25. **Representation theory of semisimple Lie algebras**

25.1. **Representations of semisimple Lie algebras.** We will now develop representation theory of complex semisimple Lie algebras. The representation theory of semisimple Lie algebras over an algebraically closed field of characteristic zero is completely parallel, so we will stick to the complex case. So all representations will be over $\mathbb{C}$. We will mostly be interested in finite dimensional representations; as we know, they can be exponentiated to holomorphic representations of the corresponding simply connected Lie group G, which defines a bijection between isomorphism classes of such representations of $\mathfrak{g}$ and G.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Recall that by Theorem 18.9, every finite dimensional representation of $\mathfrak{g}$ is completely reducible, so to classify finite dimensional representations it suffices to classify irreducible representations.

As in the simplest case of $\mathfrak{sl}_2$, a crucial tool is the decomposition of a representation in a direct sum of eigenspaces of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Definition 25.1. Let $\lambda \in \mathfrak{h}^*$, and V a representation of $\mathfrak{g}$ (possibly infinite dimensional). Then a vector $v \in V$ is said to have **weight** λ if $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$; such vectors are called **weight vectors**. The subspace of such vectors is called the **weight subspace of** V **of weight** λ and denoted by $V[\lambda]$. If $V[\lambda] \neq 0$, we say that λ is a weight of V, and the set of weights of V is denoted by $P(V)$.

It is easy to see that $\mathfrak{g}_\alpha V[\lambda] \subset V[\lambda + \alpha]$.

Let $V' \subset V$ be the span of all weight vectors in V. Then it is clear that $V' = \oplus_{\lambda \in \mathfrak{h}} V[\lambda]$.

Definition 25.2. We say that V **has a weight decomposition** (with respect to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$) if $V' = V$, i.e., if $V = \oplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$.

Note that not every representation of $\mathfrak{g}$ has a weight decomposition (e.g., for $V = U(\mathfrak{g})$ with $\mathfrak{g}$ acting by left multiplication all weight subspaces are zero).

Proposition 25.3. *Any finite dimensional representation V of $\mathfrak{g}$ has a weight decomposition. Moreover, all weights of V are* **integral**, *i.e., $P(V)$ is a finite subset of the weight lattice $P \subset \mathfrak{h}^*$ of $\mathfrak{g}$.*

Proof. For each $i = 1, ..., r$, V is a finite dimensional representation of the root subalgebra $(\mathfrak{sl}_2)_i$, so its element h_i acts semisimply on V. Thus $\mathfrak{h}$ acts semisimply on V, hence V has a weight decomposition. Also eigenvalues of h_i are integers, so for any $\lambda \in P(V)$ we have $\lambda(h_i) = (\lambda, \alpha_i^\vee) \in \mathbb{Z}$, hence $\lambda \in P$. $\square$

Definition 25.4. A vector v in $V[\lambda]$ is called a **highest weight vector of weight** λ if $e_i v = 0$ for all i, i.e., if $\mathfrak{n}_+ v = 0$. A representation V of $\mathfrak{g}$ is a **highest weight representation with highest weight** λ if it is generated by such a nonzero vector.

Proposition 25.5. *Any finite dimensional representation $V \neq 0$ contains a nonzero highest weight vector of some weight λ. Thus every irreducible finite dimensional representation of $\mathfrak{g}$ is a highest weight representation.*

Proof. Note that $P(V)$ is a finite set. Let $\rho^\vee = \sum_{i=1}^r \omega_i^\vee$. Pick $\lambda \in P(V)$ so that $(\lambda, \rho^\vee)$ is maximal. Then $\lambda + \alpha_i \notin P(V)$ for any i, since $(\lambda+\alpha_i, \rho^\vee) = (\lambda, \rho^\vee)+1$. Hence for any nonzero $v \in V[\lambda]$ (which exists as $\lambda \in P(V)$) we have $e_i v = 0$.

The second statement follows since an irreducible representation is generated by each its nonzero vector. □

25.2. **Verma modules.** Even though we are mostly interested in finite dimensional representations of $\mathfrak{g}$, it is useful to consider some infinite dimensional representations, which are called **Verma modules**.

The Verma module M_λ is defined as "the largest highest weight representation with highest weight λ". Namely, it is generated by a single highest weight vector v_λ with **defining relations** $hv = \lambda(h)v$ for $h \in \mathfrak{h}$ and $e_i v = 0$. More formally speaking, we make the following definition.

Definition 25.6. Let $I_\lambda \subset U(\mathfrak{g})$ be the left ideal generated by the elements $h - \lambda(h), h \in \mathfrak{h}$ and e_i, $i = 1, ..., r$. Then the **Verma module** M_λ is the quotient $U(\mathfrak{g})/I_\lambda$.

In this realization, the highest weight vector v_λ is just the class of the unit 1 of $U(\mathfrak{g})$.

Proposition 25.7. *The map $\phi : U(\mathfrak{n}_-) \to M_\lambda$ given by $\phi(x) = xv_\lambda$ is an isomorphism of left $U(\mathfrak{n}_-)$-modules.*

Proof. By the PBW theorem, the multiplication map

$$\xi : U(\mathfrak{n}_-) \otimes U(\mathfrak{h} \oplus \mathfrak{n}_+) \to U(\mathfrak{g})$$

is a linear isomorphism. It is easy to see that $\xi^{-1}(I_\lambda) = U(\mathfrak{n}_-) \otimes K_\lambda$, where

$$K_\lambda := \sum_i U(\mathfrak{h} \oplus \mathfrak{n}_+)(h_i - \lambda(h_i)) + \sum_i U(\mathfrak{h} \oplus \mathfrak{n}_+)e_i$$

is the kernel of the homomorphism $\lambda_+ : U(\mathfrak{h} \oplus \mathfrak{n}_+) \to \mathbb{C}$ given by $\lambda_+(h) = \lambda(h)$, $h \in \mathfrak{h}$, $\lambda_+(e_i) = 0$. Thus, we have a natural isomorphism of left $U(\mathfrak{n}_-)$-modules

$$U(\mathfrak{n}_-) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h} \oplus \mathfrak{n}_+)/K_\lambda \to M_\lambda,$$

as claimed. □

Remark 25.8. The definition of M_λ means that it is the **induced module** $U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h} \oplus \mathfrak{n}_+$ on which it acts via λ_+.

Recall that Q_+ denotes the set of elements $\sum_{i=1}^r k_i\alpha_i$ where $k_i \in \mathbb{Z}_{\geq 0}$. We obtain

Corollary 25.9. *M_λ has a weight decomposition with $P(M_\lambda) = \lambda - Q_+$, $\dim M_\lambda[\lambda] = 1$, and weight subspaces of M_λ are finite dimensional.*

Proposition 25.10. *(i) (Universal property of Verma modules) If V is a representation of $\mathfrak{g}$ and $v \in V$ is a vector such that $hv = \lambda(h)v$ for $h \in h$ and $e_iv = 0$ for $1 \leq i \leq r$ then there is a unique homomorphism $\eta : M_\lambda \to V$ such that $\eta(v_\lambda) = v$. In particular, if V is generated by such $v \neq 0$ (i.e., V is a highest weight representation with highest weight vector v) then V is a quotient of M_λ.*

(ii) Every highest weight representation has a weight decomposition into finite dimensional weight subspaces.

Proof. (i) Uniqueness follows from the fact that v_λ generates M_λ. To construct η, note that we have a natural homomorphism of $\mathfrak{g}$-modules $\widetilde{\eta} : U(\mathfrak{g}) \to V$ given by $\widetilde{\eta}(x) = xv$. Moreover, $\widetilde{\eta}|_{I_\lambda} = 0$ thanks to the relations satisfied by v, so $\widetilde{\eta}$ descends to a map $\eta : U(\mathfrak{g})/I_\lambda = M_\lambda \to V$. Moreover, if V is generated by v then this map is surjective, as desired.

(ii) This follows from (i) since a quotient of any representation with a weight decomposition must itself have a weight decomposition. □

Corollary 25.11. *Every highest weight representation V has a unique highest weight generator, up to scaling.*

Proof. Suppose v, w are two highest weight generators of V of weights λ, μ. If $\lambda = \mu$ then they are proportional since $\dim V[\lambda] \leq \dim M_\lambda[\lambda] = 1$, as V is a quotient of M_λ. On the other hand, if $\lambda \neq \mu$, then we can assume without loss of generality that $\lambda - \mu \notin Q_+$ (otherwise switch λ, μ). Then $\mu \notin \lambda - Q_+$, hence $\mu \notin P(V)$, a contradiction. □

Proposition 25.12. *For every $\lambda \in \mathfrak{h}^*$, the Verma module M_λ has a unique irreducible quotient L_λ. Moreover, L_λ is a quotient of every highest weight $\mathfrak{g}$-module V with highest weight λ.*

Proof. Let $Y \subset M_\lambda$ be a proper submodule. Then Y has a weight decomposition, and cannot contain a nonzero multiple of v_λ (as otherwise

$Y = M_\lambda$), so $P(Y) \subset (\lambda - Q_+) \setminus \{\lambda\}$. Now let J_λ be the sum of all proper submodules $Y \subset M_\lambda$. Then $P(J_\lambda) \subset (\lambda - Q_+) \setminus \{\lambda\}$, so J_λ is also a proper submodule of M_λ (the maximal one). Thus, $L_\lambda := M_\lambda / J_\lambda$ is an irreducible highest weight module with highest weight λ. Moreover, if V is any nonzero quotient of M_λ then the kernel K of the map $M_\lambda \to V$ is a proper submodule, hence contained in J_λ. Thus the surjective map $M_\lambda \to L_\lambda$ descends to a surjective map $V \to L_\lambda$. The kernel of this map is a proper submodule of V, hence zero if V is irreducible. Thus in the latter case $V \cong L_\lambda$. □

Corollary 25.13. *Irreducible highest weight $\mathfrak{g}$-modules are classified by their highest weight $\lambda \in \mathfrak{h}^*$, via the bijection $\lambda \mapsto L_\lambda$.*

25.3. **Finite dimensional modules.** Since every finite dimensional irreducible $\mathfrak{g}$-module is highest weight, it is of the form L_λ for λ belonging to some subset $P_F \subset P$, the set of weights λ such that L_λ is finite dimensional. So to obtain a final classification of finite dimensional irreducible representations of $\mathfrak{g}$, we should determine the subset P_F.

Let $P_+ \subset P$ be the intersection of P with the closure of the dominant Weyl chamber C_+; i.e., P_+ is the set of nonnegative integer linear combinations of the fundamental weights ω_i. In other words, P_+ is the set of $\lambda \in P$ such that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}_+$ for $1 \le i \le r$. Weights belonging to P_+ are called **dominant integral**.

Proposition 25.14. *We have $P_F \subset P_+$.*

Proof. The vector v_λ is highest weight for $(\mathfrak{sl}_2)_i$ with highest weight $\lambda(h_i) = (\lambda, \alpha_i^\vee)$. This must be a nonnegative integer for the corresponding $\mathfrak{sl}_2$-module to be finite dimensional. □

Lemma 25.15. *If $\lambda \in P_+$ then in L_λ, we have $f_i^{\lambda(h_i)+1} v_\lambda = 0$.*

Proof. By the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4), we have $e_i f_i^{\lambda(h_i)+1} v_\lambda = 0$. Also $e_j f_i^{\lambda(h_i)+1} v_\lambda = 0$ for $j \neq i$ since $[e_j, f_i] = 0$. Thus, $w := f_i^{\lambda(h_i)+1} v_\lambda$ is a highest weight vector in L_λ. So w cannot be a generator (as the highest weight generator is unique up to scaling). Thus w generates a proper submodule in L_λ, which must be zero since L_λ is irreducible. □

Lemma 25.16. *Let V be a $\mathfrak{g}$-module with weight decomposition into finite dimensional weight subspaces. If V is a sum of finite dimensional $(\mathfrak{sl}_2)_i$-modules for each $i = 1, ..., r$, then for each $\lambda \in P$ and $w \in W$, $\dim V[\lambda] = \dim V[w\lambda]$. In particular, $P(V)$ is W-invariant.*

Proof. Since the Weyl group W is generated by the simple reflections s_i, it suffices to prove the statement for $w = s_i$, and in fact to prove that $\dim V[\lambda] \le \dim V[s_i\lambda]$ (as $s_i^2 = 1$).

If $(\lambda, \alpha_i^\vee) = m \ge 0$ then consider the operator $f_i^m : V[\lambda] \to V[s_i\lambda]$. We claim that this operator is injective, which implies the desired inequality. Indeed, let $v \in V[\lambda]$ be a nonzero vector and E be the representation of $(\mathfrak{sl}_2)_i$ generated by v. Then E is finite dimensional, and $v \in E[m]$, so by the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4), $f_i^m v \neq 0$, as claimed.

Similarly, if $(\lambda, \alpha_i^\vee) = -m \le 0$ then the operator $e_i^m : V[\lambda] \to V[s_i\lambda]$ is injective. This proves the lemma. □

Now we are ready to state the main classification theorem.

Theorem 25.17. *For any $\lambda \in P_+$, L_λ is finite dimensional; i.e., $P_F = P_+$. Thus finite dimensional irreducible representations of $\mathfrak{g}$ are classified, up to an isomorphism, by their highest weight $\lambda \in P_+$, via the bijection $\lambda \mapsto L_\lambda$. Moreover, for any $\mu \in P$ and $w \in W$,* $\dim L_\lambda[\mu] = \dim L_\lambda[w\mu]$.

Proof. Since $f_i^{\lambda(h_i)+1} v_\lambda = 0$, we see that v_λ generates the irreducible finite dimensional $(\mathfrak{sl}_2)_i$-module of highest weight $\lambda(h_i)$. Also, every nonzero element of $\mathfrak{g}$ generates a finite dimensional $(\mathfrak{sl}_2)_i$-module. Hence every vector in L_λ generates a finite dimensional $(\mathfrak{sl}_2)_i$-module. Thus by Lemma 25.16, $P(L_\lambda)$ is W-invariant.

Now let $\mu \in P(L_\lambda) \cap P_+$. Then $\mu = \lambda - \beta$, $\beta \in Q_+$, so

$$(\mu, \rho^\vee) = (\lambda, \rho^\vee) - (\beta, \rho^\vee) \le (\lambda, \rho^\vee).$$

So if $\mu = \sum_i m_i \omega_i$, $m_i \in \mathbb{Z}_+$ then $\sum_i m_i(\omega_i, \rho^\vee) \le (\lambda, \rho^\vee)$. Since $(\omega_i, \rho^\vee) \ge \frac{1}{2}$, this implies that $P(L_\lambda) \cap P_+$ is finite. But we know that $WP_+ = P$, hence $W(P(L_\lambda) \cap P_+) = P(L_\lambda)$, as $P(L_\lambda)$ is W-invariant. It follows that $P(L_\lambda)$ is finite, hence L_λ is finite dimensional. □

Example 25.18. For $\mathfrak{g} = \mathfrak{sl}_2$ the dominant integral weights are positive integers $n \in \mathbb{Z}_{\ge 0}$, and it is easy to see that $L_n = V_n$.

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