e}_3 - \mathbf{e}_4$, $\alpha_4^\vee = \mathbf{e}_2 - \mathbf{e}_3$. So the Cartan matrix has the form

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

which gives the Dynkin diagram of F_4.

23.4. **The root system E_8.**

Definition 23.11. The root system E_8 is the union of the root system $D_8 \subset \mathbb{R}^8$ with the vectors $\sum_{i=1}^{8}(\pm\frac{1}{2}\mathbf{e}_i)$, for all choices of signs with even number of minuses.

Thus besides the roots of D_8, $\pm\mathbf{e}_i \pm \mathbf{e}_j$ (112 of them), we have 128 new roots $\sum_{i=1}^{8}(\pm\frac{1}{2}\mathbf{e}_i)$. So in total we have 240 roots. All roots have squared length 2.

Exercise 23.12. Show that it is an irreducible root system.

To give a polarization of the E_8 root system, pick t so that $t_i \gg t_{i+1}$.

Exercise 23.13. Check that for this polarization, the simple positive roots are, $\alpha_1 = \frac{1}{2}(\mathbf{e}_1+\mathbf{e}_8-\sum_{i=2}^{7}\mathbf{e}_i)$, $\alpha_2 = \mathbf{e}_7+\mathbf{e}_8$ and $\alpha_i = \mathbf{e}_{10-i}-\mathbf{e}_{11-i}$ for $3 \le i \le 8$. Thus the roots $\alpha_2, ..., \alpha_8$ generate the root system D_7, while $a_{13} = -1$ and $a_{1i} = 0$ for all $i \neq 1, 3$. In other words, the Cartan matrix has the form

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

This recovers the Dynkin diagram E_8.

23.5. **The root system E_7.**

Definition 23.14. The root system E_7 is the subsystem of E_8 generated by $\alpha_1, ..., \alpha_7$.

Note that these roots (unlike $\alpha_8 = \mathbf{e}_2 - \mathbf{e}_3$) satisfy the equation $x_1 + x_2 = 0$. Thus E_7 is the intersection of E_8 with this subspace. So it includes the roots $\pm\mathbf{e}_i \pm \mathbf{e}_j$ with $3 \le i, j \le 8$ distinct (60 roots), $\pm(\mathbf{e}_1 - \mathbf{e}_2)$ (2 roots) and $\sum_{i=1}^{8}(\pm\frac{1}{2}\mathbf{e}_i)$ with even number of minuses and the opposite signs for $\mathbf{e}_1$ and $\mathbf{e}_2$ (64 roots). Altogether we get 126 roots. The Cartan matrix is the upper left corner 7 by 7 submatrix of the Cartan matrix of E_8, so it is

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

23.6. **The root system E_6.**

Definition 23.15. The root system E_6 is the subsystem of E_8 and E_7 generated by $\alpha_1, ..., \alpha_6$.

Note that these roots (unlike $\alpha_8 = \mathbf{e}_2 - \mathbf{e}_3$ and $\alpha_7 = \mathbf{e}_3 - \mathbf{e}_4$) satisfy the equations $x_1 + x_2 = 0, x_2 - x_3 = 0$. Thus E_6 is the intersection of E_8 with this subspace. So it includes the roots $\pm\mathbf{e}_i \pm \mathbf{e}_j$ with $4 \le i, j \le 8$ distinct (40 roots), and $\sum_{i=1}^{8}(\pm\frac{1}{2}\mathbf{e}_i)$ with even number of minuses and the opposite signs for $\mathbf{e}_1$ and $\mathbf{e}_2$ and for $\mathbf{e}_1$ and $\mathbf{e}_3$ (32 roots). Altogether we get 72 roots. The Cartan matrix is the upper left corner 6 by 6 submatrix of the Cartan matrix of E_8, so it is

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

This recovers the Dynkin diagram E_6.

23.7. **The elements ρ and $\rho^\vee$.** Recall that the elements $\rho \in \mathfrak{h}^*$ and $\rho^\vee \in \mathfrak{h}$ for a simple Lie algebra $\mathfrak{g}$ are defined by the conditions $(\rho, \alpha_i^\vee) = (\rho^\vee, \alpha_i) = 1$ for all i. So for classical Lie algebras they can be computed from Example 21.18. Namely, we get

$$\rho_{A_{n-1}} = \rho^\vee_{A_{n-1}} = (\tfrac{n-1}{2}, \tfrac{n-3}{2}, ..., -\tfrac{n-1}{2}),$$
$$\rho_{B_n} = \rho^\vee_{C_n} = (\tfrac{2n-1}{2}, ..., \tfrac{3}{2}, \tfrac{1}{2}),$$
$$\rho_{C_n} = \rho^\vee_{B_n} = (n, n-1, ..., 1),$$
$$\rho_{D_n} = \rho^\vee_{D_n} = (n-1, n-2, ..., 0).$$

Exercise 23.16. Show that the elements ρ and $\rho^\vee$ for exceptional root systems (in the above realizations) are as follows:

$$\rho_{G_2} = 3\alpha + 5\beta, \ \rho^\vee_{G_2} = 5\alpha^\vee + 3\beta^\vee,$$

$$\rho_{F_4} = (\tfrac{11}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}),\ \rho_{F_4}^\vee = (8,3,2,1),$$
$$\rho_{E_8} = \rho_{E_8}^\vee = (23,6,5,4,3,2,1,0),$$
$$\rho_{E_7} = \rho_{E_7}^\vee = (\tfrac{17}{2}, -\tfrac{17}{2}, 5,4,3,2,1,0),$$
$$\rho_{E_6} = \rho_{E_6}^\vee = (4,-4,-4,4,3,2,1,0).$$

(recall that we realized E_6, E_7, E_8 inside $\mathbb{R}^8$).

23.8. **Proof of Theorem 23.7.** Now that we have shown that there exist root systems attached to all Cartan matrices, it remains to classify Cartan matrices (or Dynkin diagrams), i.e. show that there are no others than those we have considered. For this purpose we consider Dynkin diagrams as graphs with certain kind of special edges (with one, two or three lines and a possible orientation). Note first that any subgraph of a Dynkin diagram must itself be a Dynkin diagram, since a principal submatrix of a positive definite symmetric matrix is itself positive definite. On the other hand, consider **untwisted and twisted affine Dynkin diagrams** depicted on the first picture at `https://en.wikipedia.org/wiki/Affine_Lie_algebra`. These are not Dynkin diagrams since the corresponding matrix A is degenerate, hence not positive definite.

Exercise 23.17. Prove this by showing that in each case there exists a nonzero vector v such that $Av = 0$. For example, in the simply laced case (only simple edges), this amounts to finding a labeling of the vertices by nonzero numbers such that the sum of labels of the neighbors to each vertex is twice the label of that vertex, and in the non-simply laced case it's a weighted version of that.

Thus they cannot occur inside a Dynkin diagram.

We conclude that a Dynkin diagram is a tree. Indeed, it cannot have a loop with simple edges, since this is the affine diagram $\widetilde{A}_{n-1}$, which has a null vector $(1, ..., 1)$. If there is a loop with non-simple edges, this is even worse - this vector will have a negative inner product with itself.

Further, it cannot have vertices with more than four simple edges coming out since it cannot have a subdiagram $\widetilde{D}_4$ (and for non-simple edges it is even worse, as before). Thus all the vertices of our tree are i-valent for $i \le 3$.

Also we cannot have a subdiagram $\widetilde{D}_n$, $n \ge 5$, which implies that there is at most one trivalent vertex.

Further, if there is a triple edge then the diagram is G_2. There is no way to attach any edge to the G_2 diagram because $D_4^{(3)}$ and $\widetilde{G}_2$ are forbidden.

Next, if there is a trivalent vertex then there cannot be a non-simple edge anywhere in the diagram (as we have forbidden affine diagrams $A^{(2)}_{2k-1}, \widetilde{B}_n$). So in this case the diagram is simply laced, so it must be on our list (D_n, E_6, E_7, E_8) since it cannot contain affine diagrams $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$.

It remains to consider chain-shaped diagrams. They can't contain two double edges (affine diagrams $A^{(2)}_{2k}, D^{(2)}_{k+1}, \widetilde{C}_n$). Thus if the double edge is at the end, we can only get B_n and C_n.

Finally, if the double edge is in the middle, we can't have affine subdiagram $\widetilde{F}_4$ or $E^{(2)}_6$, so our diagram must be F_4. Theorem 23.7 is proved.

Remark 23.18. Note that we have **exceptional isomorphisms** $D_2 \cong A_1 \times A_1$, $D_3 \cong A_3$, $B_2 \cong C_2$. Otherwise the listed root systems are distinct.

23.9. **Simply laced and non-simply laced diagrams.** As we already mentioned, a Dynkin diagram (or the corresponding root system) is called **simply laced** if all the edges are simple, i.e. $a_{ij} = 0, -1$ for $i \neq j$. This is equivalent to the Cartan matrix being symmetric, or to all roots having the same length. The connected simply-laced diagrams are $A_n, n \geq 1; D_n, n \geq 4; E_6, E_7, E_8$. The remaining diagrams B_n, C_n, F_4, G_2 are not simply laced, but they contain roots of only two squared lengths, whose ratio is 2 for double edge (B_n, C_n, F_4) and 3 for triple edge (G_2). The roots of the bigger length are called **long** and of the smaller length are called **short**.

It is easy to see that long and short roots form a root system of the same rank (but not necessarily irreducible). For instance, in G_2 both form a root system of type A_2, and in B_2 both are $A_1 \times A_1$. In B_3 long roots form D_3 and short ones form $A_1 \times A_1 \times A_1$. However, only long roots form a root subsystem, since a long positive root can be the sum of two short ones, but not vice versa.