26. **The Weyl character formula**

26.1. **Characters.** Let V be a finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$. Recall that the action of $\mathfrak{g}$ on V can be exponentiated to the action of the corresponding simply connected complex Lie group G. Recall also that the **character** of a finite dimensional representation V of any group G is the function

$$\chi_V(g) = \mathrm{Tr}|_V(g).$$

Let us compute this character in our case. To this end, let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $h \in \mathfrak{h}$, and let us compute $\chi_V(e^h)$. Note that this completely determines χ_V since it determines $\chi_V(e^x)$ for any semisimple element $x \in \mathfrak{g}$, and semisimple elements form a dense open set in $\mathfrak{g}$ (complement of zeros of some polynomial). So elements of the form e^x as above form a dense open set at least in some neighborhood of 1 in G, and an analytic function on G is determined by its values on any nonempty open set.

We know that V has a weight decomposition: $V = \oplus_{\mu \in P} V[\mu]$. Thus we have

$$\chi_V(e^h) = \sum_{\mu \in P} \dim V[\mu] e^{\mu(h)}.$$

Consider the group algebra $\mathbb{Z}[P]$. It sits naturally inside the algebra of analytic functions on $\mathfrak{h}$ via $\lambda \mapsto e^\lambda$, where $e^\lambda(h) := e^{\lambda(h)}$, and we see that $\chi_V \in \mathbb{Z}[P]$, namely

$$\chi_V = \sum_{\mu \in P} \dim V[\mu] e^{\mu}.$$

We will call the element χ_V the **character** of V.

26.2. **Category $\mathcal{O}$.** Note that the above definition of character is a purely formal algebraic definition, i.e., χ_V is simply the generating function of dimensions of weight subspaces of V. So it makes sense for any (possibly infinite dimensional) representation V with a weight decomposition into finite dimensional weight subspaces, except we may obtain an infinite sum. More precisely, we make the following definition.

Definition 26.1. The category $\mathcal{O}_{\mathrm{int}}$ is the category of representations V of $\mathfrak{g}$ with weight decomposition into finite dimensional weight spaces $V = \oplus_{\mu \in P} V[\mu]$, such that $P(V)$ is contained in the union of sets $\lambda^i - Q_+$ for a finite collection of weights $\lambda^1, ..., \lambda^N \in P$ (depending on V).[13]

[13] Usually one also adds the condition that V is a finitely generated $U(\mathfrak{g})$-module, but we don't need this condition here, so we won't impose it.

Here the subscript "int" indicates that we consider only integral weights (i.e., ones in P). However, for brevity we will drop this subscript in this section and just denote this category by $\mathcal{O}$.

For example, any highest weight module belongs to $\mathcal{O}$.

Let $\mathcal{R}$ be the ring of series $a := \sum_{\mu\in P} a_\mu e^\mu$ ($a_\mu \in \mathbb{Z}$) such that the set $P(a)$ of μ with $a_\mu \neq 0$ is contained in the union of sets $\lambda^i - Q_+$ for a finite collection of weights $\lambda^1, ..., \lambda^N \in P$. Then for every $V \in \mathcal{O}$ we can define the character $\chi_V \in \mathcal{R}$. Moreover, it is easy to see that if

$$0 \to X \to Y \to Z \to 0$$

is a short exact sequence in $\mathcal{O}$ then $\chi_Y = \chi_X + \chi_Z$, and that for any $V, U \in \mathcal{O}$ we have $V \otimes U \in \mathcal{O}$ and $\chi_{V\otimes U} = \chi_V \chi_U$.

Example 26.2. Let $V = M_\lambda$ be the Verma module. Recall that as a vector space $M_\lambda = U(\mathfrak{n}_-)v_\lambda$, and that $U(\mathfrak{n}_-) = \otimes_{\alpha\in R_+}\mathbb{C}[e_{-\alpha}]$ (using the PBW theorem). Thus

$$\sum_\mu U(\mathfrak{n}_-)[\mu]e^\mu = \frac{1}{\prod_{\alpha\in R_+}(1-e^{-\alpha})}$$

and hence

$$\chi_{M_\lambda} = \frac{e^\lambda}{\prod_{\alpha\in R_+}(1-e^{-\alpha})}.$$

It is convenient to rewrite this formula as follows:

$$\chi_{M_\lambda} = \frac{e^{\lambda+\rho}}{\Delta},\ \Delta := \prod_{\alpha\in R_+}(e^{\alpha/2} - e^{-\alpha/2}).$$

The (trigonometric) polynomial Δ is called the **Weyl denominator**.

Note that we have a homomorphism $\varepsilon : W \to \mathbb{Z}/2$ given by the formula $w \mapsto \det(w|_{\mathfrak{h}})$, i.e. $w \mapsto (-1)^{\ell(w)}$; it is defined on simple reflections by $s_i \mapsto -1$. This homomorphism is called the **sign character**. For example, for type A_{n-1} this is the sign of a permutation in S_n. We will say that an element of $f \in \mathbb{C}[P]$ is **anti-invariant** under W if $w(f) = (-1)^{\ell(w)} f$ for all $w \in W$.

Proposition 26.3. *The Weyl denominator Δ is anti-invariant under W.*

Proof. Since s_i permutes positive roots not equal to α_i and sends α_i to $-\alpha_i$, it follows that $s_i\Delta = -\Delta$. □

26.3. **The Weyl character formula.**

Theorem 26.4. *(Weyl character formula) For any $\lambda \in P_+$ the character $\chi_\lambda := \chi_{L_\lambda}$ of the irreducible finite dimensional representation L_λ is given by*

$$\chi_\lambda = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\Delta}.$$

The proof of this theorem is in the next subsection.

Corollary 26.5. *(Weyl denominator formula) One has*

$$\Delta = \sum_{w\in W}(-1)^{\ell(w)}e^{w\rho}.$$

Proof. This follows from the Weyl character formula by setting $\lambda = 0$ (as $L_0 = \mathbb{C}$ is the trivial representation). □

For example, for $\mathfrak{g} = \mathfrak{sl}_n$ Corollary 26.5 reduces to the usual product formula for the Vandermonde determinant.

26.4. **Proof of the Weyl character formula.** Consider the product $\Delta\chi_\lambda \in \mathbb{Z}[P]$. We know that χ_λ is W-invariant, so this product is W-anti-invariant. Thus,

$$\Delta\chi_\lambda = \sum_{\mu\in P} c_\mu e^\mu,$$

where $c_{w\mu} = (-1)^{\ell(w)}c_\mu$. Moreover, $c_\mu = 0$ unless $\mu \in \lambda+\rho-Q_+$, and $c_{\lambda+\rho} = 1$. Thus to prove the Weyl character formula, we need to show that $c_\mu = 0$ if $\mu \in P_+ \cap (\lambda+\rho-Q_+)$ and $\mu \neq \lambda+\rho$.

To this end, we will construct the above decomposition $\Delta\chi_\lambda$ using representation theory, so that this vanishing property is apparent from the construction.

First recall from Subsection 18.3 that we have the Casimir element C of $U(\mathfrak{g})$ given by the formula $C = \sum_i a_i a^i$ for a basis $a_i \in \mathfrak{g}$ with dual basis a^i of $\mathfrak{g}$ under the Killing form. This element is central, so acts by a scalar on every highest weight (in particular, finite dimensional irreducible) representation. We can write C in the form

$$C = \sum_j x_j^2 + \sum_{\alpha\in R_+}(e_{-\alpha}e_\alpha + e_\alpha e_{-\alpha}),$$

for an orthonormal basis x_j of $\mathfrak{h}$. Since $[e_\alpha, e_{-\alpha}] = h_\alpha$, we find that

$$C = \sum_j x_j^2 + 2\sum_{\alpha\in R_+} e_{-\alpha}e_\alpha + \sum_{\alpha\in R_+} h_\alpha.$$

Thus we get

Lemma 26.6. *If V is a highest weight representation with highest weight λ then $C|_V = (\lambda, \lambda + 2\rho) = |\lambda + \rho|^2 - |\rho|^2$.*

Now we will define a sequence of modules $K(b)$ from category $\mathcal{O}$ parametrized by some binary strings b. This is done inductively. We set $K(\emptyset) = L_\lambda$. Now suppose $K(b)$ is already defined. If $K(b) = 0$ then we set $K(b0) = K(b1) = 0$. Otherwise, pick a nonzero vector $v_b \in K(b)$, of some weight $\nu(b) \in \lambda - Q_+$ such that the height of $\lambda - \nu(b)$ takes the minimal possible value. Then v_b is a highest weight vector, and we can consider the corresponding homomorphism

$$\xi_b : M_{\nu_b} \to K(b).$$

Let $K(b1), K(b0)$ be the kernel and cokernel of ξ_b. We have

$$\chi_{K(b1)} - \chi_{M_{\nu(b)}} + \chi_{K(b)} - \chi_{K(b0)} = 0.$$

Thus we have

$$\chi_{K(b)} = \chi_{M_{\nu(b)}} - \chi_{K(b1)} + \chi_{K(b0)}.$$

Now, it is clear that for every μ, every sufficiently long sequence b satisfies $K(b)[\mu] = 0$. So iterating this formula starting with $b = \emptyset$, we will get

$$\chi_\lambda = \sum_b (-1)^{\Sigma(b)} \chi_{M_{\nu(b)}} \qquad (26.1)$$

where $\Sigma(b)$ is the sum of digits of b (which could a priori be an infinite sum). So

$$\Delta\chi_\lambda = \sum_b (-1)^{\Sigma(b)} e^{\nu(b)+\rho}.$$

Also note that by induction in the length of b we can conclude that the eigenvalue of C on $M_{\nu(b)}$ is $|\lambda + \rho|^2 - |\rho|^2$ regardless of b, which implies that

$$|\nu(b) + \rho|^2 = |\lambda + \rho|^2$$

for all b; in particular, this shows that the sum (26.1) is finite.

So it remains to show that if $\mu = \lambda + \rho - \beta \in P_+$ with $\beta \in Q_+$ and $\beta \neq 0$ then $|\mu|^2 < |\lambda + \rho|^2$. Indeed,

$$|\lambda + \rho|^2 - |\mu|^2 = |\lambda + \rho|^2 - |\lambda - \beta + \rho|^2 =$$

$$2(\lambda + \rho, \beta) - |\beta|^2 > (\lambda + \rho, \beta) - |\beta|^2 = (\lambda + \rho - \beta, \beta) \geq 0.$$

This completes the proof of the Weyl character formula.

Exercise 26.7. Let Q be the root lattice of a simple Lie algebra $\mathfrak{g}$, Q_+ its positive part. Define the **Kostant partition function** to be the function $p : Q \to \mathbb{Z}_{\geq 0}$ which attaches to $\beta \in Q_+$ the number of ways to write β as a sum of positive roots of $\mathfrak{g}$ (where the order does not matter), and $p(\beta) = 0$ if $\beta \notin Q_+$.

(i) Show that

$$\sum_{\beta \in Q_+} p(\beta) e^{-\beta} = \frac{1}{\prod_{\alpha \in R_+}(1 - e^{-\alpha})}.$$

(ii) Prove the **Kostant multiplicity formula**

$$\dim L_\lambda[\gamma] = \sum_{w \in W} (-1)^{\ell(w)} p(w(\lambda + \rho) - \rho - \gamma).$$

(iii) Compute $p(k_1\alpha_1 + k_2\alpha_2)$ for $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{sp}_4$.

(iv) Use (iii) to compute explicitly the weight multiplicities of the irreducible representations L_λ for $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{sp}_4$. (You should get a sum of 6, respectively 8 terms, not particularly appealing, but easily computable in each special case).

26.5. **The Weyl dimension formula.** Recall that the Weyl character formula can be written as a trace formula: for $h \in \mathfrak{h}$

$$\chi_\lambda(e^h) = \mathrm{Tr}|_{L_\lambda}(e^h) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{(w(\lambda+\rho),h)}}{\prod_{\alpha \in R_+}(e^{\frac{1}{2}(\alpha,h)} - e^{-\frac{1}{2}(\alpha,h)})}.$$

The dimension of L_λ should be obtained from this formula when $h = 0$. However, we do not immediately get the answer since this formula gives the character as a ratio of two trigonometric polynomials which both vanish at $h = 0$, giving an indeterminacy. We know the limit exists since the character is a trigonometric polynomial, but we need to compute it. This can be done as follows.

Let us restrict attention to $h = 2th_\rho$ where $t \in \mathbb{R}$ and $h_\rho \in \mathfrak{h}$ corresponds to $\rho \in \mathfrak{h}^*$ using the identification induced by the invariant form. We have

$$\chi_\lambda(e^{2th_\rho}) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{2t(w(\lambda+\rho),\rho)}}{\prod_{\alpha \in R_+}(e^{t(\alpha,\rho)} - e^{-t(\alpha,\rho)})}.$$

The key idea is that for this specialization the numerator can also be factored using the denominator formula, which will allow us to resolve the indeterminacy. Namely, we have

$$\chi_{L_\lambda}(e^{2th_\rho}) = \frac{\prod_{\alpha \in R_+}(e^{t(\alpha,\lambda+\rho)} - e^{-t(\alpha,\lambda+\rho)})}{\prod_{\alpha \in R_+}(e^{t(\alpha,\rho)} - e^{-t(\alpha,\rho)})}. \qquad (26.2)$$

Now sending $t \to 0$, we obtain

Proposition 26.8. *We have*

$$\dim L_\lambda = \frac{\prod_{\alpha\in R_+}(\alpha, \lambda+\rho)}{\prod_{\alpha\in R_+}(\alpha,\rho)}.$$

Note that this number is an integer, but this is not obvious without its interpretation as the dimension of a representation.

Formula (26.2) has a meaning even before taking the limit. Namely, the eigenvalues of the element $2h_\rho$ define a $\mathbb{Z}$-grading on the representation L_λ called the **principal grading**, and we obtain a product formula for the Poincaré polynomial of this grading.

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