24. **Construction of a semisimple Lie algebra from a Dynkin diagram**

24.1. **Serre relations.** Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. We would like to show that any reduced root system gives rise to a semisimple Lie algebra over $\mathbf{k}$, and moreover a unique one. To this end, it suffices to show that any reduced *irreducible* root system gives rise to a unique (finite dimensional) *simple* Lie algebra.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbf{k}$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and root system $R \subset \mathfrak{h}^*$ (which is thus reduced and irreducible). Fix a polarization of R with the set of simple roots $\Pi = (\alpha_1, ..., \alpha_r)$, and let $A = (a_{ij})$ be the Cartan matrix of R. We have a decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{n}_\pm := \oplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$ are the Lie subalgebras spanned by positive, respectively negative root vectors. Pick elements $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ so that $e_i, f_i, h_i = [e_i, f_i]$ form an $\mathfrak{sl}_2$-triple.

Theorem 24.1. *(Serre relations) (i) The elements e_i, f_i, h_i, $i = 1, ..., r$ generate $\mathfrak{g}$.*

(ii) These elements satisfy the following relations:

$$[h_i, h_j] = 0,\ [h_i, e_j] = a_{ij} e_j,\ [h_i, f_j] = -a_{ij} f_j,\ [e_i, f_j] = \delta_{ij} h_i,$$

$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0,\ (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0,\ i \neq j.$$

The last two sets of relations are called **Serre relations**. Note that if $a_{ij} = 0$ then the Serre relations just say that $[e_i, e_j] = [f_i, f_j] = 0$.

Proof. (i) We know that h_i form a basis of $\mathfrak{h}$, so it suffices to show that e_i generate $\mathfrak{n}_+$ and f_i generate $\mathfrak{n}_-$. We only prove the first statement, the second being the same for the opposite polarization.

Let $\mathfrak{n}_+' \subset \mathfrak{n}_+$ be the Lie subalgebra generated by e_i. It is clear that $\mathfrak{n}_+' = \oplus_{\alpha \in R_+'} \mathfrak{g}_\alpha$ where $R_+' \subset R_+$. Assume the contrary, that $R_+' \neq R_+$. Pick $\alpha \in R_+ \setminus R_+'$ with the smallest height (it is not a simple root). Then $\mathfrak{g}_{\alpha - \alpha_i} \subset \mathfrak{n}_+'$, so $[e_i, \mathfrak{g}_{\alpha-\alpha_i}] = 0$. Let $x \in \mathfrak{g}_{-\alpha}$ be a nonzero element. We have

$$([x, e_i], y) = (x, [e_i, y]) = 0$$

for any $y \in \mathfrak{g}_{\alpha - \alpha_i}$. Thus $[x, e_i] = 0$ for all i, which implies, by the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4), that $(\alpha, \alpha_i^\vee) \leq 0$ for all i, hence $(\alpha, \alpha_i) \leq 0$ for all i. This would imply that $(\alpha, \alpha) \leq 0$, a contradiction. This proves (i).

(ii) All the relations except the Serre relations follow from the definition and properties of root systems. So only the Serre relations require proof. We prove only the relation involving f_i, the other one being the same for the opposite polarization. Consider the $(\mathfrak{sl}_2)_i$-submodule M_{ij} of $\mathfrak{g}$ generated by f_j. It is finite dimensional and we have $[h_i, f_j] = -a_{ij} f_j$, $[e_i, f_j] = 0$. Thus by the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4) we must have $M_{ij} \cong V_{-a_{ij}}$. Hence $(\mathrm{ad} f_i)^{-a_{ij}+1} f_j = 0$. □

24.2. **The Serre presentation for semisimple Lie algebras.** Now for any reduced root system R let $\mathfrak{g}(R)$ be the Lie algebra generated by $e_i, f_i, h_i, i = 1, ..., r$, with **defining relations** being the relations of Theorem 24.1. Precisely, this means that $\mathfrak{g}(R)$ is the quotient of the free Lie algebra FL_{3r} with generators e_i, f_i, h_i modulo the Lie ideal generated by the differences of the left and right hand sides of these relations.

Theorem 24.2. *(Serre) (i) The Lie subalgebra $\mathfrak{n}_+$ of $\mathfrak{g}(R)$ generated by e_i has the Serre relations $(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0$ as the defining relations. Similarly, the Lie subalgebra $\mathfrak{n}_-$ of $\mathfrak{g}(R)$ generated by f_i has the Serre relations $(\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0$ as the defining relations. In particular, $e_i, f_i \neq 0$ in $\mathfrak{g}(R)$. Moreover, h_i are linearly independent.*

(ii) $\mathfrak{g}(R)$ is a sum of finite dimensional modules over every simple root subalgebra $(\mathfrak{sl}_2)_i = (e_i, f_i, h_i)$.

(iii) $\mathfrak{g}(R)$ is finite dimensional.

(iv) $\mathfrak{g}(R)$ is semisimple and has root system R.

Proof. It is easy to see that $\mathfrak{g}(R_1 \sqcup R_2) = \mathfrak{g}(R_1) \oplus \mathfrak{g}(R_2)$, so it suffices to prove the theorem for irreducible root systems.

(i) Consider the (in general, infinite dimensional) Lie algebra $\widetilde{\mathfrak{g}(R)}$ generated by e_i, f_i, h_i with the defining relations of Theorem 24.1 without the Serre relations. This Lie algebra is $\mathbb{Z}$-graded, with $\deg(e_i) = 1$, $\deg(f_i) = -1$, $\deg(h_i) = 0$. Thus we have a decomposition

$$\widetilde{\mathfrak{g}(R)} = \widetilde{\mathfrak{n}_+} \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}_-},$$

where $\widetilde{\mathfrak{n}_+}$, $\widetilde{\mathfrak{h}}$ and $\widetilde{\mathfrak{n}_-}$ are Lie subalgebras spanned by elements of positive, zero and negative degree, respectively. Moreover, it is easy to see that $\widetilde{\mathfrak{n}_+}$ is generated by e_i, $\widetilde{\mathfrak{n}_-}$ is generated by f_i, and $\widetilde{\mathfrak{h}}$ is spanned by h_i (indeed, any commutator can be simplified to have only e_i, only f_i, or only a single h_i).

Lemma 24.3. *(i) The Lie algebra $\widetilde{\mathfrak{n}_+}$ is free on the generators e_i and $\widetilde{\mathfrak{n}_-}$ is free on the generators f_i.*

(ii) h_i are linearly independent in $\widetilde{\mathfrak{h}}$ (i.e., $\widetilde{\mathfrak{h}} \cong \mathfrak{h}$).

Proof. (i) We prove only the second statement, the first one being the same for the opposite polarization. Let $\mathfrak{h}'$ be a vector space with basis

h'_i, $i = 1, ..., r$ and consider the Lie algebra $\mathfrak{a} := \mathfrak{h}' \ltimes FL_r$, where FL_r is freely generated by $f'_1, ..., f'_r$ and

$$[h'_i, f'_j] = -a_{ij} f'_j,\ [h'_i, h'_j] = 0.$$

Consider the universal enveloping algebra

$$U = U(\mathfrak{a}) = \mathbf{k}[h'_1, ..., h'_r] \ltimes \mathbf{k}\langle f'_1, ..., f'_r \rangle,$$

which as a vector space is naturally identified with the tensor product $\mathbf{k}\langle f_1, ..., f_r\rangle \otimes \mathbf{k}[h'_1, ..., h'_r]$, via $f \otimes h \mapsto fh$ (by Proposition 14.4). Now define an action of $\widetilde{\mathfrak{g}(R)}$ on the space U as follows. For $P \in \mathbf{k}[h'_1, ..., h'_r]$ and w a word in f'_i of weight $-\alpha$, we set

$$h_i(w \otimes P) = w \otimes (h'_i - \alpha(h_i))P,\ f_i(w \otimes P) = f'_i w \otimes P,$$

$$e_i(f'_{j_1}...f'_{j_s} \otimes P) = \sum_{k: j_k = i} f'_{j_1}....\widehat{f'_{j_k}}...f'_{j_s} \otimes (h'_i - (\alpha_{j_{k+1}} + ... + \alpha_{j_s})(h_i))P$$

(where the hat means that the corresponding factor is omitted). It is easy to check that this indeed defines an action, i.e., the relations of $\widetilde{\mathfrak{g}(R)}$ are satisfied (check it!). Thus we have a linear map $\widetilde{\mathfrak{g}(R)} \to U$ given by $x \mapsto x(1)$. The restriction of this map to the Lie subalgebra $\widetilde{\mathfrak{n}_-}$ is a map $\phi : \widetilde{\mathfrak{n}}_- \to FL_r$ which sends every iterated commutator of f_i to itself. This implies that ϕ is an isomorphism, i.e., $\widetilde{\mathfrak{n}_-}$ is free.

(ii) The elements $h_i(1) = h'_i$ are linearly independent, hence so are h_i. □

Now consider the element $S^+_{ij} := (\mathrm{ad} e_i)^{1-a_{ij}} e_j$ in $\widetilde{\mathfrak{n}_+}$ and $S^-_{ij} := (\mathrm{ad} f_i)^{1-a_{ij}} f_j$ in $\widetilde{\mathfrak{n}_-}$. It is easy to check that $[f_k, S^+_{ij}] = 0$ (this follows easily from the representation theory of $\mathfrak{sl}_2$, Subsection 11.4,–check it!). Therefore, setting I_+ to be the ideal in the Lie algebra $\widetilde{\mathfrak{n}_+}$ generated by S^+_{ij}, and I_- to be the ideal in the Lie algebra $\widetilde{\mathfrak{n}_-}$ generated by S^-_{ij}, we see that the ideal of Serre relations in $\widetilde{\mathfrak{g}(R)}$ is $I_+ \oplus I_-$. Lemma 24.3 now implies (i).

(ii) The Serre relations imply that e_j generates the representation $V_{-a_{ij}}$ of $(\mathfrak{sl}_2)_i$ for $j \neq i$, and so does f_j. Also any element of $\mathfrak{h}$ generates V_0 or V_2 or the sum of the two, and e_i, f_i generate V_2. This implies (ii) since $\mathfrak{g}(R)$ is generated by e_i, f_i, h_i, and if x generates a representation X of $(\mathfrak{sl}_2)_i$ and y generates a representation Y then $[x, y]$ generates a quotient of $X \otimes Y$.

(iii) We have $\mathfrak{g}(R) = \oplus_{\alpha \in Q} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ are the subspaces of $\mathfrak{g}(R)$ of weight α, and $\mathfrak{g}_0 = \mathfrak{h}$. Let Q_+ be the $\mathbb{Z}_+$-span of α_i. Then $\mathfrak{g}_\alpha$ is zero unless $\alpha \in Q_+$ or $-\alpha \in Q_+$, and is finite dimensional for any α.

We will now show that if $\mathfrak{g}_\alpha \neq 0$ then $\alpha \in R$ or $\alpha = 0$, which implies (iii). It suffices to consider $\alpha \in Q_+$. We prove the statement by induction in the height $\mathrm{ht}(\alpha) = \sum_i k_i$ where $\alpha = \sum_i k_i \alpha_i$. The base case (height 1) is obvious, so we only need to justify the inductive step. We have $(\alpha, \omega_i^\vee) = k_i \ge 0$ for all i. If there is only one i with $k_i \ge 0$ then the statement is clear since $\mathfrak{g}_{m\alpha_i} = 0$ if $m \ge 2$. (as $\mathfrak{n}_+$ is generated by e_i). So assume that there are at least two such indices i. Since $(\alpha, \alpha) > 0$, there exists i such that $(\alpha, \alpha_i^\vee) > 0$. By the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4), $\mathfrak{g}_{s_i\alpha} \neq 0$. Clearly, $s_i\alpha = \alpha - (\alpha, \alpha_i^\vee)\alpha_i \notin -Q_+$ (since $k_j > 0$ for at least two indices j), so $s_i\alpha \in Q_+$ but has height smaller than α (as $(\alpha, \alpha_i^\vee) > 0$). So by the induction assumption $s_i\alpha \in R$, which implies $\alpha \in R$. This proves (iii).

(iv) We see that $\mathfrak{g}(R) = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ are 1-dimensional (this follows from (ii),(iii) since every root can be mapped to a simple root by a composition of simple reflections). Let I be a nonzero ideal in $\mathfrak{g}$. Then $I \supset \mathfrak{g}_\alpha$ for some $\alpha \neq 0$. Also, by the representation theory of $\mathfrak{sl}_2$, $I_\beta \neq 0$ implies $I_{w\beta} \neq 0$ for all $w \in W$. Thus $I_{\alpha_i} \neq 0$ for some i, i.e., $e_i \in I$. Hence $h_i, f_i \in I$. Now let J be the set of indices j for which $e_j, f_j, h_j \in I$ (or, equivalently, just $e_j \in I$); we have shown it is nonempty. Since $[h_j, e_k] = a_{jk} e_k$, we find that if $j \in J$ and $a_{jk} \neq 0$ (i.e., k is connected to j in the Dynkin diagram) then $k \in J$. Since the Dynkin diagram is connected, $J = [1, ..., r]$ and $I = \mathfrak{g}$. Thus $\mathfrak{g}$ is simple and clearly has root system R. This proves (iv) and completes the proof of Serre's theorem. □

Corollary 24.4. *Isomorphism classes of simple Lie algebras over* $\mathbf{k}$ *are in bijection with Dynkin diagrams* A_n, $n \ge 1$, B_n, $n \ge 2$, C_n, $n \ge 3$, D_n, $n \ge 4$, E_6, E_7, E_8, F_4 *and* G_2.

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