26. The Weyl character formula

26.1. Characters. Let \( V \) be a finite dimensional representation of a semisimple Lie algebra \( g \). Recall that the action of \( g \) on \( V \) can be exponentiated to the action of the corresponding simply connected complex Lie group \( G \). Recall also that the character of a finite dimensional representation \( V \) of any group \( G \) is the function

\[
\chi_V(g) = \text{Tr}_V(g).
\]

Let us compute this character in our case. To this end, let \( h \subset g \) be a Cartan subalgebra, \( h \in \mathfrak{h} \), and let us compute \( \chi_V(e^h) \). Note that this completely determines \( \chi_V \) since it determines \( \chi_V(e^x) \) for any semisimple element \( x \in g \), and semisimple elements form a dense open set in \( g \) (complement of zeros of some polynomial). So elements of the form \( e^x \) as above form a dense open set at least in some neighborhood of 1 in \( G \), and an analytic function on \( G \) is determined by its values on any nonempty open set.

We know that \( V \) has a weight decomposition: \( V = \bigoplus_{\mu \in P} V[\mu] \). Thus we have

\[
\chi_V(e^h) = \sum_{\mu \in P} \dim V[\mu] e^{\mu(h)}.
\]

Consider the group algebra \( \mathbb{Z}[P] \). It sits naturally inside the algebra of analytic functions on \( \mathfrak{h} \) via \( \lambda \mapsto e^\lambda \), where \( e^\lambda(h) := e^{\lambda(h)} \), and we see that \( \chi_V \in \mathbb{Z}[P] \), namely

\[
\chi_V = \sum_{\mu \in P} \dim V[\mu] e^\mu.
\]

We will call the element \( \chi_V \) the **character** of \( V \).

26.2. Category \( \mathcal{O} \). Note that the above definition of character is a purely formal algebraic definition, i.e., \( \chi_V \) is simply the generating function of dimensions of weight subspaces of \( V \). So it makes sense for any (possibly infinite dimensional) representation \( V \) with a weight decomposition into finite dimensional weight subspaces, except we may obtain an infinite sum. More precisely, we make the following definition.

**Definition 26.1.** The category \( \mathcal{O}_{\text{int}} \) is the category of representations \( V \) of \( g \) with weight decomposition into finite dimensional weight spaces \( V = \bigoplus_{\mu \in P} V[\mu] \), such that \( P(V) \) is contained in the union of sets \( \lambda^i - Q_+ \) for a finite collection of weights \( \lambda^1, ..., \lambda^N \in P \) (depending on \( V \)).

\[\text{[11]}\] Usually one also adds the condition that \( V \) is a finitely generated \( U(g) \)-module, but we don’t need this condition here, so we won’t impose it.
Here the subscript “int” indicates that we consider only integral weights (i.e., ones in $P$). However, for brevity we will drop this subscript in this section and just denote this category by $O$.

For example, any highest weight module belongs to $O$.

Let $R$ be the ring of series $a := \sum_{\mu \in P} a_\mu e^\mu$ ($a_\mu \in \mathbb{Z}$) such that the set $P(a)$ of $\mu$ with $a_\mu \neq 0$ is contained in the union of sets $\lambda^i - Q_+$ for a finite collection of weights $\lambda^1, ..., \lambda^N \in P$. Then for every $V \in O$ we can define the character $\chi_V \in R$. Moreover, it is easy to see that if $0 \to X \to Y \to Z \to 0$ is a short exact sequence in $O$ then $\chi_Y = \chi_X + \chi_Z$, and that for any $V, U \in O$ we have $V \otimes U \in O$ and $\chi_{V \otimes U} = \chi_V \chi_U$.

**Example 26.2.** Let $V = M_\lambda$ be the Verma module. Recall that as a vector space $M_\lambda = U(n_-)v_\lambda$, and that $U(n_-) = \otimes_{\alpha \in R_+} \mathbb{C}[e^{-\alpha}]$ (using the PBW theorem). Thus

$$\sum_{\mu} U(n_-)[\mu]e^\mu = \frac{1}{\prod_{\alpha \in R_+}(1 - e^{-\alpha})}$$

and hence

$$\chi_{M_\lambda} = \frac{e^\lambda}{\prod_{\alpha \in R_+}(1 - e^{-\alpha})}.$$ 

It is convenient to rewrite this formula as follows:

$$\chi_{M_\lambda} = \frac{e^{\lambda + \rho}}{\Delta}, \Delta := \prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2}).$$

The (trigonometric) polynomial $\Delta$ is called the **Weyl denominator**.

Note that we have a homomorphism $\varepsilon : W \to \mathbb{Z}/2$ given by the formula $w \mapsto \det(w|_h)$, i.e. $w \mapsto (-1)^{l(w)}$; it is defined on simple reflections by $s_i \mapsto -1$. This homomorphism is called the **sign character**. For example, for type $A_{n-1}$ this is the sign of a permutation in $S_n$. We will say that an element of $f \in \mathbb{C}[P]$ is **anti-invariant** under $W$ if $w(f) = (-1)^{l(w)}f$ for all $w \in W$.

**Proposition 26.3.** The Weyl denominator $\Delta$ is anti-invariant under $W$.

**Proof.** Since $s_i$ permutes positive roots not equal to $\alpha_i$ and send $\alpha_i$ to $-\alpha_i$, it follows that $s_i \Delta = -\Delta$. □
26.3. The Weyl character formula.

**Theorem 26.4. (Weyl character formula)** For any \( \lambda \in P_+ \) the character \( \chi_\lambda := \chi_{L_\lambda} \) of the irreducible finite dimensional representation \( L_\lambda \) is given by
\[
\chi_\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda + \rho)}}{\Delta}.
\]

The proof of this theorem is in the next subsection.

**Corollary 26.5. (Weyl denominator formula)** One has
\[
\Delta = \sum_{w \in W} (-1)^{\ell(w)} e^{w\rho}.
\]

**Proof.** This follows from the Weyl character formula by setting \( \lambda = 0 \) (as \( L_0 = \mathbb{C} \) is the trivial representation). \( \square \)

For example, for \( g = sl_n \) Corollary \textsuperscript{26.5} reduces to the usual product formula for the Vandermonde determinant.

26.4. **Proof of the Weyl character formula.** Consider the product \( \Delta \chi_\lambda \in \mathbb{Z}[P] \). We know that \( \chi_\lambda \) is \( W \)-invariant, so this product is \( W \)-anti-invariant. Thus,
\[
\Delta \chi_\lambda = \sum_{\mu \in P} c_\mu e^\mu,
\]
where \( c_{\mu \mu} = (-1)^{\ell(w)} c_\mu \). Moreover, \( c_\mu = 0 \) unless \( \mu \in \lambda + \rho - Q_+ \), and \( c_{\lambda + \rho} = 1 \). Thus to prove the Weyl character formula, we need to show that \( c_\mu = 0 \) if \( \mu \in P_+ \cap (\lambda + \rho - Q_+) \) and \( \mu \neq \lambda + \rho \).

To this end, we will construct the above decomposition \( \Delta \chi_\lambda \) using representation theory, so that this vanishing property is apparent from the construction.

First recall from Subsection 18.3 that we have the Casimir element \( C \) of \( U(g) \) given by the formula \( C = \sum_i a_i a^i \) for a basis \( a_i \in g \) with dual basis \( a^i \) of \( g \) under the Killing form. This element is central, so acts by a scalar on every highest weight (in particular, finite dimensional irreducible) representation. We can write \( C \) in the form
\[
C = \sum_j x_j^2 + \sum_{\alpha \in R_+} (e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha),
\]
for an orthonormal basis \( x_j \) of \( \mathfrak{h} \). Since \( [e_\alpha, e_{-\alpha}] = h_\alpha \), we find that
\[
C = \sum_j x_j^2 + 2 \sum_{\alpha \in R_+} e_{-\alpha} e_\alpha + \sum_{\alpha \in R_+} h_\alpha.
\]
Thus we get
Lemma 26.6. If \( V \) is a highest weight representation with highest weight \( \lambda \) then \( C|_V = (\lambda, \lambda + 2\rho) = |\lambda + \rho|^2 - |\rho|^2 \).

Now we will define a sequence of modules \( K(b) \) from category \( \mathcal{O} \) parametrized by some binary strings \( b \). This is done inductively. We set \( K(\emptyset) = L_\lambda \). Now suppose \( K(b) \) is already defined. If \( K(b) = 0 \), we do not define \( K(b') \) for any longer string \( b' \) starting with \( b \). Otherwise, pick a nonzero vector \( v_b \in K(b) \), of some weight \( \nu(b) \in \lambda - Q_+ \) such that the height of \( \lambda - \nu(b) \) is minimal possible. Then \( v_b \) is a highest weight vector, and we can consider the corresponding the homomorphism \( \xi_b : M_{\nu(b)} \to K(b) \).

Let \( K(b_1), K(b_0) \) be the kernel and cokernel of \( \xi_b \). We have
\[
\chi_{K(b_1)} - \chi_{M_{\nu(b)}} + \chi_{K(b)} - \chi_{K(b_0)} = 0.
\]
Thus we have
\[
\chi_{K(b)} = \chi_{M_{\nu(b)}} - \chi_{K(b_1)} + \chi_{K(b_0)}.
\]
It is clear that for every \( b \) and \( \mu \), there is \( b' \) starting with \( b \) such that \( K_{b'}[\mu] = 0 \). So iterating this formula starting with \( b = \emptyset \), we will get
\[
\chi_\lambda = \sum_b (-1)^{\Sigma(b)} \chi_{M_{\nu(b)}}
\]
where \( \Sigma(b) \) is the sum of digits of \( b \). So
\[
\Delta \chi_\lambda = \sum_b (-1)^{\Sigma(b)} e^{\nu(b) + \rho}.
\]
Also note that by induction in the length of \( b \) we can conclude that the eigenvalue of \( C \) on \( M_{\nu(b)} \) is \( |\lambda + \rho|^2 - |\rho|^2 \) regardless of \( b \), which implies that
\[
|\nu(b) + \rho|^2 = |\lambda + \rho|^2
\]
for all \( b \).

So it remains to show that if \( \mu = \lambda + \rho - \beta \in P_+ \) with \( \beta \in Q_+ \) and \( \beta \neq 0 \) then \( |\mu|^2 < |\lambda + \rho|^2 \). Indeed,
\[
|\lambda + \rho|^2 - |\mu|^2 = |\lambda + \rho|^2 - |\lambda - \beta + \rho|^2 = 2(\lambda + \rho, \beta) - |\beta|^2 > (\lambda + \rho, \beta) - |\beta|^2 = (\lambda + \rho - \beta, \beta) \geq 0.
\]
This completes the proof of the Weyl character formula.

Exercise 26.7. Let \( Q \) be the root lattice of a simple Lie algebra \( g \), \( Q_+ \) its positive part. Define the Kostant partition function to be the function \( p : Q \to \mathbb{Z}_{\geq 0} \) which attaches to \( \beta \in Q_+ \) the number of ways to write \( \beta \) as a sum of positive roots of \( g \) (where the order does not matter), and \( p(\beta) = 0 \) if \( \beta \notin Q_+ \).
(i) Show that
\[ \sum_{\beta \in Q_+} p(\beta) e^{-\beta} = \frac{1}{\prod_{\alpha \in R_+} (1 - e^{-\alpha})}. \]

(ii) Prove the Kostant multiplicity formula
\[ \dim L_\lambda[\gamma] = \sum_{w \in W} (-1)^{\ell(w)} p(w(\lambda + \rho) - \rho - \gamma). \]

(iii) Compute \( p(k_1 \alpha_1 + k_2 \alpha_2) \) for \( g = \mathfrak{sl}_3 \) and \( g = \mathfrak{sp}_4 \).

(iv) Use (iii) to compute explicitly the weight multiplicities of the irreducible representations \( L_\lambda \) for \( g = \mathfrak{sl}_3 \) and \( g = \mathfrak{sp}_4 \). (You should get a sum of 6, respectively 8 terms, not particularly appealing, but easily computable in each special case).

26.5. **The Weyl dimension formula.** Recall that the Weyl character formula can be written as a trace formula: for \( h \in \mathfrak{h} \)
\[ \chi_\lambda(e^h) = \Tr |_{L_\lambda} (e^h) = \sum_{w \in W} (-1)^{\ell(w)} e^{(w(\lambda + \rho),h)} \prod_{\alpha \in R_+} (e^{\frac{1}{2}(\alpha,h)} - e^{-\frac{1}{2}(\alpha,h)}). \]
The dimension of \( L_\lambda \) should be obtained from this formula when \( h = 0 \). However, we don’t not immediately get the answer since this formula gives the character as a ratio of two trigonometric polynomials which both vanish at \( h = 0 \), giving an indeterminacy. We know the limit exists since the character is a trigonometric polynomial, but we need to compute it. This can be done as follows.

Let us restrict attention to \( h = 2t \rho \) where \( t \in \mathbb{R} \) and \( \rho \in \mathfrak{h}^* \) corresponds to \( \rho \in \mathfrak{h} \) using the identification induced by the invariant form. We have
\[ \chi_\lambda(e^{2th}) = \sum_{w \in W} (-1)^{\ell(w)} e^{2t(w(\lambda + \rho),\rho)} \prod_{\alpha \in R_+} (e^{t(\alpha,\rho)} - e^{-t(\alpha,\rho)}). \]
The key idea is that for this specialization the numerator can also be factored using the denominator formula, which will allow us to resolve the indeterminacy. Namely, we have
\[ (26.1) \quad \chi_{L_\lambda}(e^{2th}) = \prod_{\alpha \in R_+} (e^{(\alpha,\lambda + \rho)} - e^{-t(\alpha,\rho)}) \prod_{\alpha \in R_+} (e^{(\alpha,\rho)} - e^{-t(\alpha,\rho)}). \]
Now sending \( t \to 0 \), we obtain

**Proposition 26.8.** We have
\[ \dim L_\lambda = \frac{\prod_{\alpha \in R_+} (\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+} (\alpha, \rho)}. \]
Note that this number is an integer, but this is not obvious without its interpretation as the dimension of a representation.

Formula (26.1) has a meaning even before taking the limit. Namely, the eigenvalues of the element $2h_p$ define a $\mathbb{Z}$-grading on the representation $L_\Lambda$ called the **principal grading**, and we obtain a product formula for the Poincaré polynomial of this grading.
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