LIE GROUPS AND LIE ALGEBRAS

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Introduction

The purpose of **group theory** is to give a mathematical treatment of **symmetries**. For example, symmetries of a set of n elements form the symmetric group S_n, and symmetries of a regular n-gon – the dihedral group D_n. Likewise, **Lie group theory** serves to give a mathematical treatment of **continuous symmetries**, i.e., families of symmetries continuously depending on several real parameters.

The theory of Lie groups was founded in the second half of the 19th century by the Norwegian mathematician **Sophus Lie**, after whom it is named. It was then developed by many mathematicians over the last 150 years, and has numerous applications in mathematics and science, especially physics.

A prototypical example of a Lie group is the group $SO(3)$ of rotational symmetries of the 2-dimensional sphere; in this case the parameters are the Euler angles ϕ, θ, ψ.

It turns out that unlike ordinary parametrized curves and surfaces, Lie groups are determined by their linear approximation at the identity element. This leads to the notion of the **Lie algebra** of a Lie group. This notion allows one to reformulate the theory of continuous symmetries in purely algebraic terms, which provides an extremely effective way of studying such symmetries. The goal of these notes is to give a detailed study of Lie groups and Lie algebras and interactions between them, with numerous examples.

These notes are based on a year-long introductory course on Lie groups and Lie algebras given by the author at MIT in 2020-2021 (in particular, they contain no original material). The first half (Sections 1-26) corresponds to the first semester and follows rather closely the excellent book "An introduction to Lie groups and Lie algebras" by A. Kirillov Jr. ([K]), but also discusses some additional topics. Namely, after a brief review of geometry and topology of manifolds, it covers the basic theory of Lie groups and Lie algebras, including the three fundamental theorems of Lie theory (except the proof of the third theorem, which is given in the second half). Then it proceeds to nilpotent and solvable Lie algebras, theorems of Lie and Engel, representations of $\mathfrak{sl}_2$, enveloping algebras and the Poincaré-Birkhoff Witt theorem, free Lie algebras, the Baker-Campbell-Hausdorff formula, and concludes with a detailed study and classification of complex semisimple Lie algebras, their representations, and the Weyl character formula.

The second half (starting with Section 27) covers representation theory of GL_n and other classical groups, minuscule representations, spin representations and spin groups, representation theory of compact Lie groups (again following [K]) and, more generally, compact topological groups, including existence of the Haar measure and the Peter-Weyl theorem. Then it discusses applications to quantum mechanics (a fairly complete treatment of the hydrogen atom) and proceeds to real forms of semisimple Lie algebras and groups, discussing the classification of such forms in terms of Vogan diagrams, maximal tori and maximal compact subgroups, the polar and Cartan decompositions, and classification of connected compact Lie groups and complex reductive groups. Then we discuss topology of Lie groups and homogeneous spaces (in particular, their cohomology rings), cohomology of Lie algebras, prove the third fundamental theorem of Lie theory and Ado's theorem on the existence of a faithful representation for a finite dimensional Lie algebra, and conclude with the study of Borel and parabolic subgroups, the flag manifold of a complex semisimple group and the Iwasawa decomposition for real groups.

Some other sources covering the same material are [E, FH, Hu, Kn].

Each section roughly corresponds to one 80-minute lecture. Part I consists of 26 sections, which corresponds to a 1-semester course. Part II consists of 25 sections, to allow for a review of Part I. Also, a lot of material is contained in exercises, which are often provided with detailed hints. These exercises were assigned as homework problems.[1]

Finally, we note that Lie theory is an inherently synthetic subject. While the main technical tools ultimately boil down to various parts of algebra (notably linear algebra and the theory of noncommutative rings and modules, and, at more advanced stages, algebraic geometry), Lie theory also relies in important ways on analysis, differential equations, differential geometry and topology. Thus, while we try to recall basic notions from these subjects along the way, the reader will need some degree of dexterity with them, which increases as we dig deeper into the material.

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[1] During the first semester and at the beginning of the second one homework problems were also assigned from [K].

Lie Groups and Lie algebras, I

1. **Manifolds**

1.1. **Topological spaces and groups.** Recall that the mathematical notion responsible for describing continuity is that of a **topological space**. Thus, to describe continuous symmetries, we should put this notion together with the notion of a group. This leads to the concept of a **topological group**.

Recall:

• A **topological space** is a set X certain subsets of which (including $\emptyset$ and X) are declared to be **open**, so that an arbitrary union and finite intersection of open sets is open.

• The collection of open sets in X is called **the topology** of X.

• A subset $Z \subset X$ of a topological space X is **closed** if its complement is open.

• If X, Y are topological spaces then the Cartesian product $X \times Y$ has a natural **product topology** in which open sets are (possibly infinite) unions of products $U \times V$, where $U \subset X, V \subset Y$ are open.

• Every subset $Z \subset X$ of a topological space X carries a natural **induced topology**, in which open sets are intersections of open sets in X with Z.

• A map $f : X \to Y$ between topological spaces is **continuous** if for every open set $V \subset Y$, the preimage $f^{-1}(V)$ is open in X.

For example, the open sets of the usual topology of the real line $\mathbb{R}$ are (disjoint) unions of open intervals (a, b), where $-\infty \le a < b \le \infty$.

Definition 1.1. A **topological group** is a group G which is also a topological space, so that the multiplication map $m : G \times G \to G$ and the inversion map $\iota : G \to G$ are continuous.

For example, the group $(\mathbb{R}, +)$ of real numbers with the operation of addition and the usual topology of $\mathbb{R}$ is a topological group, since the functions $(x, y) \mapsto x + y$ and $x \mapsto -x$ are continuous. Also a subgroup of a topological group is itself a topological group, so another example is rational numbers with addition, $(\mathbb{Q}, +)$. This last example is not a very good model for continuity, however, and shows that general topological groups are not very well behaved. Thus, we will focus on a special class of topological groups called **Lie groups**.

Lie groups are distinguished among topological groups by the property that as topological spaces they belong to a very special class called **topological manifolds.** So we need to start with reviewing this notion.

1.2. **Topological manifolds.** Recall:

• A **neighborhood** of a point $x \in X$ in a topological space X is an open set containing x.

• A **base** for a topological space X is a collection $\mathcal{B}$ of open sets in X such that for every neighborhood U of a point $x \in X$ there exists neighborhood $V \subset U$ of x which belongs to $\mathcal{B}$. Equivalently, every open set in X is a union of members of $\mathcal{B}$.

For example, open intervals form a base of the usual topology of $\mathbb{R}$. Moreover, we may take only intervals whose endpoints have rational coordinates, which gives a *countable* base for $\mathbb{R}$. Also if X, Y are topological spaces with bases $\mathcal{B}_X, \mathcal{B}_Y$ then products $U \times V$, where $U \in \mathcal{B}_X, V \in \mathcal{B}_Y$, form a base of the product topology of $X \times Y$. Thus if X and Y have countable bases, so does $X \times Y$; in particular, $\mathbb{R}^n$ with its usual (product) topology has a countable base (boxes whose vertices have rational coordinates).

• X is **Hausdorff** if any two distinct points have disjoint neighborhoods.

• If X is Hausdorff, we say that a sequence of points $x_n \in X, n \in \mathbb{N}$ **converges** to $x \in X$ as $n \to \infty$ (denoted $x_n \to x$) if every neighborhood of x contains almost all terms of this sequence. Then one also says that the **limit** of x_n is x and writes

$$\lim_{n\to\infty} x_n = x.$$

It is easy to show that the limit is unique when exists. In a Hausdorff space with a countable base, a closed set is one that is closed under taking limits of sequences.

• A Hausdorff space X is **compact** if every open cover $\{U_\alpha, \alpha \in A\}$ of X (i.e., $U_\alpha \subset X$ for all $\alpha \in A$ and $X = \cup_{\alpha\in A} U_\alpha$) has a finite subcover.

• A continuous map $f : X \to Y$ is a **homeomorphism** if it is a bijection and $f^{-1} : Y \to X$ is continuous.

Definition 1.2. A Hausdorff topological space X is said to be an **n-dimensional topological manifold** if it has a countable base and for every $x \in X$ there is a neighborhood $U \subset X$ of x and a continuous map $\phi : U \to \mathbb{R}^n$ such that $\phi : U \to \phi(U)$ is a homeomorphism and $\phi(U) \subset \mathbb{R}^n$ is open.

The second property is often formulated as the condition that X is **locally homeomorphic to** $\mathbb{R}^n$.

It is true (although not immediately obvious) that if a nonempty open set in $\mathbb{R}^n$ is homeomorphic to one in $\mathbb{R}^m$ then $n = m$. Therefore, the number n is uniquely determined by X as long as $X \neq \emptyset$. It is called **the dimension** of X. (By convention, $\emptyset$ is a manifold of any integer dimension).

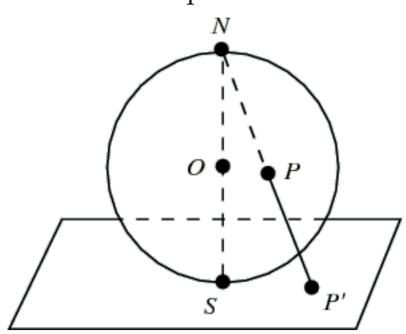
Example 1.3. 1. Obviously $X = \mathbb{R}^n$ is an n-dimensional topological manifold: we can take $U = X$ and $\phi = \mathrm{Id}$.

2. An open subset of a topological manifold is itself a topological manifold of the same dimension.

3. The circle $S^1 \subset \mathbb{R}^2$ defined by the equation $x^2 + y^2 = 1$ is a topological manifold: for example, the point $(1, 0)$ has a neighborhood $U = S^1 \setminus \{(-1, 0)\}$ and a map $\phi : U \to \mathbb{R}$ given by the stereographic projection:

$$\phi(\theta) = \tan(\tfrac{\theta}{2}), \ -\pi < \theta < \pi.$$

and similarly for every other point. More generally, the sphere $S^n \subset \mathbb{R}^{n+1}$ defined by the equation $x_0^2 + ... + x_n^2 = 1$ is a topological manifold, for the same reason. The stereographic projection for the 2-dimensional sphere is shown in the following picture.



4. The curve ∞ is not a manifold, since it is not locally homeomorphic to $\mathbb{R}$ at the self-intersection point (show it!)

A pair (U, ϕ) with the above properties is called a **local chart**. An **atlas** of local charts is a collection of charts $(U_\alpha, \phi_\alpha), \alpha \in A$ such that $\cup_{\alpha \in A} U_\alpha = X$; i.e., $\{U_\alpha, \alpha \in A\}$ is an open cover of X. Thus any topological manifold X admits an atlas labeled by points of X. There are also much smaller atlases. For instance, an open set in $\mathbb{R}^n$ has an atlas with just one chart, while the sphere S^n has an atlas with two charts. Very often X admits an atlas with finitely many charts. For example, if X is compact then there is a finite atlas, since every atlas has a finite subatlas. Moreover, there is always a countable atlas, due to the following lemma:

Lemma 1.4. *If X is a topological space with a countable base then every open cover of X has a countable subcover.*

Proof. Let $\{V_i, i \in \mathbb{N}\}$ be a countable base of X. If $\{U_\alpha\}$ is an open cover of X then for each $x \in X$ pick $\alpha(x)$ such that $x \in V_{i(x)} \subset U_{\alpha(x)}$.

Let $I \subset \mathbb{N}$ be the image of the map i. For each $i \in I$ pick $x \in X$ such that $i(x) = i$ and set $\alpha_i := \alpha(x)$. Then $\{U_{\alpha_i}, i \in I\}$ is a countable subcover of $\{U_\alpha\}$. $\square$

Now let (U, ϕ) and (V, ψ) be two charts such that $V \cap U \neq \emptyset$. Then we have the **transition map**

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V),$$

which is a homeomorphism between open subsets in $\mathbb{R}^n$. For example, consider the atlas of two charts for the circle S^1 (Example 1.3(3)), one missing the point $(-1, 0)$ and the other missing the point $(1, 0)$. Then $\phi(\theta) = \tan(\frac{\theta}{2})$ and $\psi(\theta) = \cot(\frac{\theta}{2})$, $\phi(U \cap V) = \psi(U \cap V) = \mathbb{R} \setminus 0$, and $(\phi \circ \psi^{-1})(x) = \frac{1}{x}$.

1.3. **C^k, real analytic and complex analytic manifolds.** The notion of topological manifold is too general for us, since continuous functions on which it is based in general do not admit a linear approximation. To develop the theory of Lie groups, we need more regularity. So we make the following definition.

Definition 1.5. An atlas on X is said to be **of regularity class** C^k, $1 \leq k \leq \infty$, if all transition maps between its charts are of class C^k (k times continuously differentiable). An atlas of class C^∞ is called **smooth**. Also an atlas is said to be **real analytic** if all transition maps are real analytic. Finally, if $n = 2m$ is even, so that $\mathbb{R}^n = \mathbb{C}^m$, then an atlas is called **complex analytic** if all its transition maps are complex analytic (i.e., holomorphic).

Example 1.6. The two-chart atlas for the circle S^1 defined by stereographic projections (Example 1.3(3)) is real analytic, since the function $f(x) = \frac{1}{x}$ is analytic. The same applies to the sphere S^n for any n. For example, for S^2 it is easy to see that the transition map $\mathbb{R}^2 \setminus 0 \to \mathbb{R}^2 \setminus 0$ is given by the formula

$$f(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

Using the complex coordinate $z = x + iy$, we get

$$f(z) = z/|z|^2 = 1/\overline{z}.$$

So this atlas is not complex analytic. But it can be easily made complex analytic by replacing one of the stereographic projections (ϕ or ψ) by its complex conjugate. Then we will have $f(z) = \frac{1}{z}$. On the other hand, it is known (although hard to prove) that S^n does not admit a complex analytic atlas for (even) $n \neq 2, 6$. For $n = 6$ this is a famous conjecture.

Definition 1.7. Two C^k, real analytic, or complex analytic atlases U_α, V_β are said to be **compatible** if the transition maps between U_α and V_β are of the same class (C^k, real analytic, or complex analytic).

It is clear that compatibility is an equivalence relation.

Definition 1.8. A C^k**, real analytic, or complex analytic structure** on a topological manifold X is an equivalence class of C^k, real analytic, or complex analytic atlases. If X is equipped with such a structure, it is said to be a C^k**, real analytic, or complex analytic manifold**. Complex analytic manifolds are also called **complex manifolds**, and a C^∞-manifold is also called **smooth**. A **diffeomorphism** (or **isomorphism**) between such manifolds is a homeomorphism which respects the corresponding classes of atlases.

Remark 1.9. This is really a **structure** and not a **property**. For example, consider $X = \mathbb{C}$ and $Y = D \subset \mathbb{C}$ the open unit disk, with the usual complex coordinate z. It is easy to see that X, Y are isomorphic as real analytic manifolds. But they are not isomorphic as complex analytic manifolds: a complex isomorphism would be a holomorphic function $f : \mathbb{C} \to D$, hence bounded, but by Liouville's theorem any bounded holomorphic function on $\mathbb{C}$ is a constant. Thus we have two different complex structures on $\mathbb{R}^2$ (Riemann showed that there are no others). Also, it is true, but much harder to show, that there are uncountably many different smooth structures on $\mathbb{R}^4$, and there are 28 (oriented) smooth structures on S^7.

Note that the Cartesian product $X \times Y$ of manifolds X, Y is naturally a manifold (of the same regularity type) of dimension $\dim X + \dim Y$.

Exercise 1.10. Let $f_1, ..., f_m$ be functions $\mathbb{R}^n \to \mathbb{R}$ which are C^k or real analytic. Let $X \subset \mathbb{R}^n$ be the set of points P such that $f_i(P) = 0$ for all i and $df_i(P)$ are linearly independent. Use the implicit function theorem to show that X is a topological manifold of dimension $n - m$ and equip it with a natural C^k, respectively real analytic structure. Prove the analogous statement for holomorphic functions $\mathbb{C}^n \to \mathbb{C}$, namely that in this case X is naturally a complex manifold of (complex) dimension $n - m$.

1.4. **Regular functions.** Now let $P \in X$ and (U, ϕ) be a local chart such that $P \in U$ and $\phi(P) = 0$. Such a chart is called a **coordinate chart** around P. In particular, we have **local coordinates** $x_1, ..., x_n : U \to \mathbb{R}$ (or $U \to \mathbb{C}$ for complex manifolds). Note that $x_i(P) = 0$, and $x_i(Q)$ determine Q if $Q \in U$.

Definition 1.11. A **regular function** on an open set $V \subset X$ in a C^k, real analytic, or complex analytic manifold X is a function $f : V \to \mathbb{R}, \mathbb{C}$ such that $f \circ \phi_\alpha^{-1} : \phi_\alpha(V \cap U_\alpha) \to \mathbb{R}, \mathbb{C}$ is of the corresponding regularity class, for some (and then any) atlas (U_α, ϕ_α) defining the corresponding structure on X.[2]

In other words, f is regular if it is expressed as a regular function in local coordinates near every point of V. Clearly, this is independent on the choice of coordinates.

The space (in fact, algebra) of regular functions on V will be denoted by $O(V)$.

Definition 1.12. Let V, U be neighborhoods of $P \in X$. Let us say that $f \in O(V)$, $g \in O(U)$ are **equal near** P if there exists a neighborhood $W \subset U \cap V$ of P such that $f|_W = g|_W$.

It is clear that this is an equivalence relation.

Definition 1.13. A **germ** of a regular function at P is an equivalence class of regular functions defined on neighborhoods of P which are equal near P.

The algebra of germs of regular functions at P is denoted by O_P. Thus we have $O_P = \varinjlim O(U)$, where the direct limit is taken over neighborhoods of P.

1.5. **Tangent spaces.** From now on we will only consider smooth, real analytic and complex analytic manifolds. By a **derivation at** P we will mean a linear map $D : O_P \to \mathbb{R}$ in the smooth and real analytic case and $D : O_P \to \mathbb{C}$ in the complex analytic case, satisfying the Leibniz rule

$$D(fg) = D(f)g(P) + f(P)D(g). \tag{1.1}$$

Note that for any such D we have $D(1) = 0$.

Let T_PX be the space of all such derivations. Thus T_PX is a real vector space for smooth and real analytic manifolds and a complex vector space for complex manifolds.

Lemma 1.14. *Let $x_1, ..., x_n$ be local coordinates at P. Then T_PX has basis $D_1, ..., D_n$, where*

$$D_i(f) := \frac{\partial f}{\partial x_i}(0).$$

[2]More precisely, for C^k and real analytic manifolds regular functions will be assumed real-valued, unless specified otherwise. In the complex analytic case there is, of course, no choice, and regular functions are automatically complex-valued.

Proof. We may assume $X = \mathbb{R}^n$ or $\mathbb{C}^n$, $P = 0$. Clearly, $D_1, ..., D_n$ is a linearly independent set in $T_P X$. Also let $D \in T_P X$, $D(x_i) = a_i$, and consider $D_* := D - \sum_i a_i D_i$. Then $D_*(x_i) = 0$ for all i. Now given a regular function f near 0, for small $x_1, ..., x_n$ by the fundamental theorem of calculus and the chain rule we have:

$$f(x_1, ..., x_n) = f(0) + \int_0^1 \frac{df(tx_1, ..., tx_n)}{dt} dt = f(0) + \sum_{i=1}^n x_i h_i(x_1, ..., x_n),$$

where

$$h_i(x_1, ..., x_n) := \int_0^1 (\partial_i f)(tx_1, ..., tx_n) dt$$

are regular near 0. So by the Leibniz rule $D_*(f) = 0$, hence $D_* = 0$. □

Definition 1.15. The space $T_P X$ is called the **tangent space** to X at P. Elements $v \in T_P X$ are called **tangent vectors** to X at P.

Observe that every tangent vector $v \in T_P X$ defines a derivation $\partial_v : O(U) \to \mathbb{R}, \mathbb{C}$ for every neighborhood U of P, satisfying (1.1). The number $\partial_v f$ is called the **derivative of f along** v. For usual curves and surfaces in $\mathbb{R}^3$ these coincide with the familiar notions from calculus.[3]

1.6. **Regular maps.**

Definition 1.16. A continuous map $F : X \to Y$ between manifolds (of the same regularity class) is **regular** if for any regular function h on an open set $U \subset Y$ the function $h \circ F$ on $F^{-1}(U)$ is regular. In other words, F is regular if it is expressed by regular functions in local coordinates.

It is easy to see that the composition of regular maps is regular, and that a homeomorphism F such that F, F^{-1} are both regular is the same thing as a diffeomorphism (=isomorphism).

Let $F : X \to Y$ be a regular map and $P \in X$. Then we can define the **differential** of F at P, $d_P F$, which is a linear map $T_P X \to T_{F(P)} Y$. Namely, for $f \in O_{F(P)}$ and $v \in T_P X$, the vector $d_P F \cdot v$ is defined by the formula

$$(d_P F \cdot v)(f) := v(f \circ F).$$

The differential of F is also denoted by F_*; namely, for $v \in T_P X$ one writes $dF_P \cdot v = F_* v$.

[3]Note however that $\partial_v f$ differs from the *directional derivative* $D_v f$ defined in calculus. Namely, $D_v f = \frac{\partial_v f}{|v|}$ (thus defined only for $v \neq 0$) and depends only on the direction of v.

Moreover, if $G : Y \to Z$ is another regular map, then we have the usual **chain rule**,

$$d(G \circ F)_P = dG_{F(P)} \circ dF_P.$$

In particular, if $\gamma : (a, b) \to X$ is a regular **parametrized curve** then for $t \in (a, b)$ we can define the **velocity vector**

$$d_t\gamma \cdot 1 = \gamma'(t) \in T_{\gamma(t)}X$$

(where $1 \in \mathbb{R} = T_t(a, b)$).

1.7. **Submersions and immersions, submanifolds.**

Definition 1.17. A regular map of manifolds $F : X \to Y$ is a **submersion** if $dF_P : T_PX \to T_{F(P)}Y$ is surjective for all $P \in X$.

The following proposition is a version of the implicit function theorem for manifolds.

Proposition 1.18. *If F is a submersion then for any $Q \in Y$, $F^{-1}(Q)$ is a manifold of dimension* $\dim X - \dim Y$.

Proof. This is a local question, so it reduces to the case when X, Y are open subsets in Euclidean spaces. In this case it reduces to Exercise 1.10. ☐

Definition 1.19. A regular map of manifolds $f : X \to Y$ is an **immersion** if $d_PF : T_PX \to T_{F(P)}Y$ is injective for all $P \in X$.

Example 1.20. The inclusion of the sphere S^n into $\mathbb{R}^{n+1}$ is an immersion. The map $F : S^1 \to \mathbb{R}^2$ given by

$$x(t) = \frac{\cos\theta}{1 + \sin^2\theta}, \quad y(t) = \frac{\sin\theta\cos\theta}{1 + \sin^2\theta} \tag{1.2}$$

is also an immersion; its image is the lemniscate (shaped as ∞). This shows that an immersion need not be injective. On the other hand, the map $F : \mathbb{R} \to \mathbb{R}^2$ given by $F(t) = (t^2, t^3)$ parametrizing a semicubic parabola $\prec$ is injective, but not an immersion, since $F'(0) = (0, 0)$.

Definition 1.21. An immersion $f : X \to Y$ is an **embedding** if the map $F : X \to F(X)$ is a homeomorphism (where $F(X)$ is equipped with the induced topology from Y). In this case, $F(X) \subset Y$ is said to be an **(embedded) submanifold.**[4]

[4]Recall that a subset Z of a topological space X is called **locally closed** if it is a closed subset in an open subset $U \subset X$. It is clear that embedded submanifolds are locally closed. For this reason they are often called locally closed (embedded) submanifolds.

Example 1.22. The immersion of S^n into $\mathbb{R}^{n+1}$ and of $(0,1)$ into $\mathbb{R}$ are embeddings, but the parametrization of the lemniscate by the circle given by (1.2) is not. The parametrization of the curve ρ by $\mathbb{R}$ is also not an embedding; it is injective but the inverse is not a homeomorphism.

Definition 1.23. An embedding $F : X \to Y$ of manifolds is **closed** if $F(X) \subset Y$ is a closed subset. In this case we say that $F(X)$ is a **closed (embedded) submanifold** of Y.

Example 1.24. The embedding of S^n into $\mathbb{R}^{n+1}$ is closed but of $(0,1)$ into $\mathbb{R}$ is not. Also in Proposition 1.18, $f^{-1}(Q)$ is a closed submanifold of X.

2. **Lie groups, I**

2.1. **The definition of a Lie group.**

Definition 2.1. A C^k, real or complex analytic **Lie group** is a manifold G of the same class, with a group structure such that the multiplication map $m : G \times G \to G$ is regular.

Thus, in a Lie group G for any $g \in G$ the left and right translation maps $L_g, R_g : G \to G$, $L_g(x) := gx$, $R_g(x) := xg$, are diffeomorphisms.

Proposition 2.2. *In a Lie group G, the inversion map $\iota : G \to G$ is a diffeomorphism, and $d\iota_1 = -\mathrm{Id}$.*

Proof. For the first statement it suffices to show that ι is regular near 1, the rest follows by translation. So let us pick a coordinate chart near $1 \in G$ and write the map m in this chart in local coordinates. Note that in these coordinates, $1 \in G$ corresponds to $0 \in \mathbb{R}^n$. Since $m(x, 0) = x$ and $m(0, y) = y$, the linear approximation of $m(x, y)$ at 0 is $x+y$. Thus by the implicit function theorem, the equation $m(x, y) = 0$ is solved near 0 by a regular function $y = \iota(x)$ with $d\iota(0) = -\mathrm{Id}$. This proves the proposition. □

Remark 2.3. A C^0 **Lie group** is a topological group which is a topological manifold. The **Hilbert 5th problem** was to show that any such group is actually a real analytic Lie group (i.e., the regularity class does not matter). This problem is solved by the deep **Gleason-Yamabe theorem**, proved in 1950s. So from now on we will not pay attention to regularity class and consider only real and complex Lie groups.

Note that any complex Lie group of dimension n can be regarded as a real Lie group of dimension $2n$. Also the Cartesian product of real (complex) Lie groups is a real (complex) Lie group.

2.2. **Homomorphisms.**

Definition 2.4. A **homomorphism of Lie groups** $f : G \to H$ is a group homomorphism which is also a regular map. An **isomorphism of Lie groups** is a homomorphism f which is a group isomorphism, such that $f^{-1} : H \to G$ is regular.

We will see later that the last condition is in fact redundant.

2.3. **Examples.**

Example 2.5. 1. $(\mathbb{R}^n, +)$ is a real Lie group and $(\mathbb{C}^n, +)$ is a complex Lie group (both n-dimensional).

2. $(\mathbb{R}^\times, \times)$, $(\mathbb{R}_{>0}, \times)$ are real Lie groups, $(\mathbb{C}^\times, \times)$ is a complex Lie group (all 1-dimensional)

3. $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is a 1-dimensional real Lie group under multiplication of complex numbers.

Note that $\mathbb{R}^\times \cong \mathbb{R}_{>0} \times \mathbb{Z}/2$, $\mathbb{C}^\times \cong \mathbb{R}_{>0} \times S^1$ as real Lie groups (trigonometric form of a complex number) and $(\mathbb{R}, +) \cong (\mathbb{R}_{>0}, \times)$ via $x \mapsto e^x$.

4. The groups of invertible n by n matrices: $GL_n(\mathbb{R})$ is a real Lie group and $GL_n(\mathbb{C})$ is a complex Lie group. These are open sets in the corresponding spaces of all matrices and have dimension n^2.

5. $SU(2)$, the special unitary group of size 2. This is the set of complex 2-by-2 matrices A such that

$$AA^\dagger = \mathbf{1}, \ \det A = 1.$$

So writing

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix},$$

we get

$$a\bar{a} + b\bar{b} = 1, \ a\bar{c} + b\bar{d} = 0, \ c\bar{c} + d\bar{d} = 1.$$

The second equation implies that $(c, d) = \lambda(-\bar{b}, \bar{a})$. Then we have

$$1 = \det A = ad - bc = \lambda(a\bar{a} + b\bar{b}) = \lambda,$$

so $\lambda = 1$. Thus $SU(2)$ is identified with the set of $(a, b) \in \mathbb{C}^2$ such that $a\bar{a} + b\bar{b} = 1$. Writing $a = x + iy$, $b = z + it$, we have

$$SU(2) = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 + z^2 + t^2 = 1\}.$$

Thus $SU(2)$ is a 3-dimensional real Lie group which as a manifold is the 3-dimensional sphere $S^3 \subset \mathbb{R}^4$.

6. Any countable group G with **discrete topology** (i.e., such that every set is open) is a (real and complex) Lie group.

2.4. **The connected component of** 1. Recall:

• A topological space X is **path-connected** if for any $P, Q \in X$ there is a continuous map $x : [0, 1] \to X$ such that $x(0) = P, x(1) = Q$.

• If X is any topological space, then for $P \in X$ we can define its **path-connected component** to be the set X_P of $Q \in X$ for which there is a continuous map $x : [0, 1] \to X$ such that $x(0) = P, x(1) = Q$ (such x is called **a path connecting** P **and** Q). Then X_P is the largest path-connected subset of X containing P. Clearly, the relation that Q belongs to X_P is an equivalence relation, which splits X into equivalence classes called **path-connected components.** The set of such components is denoted $\pi_0(X)$.

• A topological space X is **connected** if the only subsets of X that are both open and closed are $\emptyset$ and X. For $P \in X$, the **connected component** of X is the union X^P of all connected subsets of X containing P, which is obviously connected itself (so it is the largest connected subset of X containing P). A path-connected space X is always connected but not vice versa (the classic counterexample is the graph of the function $y = \sin(\frac{1}{x})$ together with the interval $[-1, 1]$ of the y-axis); however, a connected *manifold* is path-connected (show it!), so for manifolds the notions of connected component and path-connected component coincide.

• If Y is a topological space, X is a set and $p : Y \to X$ is a surjective map (i.e., $X = Y/\sim$ is the quotient of Y by an equivalence relation) then X acquires a topology called the **quotient topology**, in which open sets are subsets $V \subset X$ such that $p^{-1}(V)$ is open.

Now let G be a real or complex Lie group, and G° the connected component of $1 \in G$. Then the connected component of any $g \in G$ is gG°.

Proposition 2.6. *(i) G° is a normal subgroup of G.*

(ii) $\pi_0(G) = G/G^\circ$ with quotient topology is a discrete and countable group.

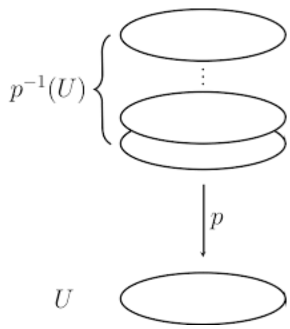
Proof. (i) Let $g \in G$, $a \in G^\circ$, and $x : [0, 1] \to G$ be a path connecting 1 to a. Then gxg^{-1} is a path connecting 1 to gag^{-1}, so $gag^{-1} \in G^\circ$, hence G° is normal.

(ii) Since G is a manifold, for any $g \in G$, there is a neighborhood of g contained in $G_g = gG^\circ$. This implies that any coset of G° in G is open, hence G/G° is discrete. Also G/G° is countable since G has a countable base. □

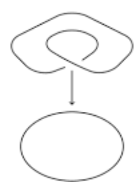
Thus we see that any Lie group is an extension of a discrete countable group by a connected Lie group. This essentially reduces studying Lie groups to studying connected Lie groups. In fact, one can further reduce to simply connected Lie groups, which is done in the next subsections.

3. **Lie groups, II**

3.1. **A crash course on coverings.** Now we need to review some more topology. Let X, Y be Hausdorff topological spaces, and $p : Y \to X$ a continuous map. Then p is called a **covering** if every point $x \in X$ has a neighborhood U such that $p^{-1}(U)$ is a union of disjoint open sets (called **sheets** of the covering) each of which is mapped homeomorphically onto U by p:



In other words, there exists a homeomorphism $h : U \times F \to p^{-1}(U)$ for some discrete space F with $(p \circ h)(u, f) = u$ for all $u \in U$, $f \in F$. I.e., informally speaking, a covering is a map that locally on X looks like the projection $X \times F \to X$ for some discrete F. It is clear that a covering of a manifold (C^k, real or complex analytic) is a manifold of the same type, and the covering map is regular.



Two paths $x_0, x_1 : [0, 1] \to X$ such that $x_i(0) = P, x_i(1) = Q$ are said to be **homotopic** if there is a continuous map

$$x : [0, 1] \times [0, 1] \to X,$$

called a **homotopy** between x_0 and x_1, such that $x(t, 0) = x_0(t)$ and $x(t, 1) = x_1(t)$, $x(0, s) = P, x(1, s) = Q$. See a movie here:
`https://commons.wikimedia.org/wiki/File:Homotopy.gif#/media/File:HomotopySmall.gif`

For example, if $x(t)$ is a path and $g : [0, 1] \to [0, 1]$ is a change of parameter with $g(0) = 0$, $g(1) = 1$ then the paths $x_1(t) = x(t)$ and $x_2(t) = x(g(t))$ are clearly homotopic.

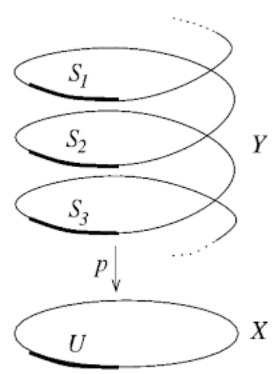
A path-connected Hausdorff space X is said to be **simply connected** if for any $P, Q \in X$ any paths $x_0, x_1 : [0, 1] \to X$ such that $x_i(0) = P, x_i(1) = Q$ are homotopic.

Example 3.1. S^1 is not simply connected but S^n is simply connected for $n \geq 2$.

It is easy to show that any covering has a **homotopy lifting property**: if $b \in X$ and $\widetilde{b} \in p^{-1}(b) \subset Y$ then any path γ starting at b admits a unique lift to a path $\widetilde{\gamma}$ starting at $\widetilde{b}$, i.e., $p(\widetilde{\gamma}) = \gamma$. Moreover, if γ_1, γ_2 are homotopic paths on X then $\widetilde{\gamma}_1, \widetilde{\gamma}_2$ are homotopic on Y (in particular, have the same endpoint). Thus, if Z is a simply connected space with a point z then any continuous map $f : Z \to X$ with $f(z) = b$ lifts to a unique continuous map $\widetilde{f} : Z \to Y$ satisfying $\widetilde{f}(z) = \widetilde{b}$; i.e., $p \circ \widetilde{f} = f$. Namely, to compute $\widetilde{f}(w)$, pick a path β from z to w, let $\gamma = f(\beta)$ and consider the path $\widetilde{\gamma}$. Then the endpoint of $\widetilde{\gamma}$ is $\widetilde{f}(w)$, and it does not depend on the choice of β.

If Z, X are manifolds (of any regularity type), Z is simply connected, and $f : Z \to X$ is a regular map then the lift $\widetilde{f} : Z \to Y$ is also regular. Indeed, if we introduce local coordinates on Y using the homeomorphism between sheets of the covering and their images then $\widetilde{f}$ and f will be locally expressed by the same functions.

A covering $p : Y \to X$ of a path-connected space X is called **universal** if Y is simply connected.



If X is a sufficiently nice space, e.g., a manifold, its universal covering can be constructed as follows. Fix $b \in X$ and let $\widetilde{X}_b$ be the set of homotopy classes of paths on X starting at b. We have a natural map $p : \widetilde{X}_b \to X$, $p(\gamma) = \gamma(1)$. If $U \subset X$ is a small enough neighborhood of a point $x \in X$ then U is simply connected, so we have a natural identification $h : U \times F \to p^{-1}(U)$ with $(p \circ h)(u, f) = u$, where $F = p^{-1}(x)$ is the set of homotopy classes of paths from b to x; namely, $h(u, f)$ is the concatenation of f with any path connecting x with u inside U. Here the **concatenation** $\gamma_1 \circ \gamma_2$ of paths $\gamma_1, \gamma_2 : [0, 1] \to X$ with $\gamma_2(1) = \gamma_1(0)$ is the path $\gamma = \gamma_1 \circ \gamma_2 : [0, 1] \to X$ such that $\gamma(t) = \gamma_2(2t)$ for $t \leq 1/2$ and $\gamma(t) = \gamma_1(2t - 1)$ for $t \geq 1/2$.

The topologies on all such $p^{-1}(U)$ induced by these identifications glue together into a topology on Y, and the map $p : Y \to X$ is then a covering. Moreover, the homotopy lifting property implies that Y is simply connected, so this covering is universal.

It is easy to see that a universal covering $p : Y \to X$ covers any path-connected covering $p' : Y' \to X$, i.e., there is a covering $q : Y \to Y'$ such that $p = p' \circ q$; this is why it is called universal. Therefore a universal covering is unique up to an isomorphism (indeed, if Y, Y' are universal then we have coverings $q_1 : Y \to Y'$ and $q_2 : Y' \to Y$ and $q_1 \circ q_2 = q_2 \circ q_1 = \mathrm{Id}$).

Example 3.2. 1. The map $z \mapsto z^n$ defines an n-sheeted covering $S^1 \to S^1$.

2. The map $x \to e^{ix}$ defines the universal covering $\mathbb{R} \to S^1$.

Now denote by $\pi_1(X, x)$ the set of homotopy classes of *closed* paths on a path-connected space X, starting and ending at x. Then $\pi_1(X, x)$ is a group under concatenation of paths (concatenation is associative since the paths $a(bc)$ and $(ab)c$ differ only by parametrization and hence homotopic). This group is called the **fundamental group** of X relative to the point x. It acts on the fiber $p^{-1}(x)$ for every covering $p : Y \to X$ (by lifting $\gamma \in \pi_1(X, x)$ to Y), which is called the action by **deck transformations**. This action is transitive iff Y is path-connected and moreover free iff Y is universal.

Finally, the group $\pi_1(X, x)$ does not depend on x up to an isomorphism. More precisely, conjugation by any path from x_1 to x_2 defines an isomorphism $\pi_1(X, x_1) \to \pi_1(X, x_2)$ (although two non-homotopic paths may define different isomorphisms if π_1 is non-abelian).

Example 3.3. 1. $\pi_1(S^1) = \mathbb{Z}$.

2. $\pi_1(\mathbb{C} \setminus \{z_1, ..., z_n\}) = F_n$ is a free group in n generators.

3. We have a 2-sheeted universal covering $S^n \to \mathbb{RP}^n$ (real projective space) for $n \geq 2$. Thus $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2$ for $n \geq 2$.

Exercise 3.4. Make sure you can fill all the details in this subsection!

3.2. **Coverings of Lie groups.** Let G be a connected (real or complex) Lie group and $\widetilde{G} = \widetilde{G}_1$ be the universal covering of G, consisting of homotopy classes of paths $x : [0, 1] \to G$ with $x(0) = 1$. Then $\widetilde{G}$ is a group via $(x \cdot y)(t) = x(t)y(t)$, and also a manifold.

Proposition 3.5. *(i) $\widetilde{G}$ is a simply connected Lie group. The covering $p : \widetilde{G} \to G$ is a homomorphism of Lie groups.*

(ii) $\mathrm{Ker}(p)$ *is a central subgroup of* $\widetilde{G}$ *naturally isomorphic to* $\pi_1(G) = \pi_1(G, 1)$*. Thus,* $\widetilde{G}$ *is a central extension of* G *by* $\pi_1(G)$*. In particular,* $\pi_1(G)$ *is abelian.*

Proof. We will only prove (i). We only need to show that $\widetilde{G}$ is a Lie group, i.e., that the multiplication map $\widetilde{m} : \widetilde{G} \times \widetilde{G} \to \widetilde{G}$ is regular. But $\widetilde{G} \times \widetilde{G}$ is simply connected, and $\widetilde{m}$ is a lifting of the map

$$m' := m \circ (p \times p) : \widetilde{G} \times \widetilde{G} \to G \times G \to G,$$

so it is regular. In other words, $\widetilde{m}$ is regular since in local coordinates it is defined by the same functions as m. $\square$

Exercise 3.6. Prove Proposition 3.5(ii).

Remark 3.7. The same argument shows that more generally, the fundamental group of any path-connected topological group is abelian.

Example 3.8. 1. The map $z \mapsto z^n$ defines an n-sheeted covering of Lie groups $S^1 \to S^1$.

2. The map $x \to e^{ix}$ defines the universal covering of Lie groups $\mathbb{R} \to S^1$.

Exercise 3.9. Consider the action of $SU(2)$ on the 3-dimensional real vector space of traceless Hermitian 2-by-2 matrices by conjugation.

(i) Show that this action preserves the positive inner product $(A, B) = \mathrm{Tr}(AB)$ and has determinant 1. Deduce that it defines a homomorphism $\phi : SU(2) \to SO(3)$.

(ii) Show that ϕ is surjective, with kernel ± 1, and is a universal covering map (use that $SU(2) = S^3$ is simply connected). Deduce that $\pi_1(SO(3)) = \mathbb{Z}/2$ and that $SO(3) \cong \mathbb{RP}^3$ as a manifold.

This is demonstrated by the famous **Dirac belt trick**, which illustrates the notion of a **spinor**; namely, spinors are vectors in $\mathbb{C}^2$ acted upon by matrices from $SU(2)$. Here are some videos of the belt trick:
`https://www.youtube.com/watch?v=17Q0tJZcsnY`
`https://www.youtube.com/watch?v=Vfh21o-JW9Q`

3.3. **Closed Lie subgroups.**

Definition 3.10. A **closed Lie subgroup** of a (real or complex) Lie group G is a subgroup which is also an embedded submanifold.

This terminology is justified by the following lemma.

Lemma 3.11. *A closed Lie subgroup of* G *is closed in* G*.*

Exercise 3.12. Prove Lemma 3.11.

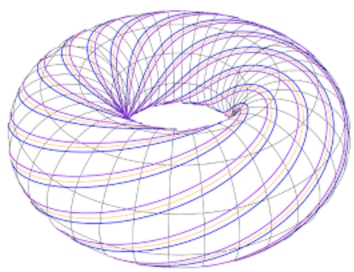
We also have

Theorem 3.13. *Any closed subgroup of a real Lie group G is a closed Lie subgroup.*

This theorem is rather nontrivial, and we will not prove it at this time (it will be proved much later in Exercise 36.13), but we will soon prove a weaker version which suffices for our purposes.

Example 3.14. 1. $SL_n(\mathbb{K})$ is a closed Lie subgroup of $GL_n(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Indeed, the equation $\det A = 1$ defines a smooth hypersurface in the space of matrices (show it!).

2. Let $\phi : \mathbb{R} \to S^1 \times S^1$ be the irrational torus winding given by the formula $\phi(x) = (e^{ix}, e^{ix\sqrt{2}})$:



Then $\phi(\mathbb{R})$ is a subgroup of $S^1 \times S^1$ but not a closed Lie subgroup, since it is not an embedded submanifold: although ϕ is an immersion, the map $\phi^{-1} : \phi(\mathbb{R}) \to \mathbb{R}$ is not continuous.

3.4. **Generation of connected Lie groups by a neighborhood of the identity.**

Proposition 3.15. *(i) If G is a connected Lie group and U a neighborhood of 1 in G then U generates G.*

(ii) If $f : G \to K$ is a homomorphism of Lie groups, K is connected, and $df_1 : T_1G \to T_1K$ is surjective, then f is surjective.

Proof. (i) Let H be the subgroup of G generated by U. Then H is open in G since $H = \cup_{h\in H} hU$. Thus H is an embedded submanifold of G, hence a closed Lie subgroup. Thus by Lemma 3.11 $H \subset G$ is closed. So $H = G$ since G is connected.

(ii) Since df_1 is surjective, by the implicit function theorem $f(G)$ contains some neighborhood of 1 in K. Thus it contains the whole K by (i). $\square$

4. **Homogeneous spaces, Lie group actions**

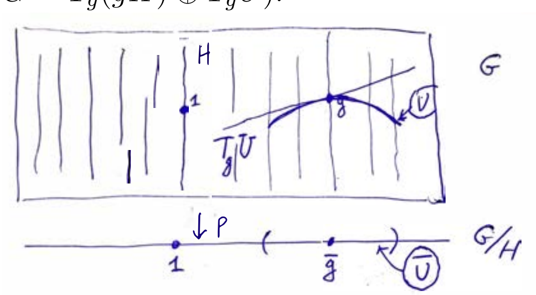
4.1. **Homogeneous spaces.** A regular map of manifolds $p : Y \to X$ is a said to be a **locally trivial fibration** (or **fiber bundle**) with **base** X, **total space** Y and **fiber** being a manifold F if every point $x \in X$ has a neighborhood U such that there is a diffeomorphism $h : U \times F \cong p^{-1}(U)$ with $(p \circ h)(u, f) = u$. In other words, locally p looks like the projection $X \times F \to X$ (the trivial fiber bundle with fiber F over X), but not necessarily globally so. This generalizes the notion of a covering, in which case F is 0-dimensional (discrete).

Theorem 4.1. *(i) Let G be a Lie group of dimension n and $H \subset G$ a closed Lie subgroup of dimension k. Then the* **homogeneous space** *G/H has a natural structure of an $n-k$-dimensional manifold, and the map $p : G \to G/H$ is a locally trivial fibration with fiber H.*

(ii) If moreover H is normal in G then G/H is a Lie group.

(iii) We have a natural isomorphism $T_1(G/H) \cong T_1G/T_1H$.

Proof. Let $\bar{g} \in G/H$ and $g \in p^{-1}(\bar{g})$. Then $gH \subset G$ is an embedded submanifold (image of H under left translation by g). Pick a sufficiently small transversal submanifold U passing through g (i.e., $T_gG = T_g(gH) \oplus T_gU$).



By the inverse function theorem, the set UH is open in G. Let $\overline{U}$ be the image of UH in G/H. Since $p^{-1}(\overline{U}) = UH$ is open, $\overline{U}$ is open in the quotient topology. Also it is clear that $p : U \to \overline{U}$ is a homeomorphism. This defines a local chart near $\bar{g} \in G/H$, and it is easy to check that transition maps between such charts are regular. So G/H acquires the structure of a manifold, which is easily checked to be independent on the choices we made. Also the multiplication map $U \times H \to UH$ is a diffeomorphism, which implies that $p : G \to G/H$ is a locally trivial fibration with fiber H. Finally, we have a surjective linear map $T_gG \to T_{\bar{g}}G/H$ whose kernel is $T_g(gH)$. So in particular for $g = 1$ we get $T_1(G/H) \cong T_1G/T_1H$. This proves all parts of the proposition. ∎

Recall that a sequence of group homomorphisms $d_i : C^i \to C^{i+1}$ is a **complex** if for all i, $d_i \circ d_{i-1}$ is the trivial homomorphism $C_{i-1} \to C_{i+1}$. (One may consider finite complexes, semi-infinite to the left or to the right, or infinite in both directions). In this case $\mathrm{Im}(d_{i-1}) \subset \mathrm{Ker}(d_i)$ is a subgroup. The i-th **cohomology** $H^i(C^\bullet)$ of the complex $C^\bullet$ is the quotient $\mathrm{Ker}(d_i)/\mathrm{Im}(d_{i-1})$. In general it is just a set but if C^i are abelian groups, it is also an abelian group. Also recall that a complex $C^\bullet$ is called **exact** in the i-th term if $\mathrm{Ker}(d_i) = \mathrm{Im}(d_{i-1})$, i.e., if $H^i(C^\bullet)$ is trivial (consists of one element). A complex exact in all its terms (except possibly first and last, where this condition makes no sense) is called an **exact sequence**.

Corollary 4.2. *Let $H \subset G$ be a closed Lie subgroup.*

(i) If H is connected then the map $p_0 : \pi_0(G) \to \pi_0(G/H)$ is a bijection.

(ii) If also G is connected then there is an exact sequence

$$\pi_1(H) \to \pi_1(G) \to \pi_1(G/H) \to 1.$$

Proof. This follows from the theory of covering spaces using that $p : G \to G/H$ is a fibration. □

Exercise 4.3. Fill in the details in the proof of Corollary 4.2.

Remark 4.4. The sequence in Corollary 4.2(ii) is the end portion of the infinite **long exact sequence of homotopy groups of a fibration**,

$$... \to \pi_i(H) \to \pi_i(G) \to \pi_i(G/H) \to \pi_{i-1}(H) \to ...,$$

where $\pi_i(X)$ is the i-th homotopy group of X.

4.2. **Lie subgroups.** We will call the image of an injective immersion of manifolds **an immersed submanifold**; it has a manifold structure coming from the source of the immersion.

Definition 4.5. A **Lie subgroup** of a Lie group G is a subgroup H which is also an *immersed* submanifold (but need not be an *embedded* submanifold, nor a closed subset).

It is clear that in this case H is still a Lie group and the inclusion $H \hookrightarrow G$ is a homomorphism of Lie groups.

Example 4.6. 1. The winding of a torus in Example 3.14(2) realizes $\mathbb{R}$ as a Lie subgroup of $S^1 \times S^1$ which is not closed.

2. Any countable subgroup of G is a 0-dimensional Lie subgroup, but not always a closed one (e.g., $\mathbb{Q} \subset \mathbb{R}$).

Proposition 4.7. *Let $f : G \to K$ be a homomorphism of Lie groups. Then $H := \mathrm{Ker} f$ is a closed normal Lie subgroup in G and $\mathrm{Im} f$ is a Lie subgroup (not necessarily closed) in K if it is an embedded submanifold. In this case we have an isomorphism of Lie groups $G/H \cong \mathrm{Im} f$.*

We will prove Proposition 4.7 in Subsection 9.1.

4.3. **Actions and representations of Lie groups.** Let X be a manifold, G a Lie group, and $a : G \times X \to X$ a set-theoretical left action of G on X.

Definition 4.8. This action is called **regular** if the map a is regular.

From now on, by an action of G on X we will always mean a regular action.

Example 4.9. 1. $GL_n(\mathbb{R})$ and any its Lie subgroup acts on $\mathbb{R}^n$ by linear transformations. Likewise, $GL_n(\mathbb{C})$ and any its Lie subgroup acts on $\mathbb{C}^n$.

2. $SO(3)$ acts on S^2 by rotations.

Definition 4.10. A (real analytic) **finite dimensional representation** of a *real* Lie group G is a linear action of G on a finite dimensional vector space V over $\mathbb{R}$ or $\mathbb{C}$. Similarly, a (complex analytic) finite dimensional representation of a *complex* Lie group G is a linear action of G on a finite dimensional vector space V over $\mathbb{C}$.

In other words, a representation is a homomorphism of Lie groups $\pi_V : G \to GL(V)$.

Definition 4.11. A **(homo)morphism of representations** (or **intertwining operator**) $A : V \to W$ is a linear map which commutes with the G-action, i.e., $A\pi_V(g) = \pi_W(g)A$, $g \in G$.

As usual, an **isomorphism of representations** is an invertible morphism. With these definitions, finite dimensional representations of G form a *category*.

Note also that we have the operations of **dual and tensor product on representations.** Namely, given a representation V of G, we can define its representation on the dual space V^* by

$$\pi_{V^*}(g) = \pi_V(g^{-1})^*,$$

and if W is another representation of G then we can define a representation of G on $V \otimes W$ (the tensor product of vector spaces) by

$$\pi_{V\otimes W}(g) = \pi_V(g) \otimes \pi_W(g).$$

Also if $V \subset W$ is a **subrepresentation** (i.e., a subspace invariant under G) then W/V is also a representation of G, called the **quotient representation**.

4.4. **Orbits and stabilizers.** As in ordinary group theory, if G acts on X and $x \in X$ then we can define the **orbit** $Gx \subset X$ of x as the set of gx, $g \in G$, and the **stabilizer**, or **isotropy group** $G_x \subset G$ to be the group of $g \in G$ such that $gx = x$.

Proposition 4.12. *(The orbit-stabilizer theorem for Lie group actions) The stabilizer $G_x \subset G$ is a closed Lie subgroup, and the natural map $G/G_x \to X$ is an injective immersion whose image is Gx.*

Proposition 4.12 will be proved in Subsection 9.1.

Corollary 4.13. *The orbit $Gx \subset X$ is an immersed submanifold, and we have a natural isomorphism $T_x(Gx) \cong T_1G/T_1G_x$. If Gx is an embedded submanifold then the map $G/G_x \to Gx$ is a diffeomorphism.*

Remark 4.14. Note that Gx need not be closed in X. E.g., let $\mathbb{C}^\times$ act on $\mathbb{C}$ by multiplication. The orbit of 1 is $\mathbb{C}^\times \subset \mathbb{C}$, which is not closed.

Example 4.15. Suppose that G acts on X transitively. Then we get that $X \cong G/G_x$ for any $x \in X$, i.e., X is a **homogeneous space**.

Corollary 4.16. *If G acts transitively on X then the map $p : G \to X$ given by $p(g) = gx$ is a locally trivial fibration with fiber G_x.*

Example 4.17. 1. $SO(3)$ acts transitively on S^2 by rotations, $G_x = S^1 = SO(2)$, so $S^2 = SO(3)/S^1$. Thus $SO(3) = \mathbb{RP}^3$ fibers over S^2 with fiber S^1.

2. $SU(2)$ acts on $S^2 = \mathbb{CP}^1$, and the stabilizer is $S^1 = U(1)$. Thus $SU(2)/S^1 = S^2$, and $SU(2) = S^3$ fibers over S^2 with fiber S^1 (the **Hopf fibration**). Here is a nice keyring model of the Hopf fibration: `http://homepages.wmich.edu/~drichter/hopffibration.htm`

3. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathcal{F}_n(\mathbb{K})$ the set of flags $0 \subset V_1 \subset ... \subset V_n = \mathbb{K}^n$ ($\dim V_i = i$). Then $G = GL_n(\mathbb{K})$ acts transitively on $\mathcal{F}_n(\mathbb{K})$ (check it!). Also let $P \in \mathcal{F}_n(\mathbb{K})$ be the flag for which $V_i = \mathbb{K}^i$ is the subspace of vectors whose all coordinates but the first i are zero. Then G_P is the subgroup $B_n(\mathbb{K}) \subset GL_n(\mathbb{K})$ of invertible upper triangular matrices. Thus $\mathcal{F}_n(\mathbb{K}) = GL_n(\mathbb{K})/B_n(\mathbb{K})$ is a homogeneous space of $GL_n(\mathbb{K})$, in particular, a $\mathbb{K}$-manifold. It is called the **flag manifold**.

4.5. **Left translation, right translation, and adjoint action.** Recall that a Lie group G acts on itself by left translations $L_g(x) = gx$ and right translations $R_{g^{-1}}(x) = xg^{-1}$ (note that both are left actions).

Definition 4.18. The **adjoint action** $\mathrm{Ad}_g : G \to G$ is the action $\mathrm{Ad}_g = L_g \circ R_{g^{-1}} = R_{g^{-1}} \circ L_g$; i.e., $\mathrm{Ad}_g(x) = gxg^{-1}$.

Note this is an action by (inner) automorphisms. Also since $\mathrm{Ad}_g(1) = 1$, we have a linear map $d_1\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$, where $\mathfrak{g} = T_1G$. We will abuse notation and denote this map just by Ad_g. This defines a representation of G on $\mathfrak{g}$ called the **adjoint representation**.

5. **Tensor fields**

5.1. **A crash course on vector bundles.** Let X be a real manifold. A **vector bundle** on X is, informally speaking, a (locally trivial) fiber bundle on X whose fibers are finite dimensional vector spaces. In other words, it is a family of vector spaces parametrized by $x \in X$ and varying regularly with x. More precisely, we have the following definition.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 5.1. A $\mathbb{K}$-**vector bundle** of rank n on X is a manifold E with a surjective regular map $p : E \to X$ and a $\mathbb{K}$-vector space structure on each fiber $p^{-1}(x)$ such that every $x \in X$ has a neighborhood U admitting a diffeomorphism $h : U \times \mathbb{K}^n \to p^{-1}(U)$ with the following properties:
(i) $(p \circ g_U)(u, v) = u$, and
(ii) the map $g_U : p^{-1}(u) \to u \times \mathbb{K}^n$ is $\mathbb{K}$-linear.

In other words, locally on X, E is isomorphic to $X \times \mathbb{K}^n$, but not necessarily globally so.

As for ordinary fiber bundles, E is called the **total space** and X the **base** of the bundle.

Note that even if X is a complex manifold and $\mathbb{K} = \mathbb{C}$, E need not be a complex manifold.

Definition 5.2. A complex vector bundle $p : E \to X$ on a complex manifold X is said to be **holomorphic** if E is a complex manifold and the diffeomorphisms g_U can be chosen holomorphic.

From now on, unless specified otherwise, all complex vector bundles on complex manifolds we consider will be holomorphic.

It follows from the definition that if $p : E \to X$ is a vector bundle then X has an open cover $\{U_\alpha\}$ such that E trivializes on each U_α, i.e., there is a diffeomorphism $g_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$ as above. In this case we have **clutching functions**

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{K})$$

(holomorphic if E is a holomorphic bundle), defined by the formula

$$(g_\alpha \circ g_\beta^{-1})(x, v) = (x, h_{\alpha\beta}(x)v)$$

which satisfy the **consistency conditions**

$$h_{\alpha\beta}(x) = h_{\beta\alpha}(x)^{-1}$$

and

$$h_{\alpha\beta}(x) \circ h_{\beta\gamma}(x) = h_{\alpha\gamma}(x)$$

for $x \in U_\alpha \cap U_\beta \cap U_\gamma$. Moreover, the bundle can be reconstructed from this data, starting from the disjoint union $\sqcup_\alpha U_\alpha \times \mathbb{K}^n$ and identifying (gluing) points according to

$$h_{\alpha\beta} : (x, v) \in U_\beta \times \mathbb{K}^n \sim (x, h_{\alpha\beta}(x)v) \in U_\alpha \times \mathbb{K}^n.$$

The consistency conditions ensure that the relation $\sim$ is symmetric and transitive, so it is an equivalence relation, and we define E to be the space of equivalence classes with the quotient topology. Then E has a natural structure of a vector bundle on X.

This can also be used for constructing vector bundles. Namely, the above construction defines a $\mathbb{K}$-vector bundle on X once we are given a cover $\{U_\alpha\}$ on X and a collection of clutching functions

$$h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL_n(\mathbb{K})$$

satisfying the consistency conditions.

Remark 5.3. All this works more generally for non-linear fiber bundles if we drop the linearity conditions along fibers.

Example 5.4. 1. The **trivial bundle** $p : E = X \times \mathbb{K}^n \to X$, $p(x, v) = x$.

2. The **tangent bundle** is the vector bundle $p : TX \to X$ constructed as follows. For the open cover we take an atlas of charts (U_α, ϕ_α) with transition maps

$$\theta_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta),$$

and we set

$$h_{\alpha\beta}(x) := d_{\phi_\beta(x)}\theta_{\alpha\beta}.$$

(Check that these maps satisfy consistency conditions!)

Thus the tangent bundle TX is a vector bundle of rank $\dim X$ whose fiber $p^{-1}(x)$ is naturally the tangent space T_xX (indeed, the tangent vectors transform under coordinate changes exactly by multiplication by $h_{\alpha\beta}(x)$). In other words, it formalizes the idea of "the tangent space T_xX varying smoothly with $x \in X$".

Definition 5.5. A **section** of a map $p : E \to X$ is a map $s : X \to E$ such that $p \circ s = \mathrm{Id}_x$.

Example 5.6. If $p : X \times Y = E \to X$, $p(x, y) = x$ is the trivial bundle then a section $s : X \to E$ is given by $s(x) = (x, f(x))$ where $y = f(x)$ is a function $X \to Y$, and the image of s is the graph of f. So the notion of a section is a generalization of the notion of a function.

In particular, we may consider sections of a vector bundle $p : E \to X$ over an open set $U \subset X$. These sections form a vector space denoted $\Gamma(U, E)$.

Exercise 5.7. Show that a vector bundle $p : E \to X$ is trivial (i.e., globally isomorphic to $X \times \mathbb{K}^n$) if and only if it admits sections $s_1, ..., s_n$ which form a basis in every fiber $p^{-1}(x)$.

5.2. **Vector fields.**

Definition 5.8. A **vector field** on X is a section of the tangent bundle TX.

Thus in local coordinates a vector field looks like

$$\mathbf{v} = \sum_i v_i \frac{\partial}{\partial x_i},$$

$v_i = v_i(\mathbf{x})$, and if $x_i \mapsto x_i'$ is a change of local coordinates then the expression for $\mathbf{v}$ in the new coordinates is

$$\mathbf{v} = \sum_i v_i' \frac{\partial}{\partial x_i'}$$

where

$$v_i' = \sum_j \frac{\partial x_i'}{\partial x_j} v_j,$$

i.e., the clutching function is the **Jacobi matrix** of the change of variable. Thus, every vector field $\mathbf{v}$ on X defines a derivation of the algebra $O(U)$ for every open set $U \subset X$ compatible with restriction maps $O(U) \to O(V)$ for $V \subset U$; in particular, a derivation $O_x \to O_x$ for all $x \in X$. Conversely, it is easy to see that such a collection of derivations gives rise to a vector field, so this is really the same thing.

A manifold X is called **parallelizable** if its tangent bundle is trivial. By Exercise 5.7, this is equivalent to having a collection of vector fields $\mathbf{v}_1, ..., \mathbf{v}_n$ which form a basis in every tangent space (such a collection is called a **frame**). For example, the circle S^1 and hence the torus $S^1 \times S^1$ are parallelizable. On the other hand, the sphere S^2 is not parallelizable, since it does not even have a single nowhere vanishing vector field (the **Hairy Ball theorem**, or **Hedgehog theorem**). The same is true for any even-dimensional sphere S^{2m}, $m \geq 1$.

5.3. **Tensor fields, differential forms.** Since vector bundles are basically just smooth families of vector spaces varying over some base manifold X, we can do with them the same things we can do with vector spaces - duals, tensor products, symmetric and exterior powers, etc. E.g., the **cotangent bundle** T^*X is dual to the tangent bundle TX.

More generally, we make the following definition.

Definition 5.9. A **tensor field** of rank (k, m) on a manifold X is a section of the tensor product $(TX)^{\otimes k} \otimes (T^*X)^{\otimes m}$.

For example, a tensor field of rank $(1, 0)$ is a vector field. Also, a skew-symmetric tensor field of rank $(0, m)$ is called a **differential** m-**form** on X. In other words, a differential m-form is a section of the vector bundle $\Lambda^m T^*X$.

For instance, if $f \in O(X)$ then we have a differential 1-form df on X, called **the differential of** f (indeed, recall that $d_x f : T_xX \to \mathbb{K}$). A general 1-form can therefore be written in local coordinates as

$$\omega = \sum_i a_i dx_i.$$

where $a_i = a_i(\mathbf{x})$. If coordinates are changed as $x_i \mapsto x_i'$, then in new coordinates

$$\omega = \sum_i a_i' dx_i'$$

where

$$a_i' = \sum_j \frac{\partial x_j}{\partial x_i'} a_j.$$

Thus the clutching function is the **inverse of the Jacobi matrix** of the change of variable. For instance,

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$

More generally, a differential m-form in local coordinates looks like

$$\omega = \sum_{1 \le i_1 < ... < i_m \le n} a_{i_1 ... i_m}(x) dx_{i_1} \wedge ... \wedge dx_{i_m}.$$

5.4. **Left and right invariant tensor fields on Lie groups.** Note that if a Lie group G acts on a manifold X, then it automatically acts on the tangent bundle TX and thus on vector and, more generally, tensor fields on X. In particular, G acts on tensor fields on itself by left and right translations; we will denote this action by L_g and R_g, respectively. We say that a tensor field T on G is **left invariant** if $L_g T = T$ for all $g \in G$, and **right invariant** if $R_g T = T$ for all $g \in G$.

Proposition 5.10. *(i) For any $\tau \in \mathfrak{g}^{\otimes k} \otimes \mathfrak{g}^{*\otimes m}$ there exists a unique left invariant tensor field $\mathbf{L}_\tau$ and a unique right invariant tensor field*

$\mathbf{R}_\tau$ whose value at 1 is τ. Thus, the spaces of such tensor fields are naturally isomorphic to $\mathfrak{g}^{\otimes k} \otimes \mathfrak{g}^{*\otimes m}$.

(ii) $\mathbf{L}_\tau$ is also right invariant iff $\mathbf{R}_\tau$ is also left invariant iff τ is invariant under the adjoint representation Ad_g.

Proof. We only prove (i). Consider the tensor fields $\mathbf{L}_\tau(g) := L_g\tau, \mathbf{R}_\tau(g) := R_{g^{-1}}\tau$ (i.e., we "spread" τ from $1 \in G$ to other points $g \in G$ by left/right translations). By construction, $R_{g^{-1}}\tau$ is right invariant, while $L_g\tau$ is left invariant, both with value τ at 1, and it is clear that these are unique. $\square$

Exercise 5.11. Prove Proposition 5.10(ii).

Corollary 5.12. *A Lie group is parallelizable.*

Proof. Given a basis $e_1, ..., e_n$ of $\mathfrak{g} = T_1G$, the vector fields $L_g e_1, ..., L_g e_n$ form a frame. $\square$

Remark 5.13. In particular, S^1 and $SU(2) = S^3$ are parallelizable. It turns out that S^n for $n \geq 1$ is parallelizable if and only if $n = 1, 3, 7$ (a deep theorem in differential topology). So spheres of other dimensions don't admit a Lie group structure. The sphere S^7 does not admit one either, although it admits a weaker structure of a "homotopy Lie group", or H-space (arising from octonions) which suffices for parallelizability. Thus the only spheres admitting a Lie group structure are $S^0 = \{1, -1\}$, S^1 and S^3. This result is fairly elementary and will be proved in Section 46.

6. **Classical Lie groups**

6.1. **First examples of classical groups.** Roughly speaking, **classical groups** are groups of matrices arising from linear algebra. More precisely, classical groups are the following subgroups of the **general linear group** $GL_n(\mathbb{K})$: $GL_n(\mathbb{K})$, $SL_n(\mathbb{K})$ (the **special linear group**), $O_n(\mathbb{K})$, $SO_n(\mathbb{K})$, $Sp_{2n}(\mathbb{K})$, $O(p,q)$, $SO(p,q)$, $U(p,q)$, $SU(p,q)$, $Sp(2p,2q) := Sp_{2n}(\mathbb{C}) \cap U(2p,2q)$ for $p+q=n$ (and also some others we'll consider later).

Namely,

- The **orthogonal group** $O_n(\mathbb{K})$ is the group of matrices preserving the nondegenerate quadratic form in n variables, $Q = x_1^2 + ... + x_n^2$ (or, equivalently, the corresponding bilinear form $x_1y_1 + ... + x_ny_n$);
- The **symplectic group** $Sp_{2n}(\mathbb{K})$ is the group of matrices preserving a nondegenerate skew-symmetric form in $2n$ variables;
- The **pseudo-orthogonal group** $O(p,q)$, $p+q=n$ is the group of real matrices preserving a nondegenerate quadratic form of signature (p,q), $Q = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_n^2$ (or, equivalently, the corresponding bilinear form);
- The **pseudo-unitary group** $U(p,q)$, $p+q=n$ is the group of complex matrices preserving a nondegenerate Hermitian quadratic form of signature (p,q), $Q = |x_1|^2 + ... + |x_p|^2 - |x_{p+1}|^2 - ... - |x_n|^2$ (or, equivalently, the corresponding sesquilinear form);
- The **special pseudo-orthogonal, pseudo-unitary, and orthogonal groups** $SO(p,q) \subset O(p,q)$, $SU(p,q) \subset U(p,q)$, $SO_n \subset O_n$ are the subgroups of matrices of determinant 1.

Note that the groups don't change under switching p,q and that $(S)O_n(\mathbb{R}) = (S)O(n,0)$; it is also denoted $(S)O(n)$. Also $(S)U(n,0)$ is denoted by $(S)U(n)$.

Exercise 6.1. Show that the special (pseudo)orthogonal groups are index 2 subgroups of the (pseudo)orthogonal groups.

Let us show that they are all Lie groups. For this purpose we'll use the **exponential map** for matrices. Namely, recall from linear algebra that we have an analytic function $\exp : \mathfrak{gl}_n(\mathbb{K}) \to GL_n(\mathbb{K})$ given by the formula

$$\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!},$$

and the matrix-valued analytic function log near $1 \in GL_n(\mathbb{K})$,

$$\log(A) = -\sum_{n=1}^{\infty} \frac{(1-A)^n}{n}.$$

Namely, this is well defined if the spectral radius of $1-A$ is < 1 (i.e., all eigenvalues are in the open unit disk). These maps have the following properties:

1. They are mutually inverse.
2. They are conjugation-invariant.
3. $d\exp_0 = d\log_1 = \mathrm{Id}$.
4. If $xy = yx$ then $\exp(x+y) = \exp(x)\exp(y)$. If $XY = YX$ then $\log(XY) = \log(X) + \log(Y)$ (for X, Y sufficiently close to 1).
5. For $x \in \mathfrak{gl}_n(\mathbb{K})$ the map $t \mapsto \exp(tx)$ is a homomorphism of Lie groups $\mathbb{K} \to GL_n(\mathbb{K})$.
6. $\det\exp(a) = \exp(\mathrm{Tr}\ a)$, $\log(\det A) = \mathrm{Tr}(\log A)$.

Now we can look at classical groups and see what happens to the equations defining them when we apply log.

1. $G = SL_n(\mathbb{K})$. We already showed that it is a Lie group in Example 3.14 but let us re-do it by a different method. The group G is defined by the equation $\det A = 1$. So for A close to 1 we have $\log(\det A) = 0$, i.e., $\mathrm{Tr}\log(A) = 0$. So $\log(A) \in \mathfrak{sl}_n(\mathbb{K}) = \mathfrak{g}$, the space of matrices with trace 0. This defines a local chart near $1 \in G$, showing that G is a manifold, hence a Lie group (namely, local charts near other points are obtained by translation).

2. $G = O_n(\mathbb{K})$. The equation is $A^T = A^{-1}$, thus $\log(A)^T = -\log(A)$, so $\log(A) \in \mathfrak{so}_n(\mathbb{K}) = \mathfrak{g}$, the space of skew-symmetric matrices.

3. $G = U(n)$. The equation is $\overline{A}^T = A^{-1}$, thus $\overline{\log(A)}^T = -\log(A)$, so $\log(A) \in \mathfrak{u}_n = \mathfrak{g}$, the space of skew-Hermitian matrices.

Exercise 6.2. Do the same for all classical groups listed above.

We obtain

Proposition 6.3. *Every classical group G from the above list is a Lie group, with $\mathfrak{g} = T_1G \subset \mathfrak{gl}_n(\mathbb{K})$. Moreover, if $\mathfrak{u} \subset \mathfrak{gl}_n(\mathbb{K})$ is a small neighborhood of 0 and $U = \exp(\mathfrak{u})$ then* exp *and* log *define mutually inverse diffeomorphisms between $\mathfrak{u} \cap \mathfrak{g}$ and $U \cap G$.*

Exercise 6.4. Which of these groups are complex Lie groups?

Exercise 6.5. Use this proposition to compute the dimensions of classical groups: $\dim SL_n = n^2 - 1$, $\dim O_n = n(n-1)/2$, $\dim Sp_{2n} = n(2n+1)$, $\dim SU_n = n^2 - 1$, etc. (Note that for complex groups we give the dimension over $\mathbb{C}$).

6.2. **Quaternions.** An important role in the theory of Lie groups is played by the **algebra of quaternions**, which is the only noncommutative finite dimensional division algebra over $\mathbb{R}$, discovered in the 19th century by W. R. Hamilton.

Definition 6.6. The **algebra of quaternions** is the $\mathbb{R}$-algebra with basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and multiplication rules

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

This algebra is associative but not commutative.

Given a quaternion

$$\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k},\ \ a, b, c, d \in \mathbb{R},$$

we define the **conjugate quaternion** by the formula

$$\overline{\mathbf{q}} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k}.$$

Thus

$$\mathbf{q}\overline{\mathbf{q}} = |\mathbf{q}|^2 = a^2 + b^2 + c^2 + d^2 \in \mathbb{R},$$

where $|\mathbf{q}|$ is the length of $\mathbf{q}$ as a vector in $\mathbb{R}^4$. So if $\mathbf{q} \neq 0$ then it is invertible and

$$\mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}.$$

Thus $\mathbb{H}$ is a **division algebra** (i.e., a skew-field). One can show that the only finite dimensional associative division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. (See Exercise 6.9).

In particular, we can do linear algebra over $\mathbb{H}$ in almost the same way as we do over ordinary fields. Namely, every (left or right) module over $\mathbb{H}$ is free and has a basis; such a module is called a (left or right) **quaternionic vector space**. In particular, any (say, right) quaternionic vector space of dimension n (i.e., with basis of n elements) is isomorphic to $\mathbb{H}^n$. Moreover, $\mathbb{H}$-linear maps between such spaces are given by left multiplication by quaternionic matrices. Finally, it is easy to see that Gaussian elimination works the same way as over ordinary fields; in particular, every invertible square matrix over $\mathbb{H}$ is a product of elementary matrices of the form $1 + (\mathbf{q} - 1)E_{ii}$ and $1 + \mathbf{q}E_{ij}$, $i \neq j$, where $\mathbf{q} \in \mathbb{H}$ is nonzero.

Also it is easy to show that

$$\overline{\mathbf{q}_1\mathbf{q}_2} = \overline{\mathbf{q}_2\mathbf{q}_1},\ \ |\mathbf{q}_1\mathbf{q}_2| = |\mathbf{q}_1| \cdot |\mathbf{q}_2|$$

(check this!). So quaternions are similar to complex numbers, except they are non-commutative. Finally, note that $\mathbb{H}$ contains a copy of $\mathbb{C}$ spanned by $1, \mathbf{i}$; however, this does not make $\mathbb{H}$ as $\mathbb{C}$-algebra since $\mathbf{i}$ is not a central element.

Proposition 6.7. *The group of unit quaternions* $\{\mathbf{q} \in \mathbb{H} : |\mathbf{q}| = 1\}$ *under multiplication is isomorphic to* $SU(2)$ *as a Lie group.*

Proof. We can realize $\mathbb{H}$ as $\mathbb{C}^2$, where $\mathbb{C} \subset \mathbb{H}$ is spanned by $1, \mathbf{i}$; namely, $(z_1, z_2) \mapsto z_1 + \mathbf{j}z_2$. Then left multiplication by quaternions on $\mathbb{H} = \mathbb{C}^2$ commutes with right multiplication by $\mathbb{C}$, i.e., is $\mathbb{C}$-linear. So it is given by complex 2-by-2 matrices. It is easy to compute that the corresponding matrix is

$$z_1 + z_2\mathbf{j} \mapsto \begin{pmatrix} z_1 & z_2 \\ -\overline{z_2} & \overline{z_1} \end{pmatrix},$$

and we showed in Example 2.3(5) that such matrices (with $|z_1|^2+|z_2|^2 = 1$) are exactly the matrices from $SU(2)$. $\square$

This is another way to see that $SU(2) \cong S^3$ as a manifold (since the set of unit quaternions is manifestly S^3).

Corollary 6.8. *The map* $\mathbf{q} \mapsto (\frac{\mathbf{q}}{|\mathbf{q}|}, |\mathbf{q}|)$ *is an isomorphism of Lie groups* $\mathbb{H}^\times \cong SU(2) \times \mathbb{R}_{>0}$.

This is the quaternionic analog of the trigonometric form of complex numbers, except the "phase" factor $\frac{\mathbf{q}}{|\mathbf{q}|}$ is now not in S^1 but in $S^3 = SU(2)$.

Exercise 6.9. Let D be a finite dimensional division algebra over $\mathbb{R}$.

(i) Show that if D is commutative then $D = \mathbb{R}$ or $D = \mathbb{C}$.

(ii) Assume that D is not commutative. Take $\mathbf{q} \in D$, $\mathbf{q} \notin \mathbb{R}$. Show that there exist $a, b \in \mathbb{R}$ such that $\mathbf{i} := a + b\mathbf{q}$ satisfies $\mathbf{i}^2 = -1$.

(iii) Decompose D into the eigenspaces $D_\pm$ of the operator of conjugation by $\mathbf{i}$ with eigenvalues ± 1 and show that $1, \mathbf{i}$ is a basis of D_+, i.e., $D_+ \cong \mathbb{C}$.

(iv) Pick $\mathbf{q} \in D_-$, $\mathbf{q} \neq 0$, and show that $D_- = D_+\mathbf{q}$, so $\{1, \mathbf{i}, \mathbf{q}, \mathbf{iq}\}$ is a basis of D over $\mathbb{R}$. Deduce that $\mathbf{q}^2$ is a central element of D.

(v) Conclude that $\mathbf{q}^2 = -\lambda$ where $\lambda \in \mathbb{R}_{>0}$ and deduce that $D \cong \mathbb{H}$.

6.3. **More classical groups.** Now we can define a new classical group $GL_n(\mathbb{H})$, a real Lie group of dimension $4n^2$, called the **quaternionic general linear group**. For example, as we just showed, $GL_1(\mathbb{H}) = \mathbb{H}^\times \cong SU(2) \times \mathbb{R}_{>0}$.

For $A \in GL_n(\mathbb{H})$, let $\det A$ be the determinant of A as a linear operator on $\mathbb{C}^{2n} = \mathbb{H}^n$.

Lemma 6.10. *We have* $\det A > 0$.

Proof. For $n = 1$, $A = \mathbf{q} \in \mathbb{H}^\times$ and $\det \mathbf{q} = |\mathbf{q}|^2 > 0$. It follows that $\det(1+(\mathbf{q}-1)E_{ii}) = |\mathbf{q}|^2 > 0$. Also it is easy to see that $\det(1+\mathbf{q}E_{ij}) = 1$ for $i \neq j$. It then follows by Gaussian elimination that for any A we have $\det(A) > 0$. $\square$

Let $SL_n(\mathbb{H}) \subset GL_n(\mathbb{H})$ be the subgroup of matrices A with $\det A = 1$, called the **quaternionic special linear group**.

Exercise 6.11. Show that $SL_n(\mathbb{H}) \subset GL_n(\mathbb{H})$ is a normal subgroup, and $GL_n(\mathbb{H}) \cong SL_n(\mathbb{H}) \times \mathbb{R}_{>0}$.

Thus $SL_n(\mathbb{H})$ is a real Lie group of dimension $4n^2 - 1$.

We can also define groups of quaternionic matrices preserving various sesquilinear forms. Namely, let $V \cong \mathbb{H}^n$ be a right quaternionic vector space.

Definition 6.12. A **sesquilinear form** on V is a biadditive function $(,) : V \times V \to \mathbb{H}$ such that

$$(\mathbf{x}\alpha, \mathbf{y}\beta) = \overline{\alpha}(\mathbf{x}, \mathbf{y})\beta, \ \mathbf{x}, \mathbf{y} \in V, \ \alpha, \beta \in \mathbb{H}.$$

Such a form is called **Hermitian** if $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$ and **skew-Hermitian** if $(\mathbf{x}, \mathbf{y}) = -\overline{(\mathbf{y}, \mathbf{x})}$.

Note that the order of factors is important here!

Proposition 6.13. *(i) Every nondegenerate Hermitian form on V in some basis takes the form*

$$(\mathbf{x}, \mathbf{y}) = \overline{x_1}y_1 + ... + \overline{x_p}y_p - \overline{x_{p+1}}y_{p+1} - ... - \overline{x_n}y_n$$

for a unique pair (p, q) with $p + q = n$.

(ii) Every nondegenerate skew-Hermitian form on V in some basis takes the form

$$(\mathbf{x}, \mathbf{y}) = \overline{x_1}\mathbf{j}y_1 + ... + \overline{x_n}\mathbf{j}y_n.$$

Exercise 6.14. Prove Proposition 6.13.

In (i), the pair (p, q) is called the **signature** of the quaternionic Hermitian form.

Exercise 6.15. Show that a nondegenerate quaternionic Hermitian form of signature (p, q) can be written as

$$(\mathbf{x}, \mathbf{y}) = B_1(\mathbf{x}, \mathbf{y}) + \mathbf{j}B_2(\mathbf{x}, \mathbf{y}),$$

with B_1, B_2 taking values in $\mathbb{C} = \mathbb{R} + \mathbb{R}\mathbf{i} \subset \mathbb{H}$, where B_1 is a usual nondegenerate Hermitian form of signature $(2p, 2q)$ and B_2 is a nondegenerate skew-symmetric bilinear form on V as a ($2n$-dimensional) $\mathbb{C}$-vector space. Show that $B_2(\mathbf{x}, \mathbf{y}) = B_1(\mathbf{x}\mathbf{j}, \mathbf{y})$. Deduce that any complex linear transformation preserving B_1 and B_2 is $\mathbb{H}$-linear.

Thus the group of symmetries of a nondegenerate quaternionic Hermitian form of signature (p, q) is $Sp(2p, 2q) = Sp_{2n}(\mathbb{C}) \cap U(2p, 2q)$. It is called the **quaternionic pseudo-unitary group**.

One also sometimes uses the notation $U(p,q,\mathbb{R}) = O(p,q)$, $U(p,q,\mathbb{C}) = U(p,q)$, $U(p,q,\mathbb{H}) = Sp(2p,2q)$, and $U(n,0,\mathbb{K}) = U(n,\mathbb{K})$ for $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H}$.

Exercise 6.16. Show that a nondegenerate quaternionic skew-Hermitian form can be written as

$$(\mathbf{x},\mathbf{y}) = B_1(\mathbf{x},\mathbf{y}) + \mathbf{j}B_2(\mathbf{x},\mathbf{y}),$$

with B_1, B_2 taking values in $\mathbb{C} = \mathbb{R} + \mathbb{R}\mathbf{i} \subset \mathbb{H}$, where B_1 is an ordinary skew-Hermitian form, while B_2 is a symmetric bilinear form (both nondegenerate). Show that $B_2(\mathbf{x},\mathbf{y}) = B_1(\mathbf{x}\mathbf{j},\mathbf{y})$. Deduce that any complex linear transformation preserving B_1 and B_2 is $\mathbb{H}$-linear. Also show that the signature of the Hermitian form iB_1 is necessarily (n,n).

Thus the group of symmetries of a nondegenerate quaternionic skew-Hermitian form is $O_{2n}(\mathbb{C}) \cap U(n,n)$. This group is denoted by $O^*(2n)$ and called the **quaternionic orthogonal group**. There is also the subgroup $SO^*(2n) \subset O^*(2n)$ of matrices of determinant 1 (having index 2).

All of these groups are Lie groups, which is shown similarly to Subsection 6.1, using the exponential map.

Exercise 6.17. Compute the dimensions of all classical groups introduced above.

7. The exponential map of a Lie group

7.1. **The exponential map.** We will now generalize the exponential and logarithm maps from matrix groups to arbitrary Lie groups.

Let G be a real Lie group, $\mathfrak{g} = T_1 G$.

Proposition 7.1. *Let $x \in \mathfrak{g}$. There is a unique morphism of Lie groups $\gamma = \gamma_x : \mathbb{R} \to G$ such that $\gamma'(0) = x$.*

Proof. For such a morphism we should have

$$\gamma(t+s) = \gamma(t)\gamma(s),\ \ t, s \in \mathbb{R},$$

so differentiating by s at $s = 0$, we get[5]

$$\gamma'(t) = \gamma(t)x.$$

Thus $\gamma(t)$ is a solution of the ODE defined by the left-invariant vector field $\mathbf{L}_x$ corresponding to $x \in \mathfrak{g}$ with initial condition $\gamma(0) = 1$. By the existence and uniqueness theorem for solutions of ODE, this equation has a unique solution with this initial condition defined for $|t| < \varepsilon$ for some $\varepsilon > 0$. Moreover, if $|s| + |t| < \varepsilon$, both $\gamma_1(t) := \gamma(s+t)$ and $\gamma_2(t) := \gamma(s)\gamma(t)$ satisfy this differential equation with initial condition $\gamma_1(0) = \gamma_2(0) = \gamma(s)$, so $\gamma_1 = \gamma_2$. Thus

$$\gamma(s+t) = \gamma(s)\gamma(t),\ \ |s| + |t| < \varepsilon;$$

hence $\gamma(t)x = x\gamma(t)$ for $|t| < \varepsilon$.

We claim that the solution $\gamma(t)$ extends to all values of $t \in \mathbb{R}$. Indeed, let us prove that it extends to $|t| < 2^n\varepsilon$ for all $n \geq 0$ by induction in n. The base of induction ($n = 0$) is already known, so we only need to justify the induction step from $n-1$ to n. Given t with $|t| < 2^n\varepsilon$, we define

$$\gamma(t) := \gamma(\tfrac{t}{2})^2.$$

This agrees with the previously defined solution for $|t| < 2^{n-1}\varepsilon$, and we have

$$\gamma'(t) = \tfrac{1}{2}(\gamma'(\tfrac{t}{2})\gamma(\tfrac{t}{2}) + \gamma(\tfrac{t}{2})\gamma'(\tfrac{t}{2})) = \tfrac{1}{2}\gamma(\tfrac{t}{2})x\gamma(\tfrac{t}{2}) + \tfrac{1}{2}\gamma(\tfrac{t}{2})^2 x = \gamma(\tfrac{t}{2})^2 x = \gamma(t)x,$$

as desired.

Thus, we have a regular map $\gamma : \mathbb{R} \to G$ with $\gamma(s+t) = \gamma(s)\gamma(t)$ and $\gamma'(0) = x$, which is unique by the uniqueness of solutions of ODE. $\square$

Definition 7.2. The **exponential map** $\exp : \mathfrak{g} \to G$ is defined by the formula $\exp(x) = \gamma_x(1)$.

Thus $\gamma_x(t) = \exp(tx)$. So we have

[5] For brevity for $g \in G$, $x \in \mathfrak{g}$ we denote $L_g x$ by gx and $R_g x$ by xg.

Proposition 7.3. *The flow defined by the right-invariant vector field $\mathbf{R}_x$ is given by $g \mapsto \exp(tx)g$, and the flow defined by the left-invariant vector field $\mathbf{L}_x$ is given by $g \mapsto g\exp(tx)$.*

Example 7.4. 1. Let $G = \mathbb{K}^n$. Then $\exp(x) = x$.

2. Let $G = GL_n(\mathbb{K})$ or its Lie subgroup. Then $\gamma_x(t)$ satisfies the matrix differential equation

$$\gamma'(t) = \gamma(t)x$$

with $\gamma(0) = 1$, so

$$\gamma_x(t) = e^{tx},$$

the matrix exponential. For example, if $n = 1$, this is the usual exponential function.

The following theorem describes the basic properties of the exponential map. Let G be a real or complex Lie group.

Theorem 7.5. *(i)* $\exp : \mathfrak{g} \to G$ *is a regular map which is a diffeomorphism of a neighborhood of* $0 \in \mathfrak{g}$ *onto a neighborhood of* $1 \in G$*, with* $\exp(0) = 1$, $\exp'(0) = \mathrm{Id}_{\mathfrak{g}}$.

(ii) $\exp((s+t)x) = \exp(sx)\exp(tx)$ *for* $x \in \mathfrak{g}$, $s, t \in \mathbb{K}$.

(iii) For any morphism of Lie groups $\phi : G \to K$ and $x \in T_1G$ we have

$$\phi(\exp(x)) = \exp(\phi_* x);$$

i.e., the exponential map commutes with morphisms.

(iv) For any $g \in G$, $x \in \mathfrak{g}$, we have

$$g\exp(x)g^{-1} = \exp(\mathrm{Ad}_g x).$$

Proof. (i) The regularity of exp follows from the fact that if a differential equation depends regularly on parameters then so do its solutions. Also $\gamma_0(t) = 1$ so $\exp(0) = 1$. We have $\exp'(0)x = \frac{d}{dt}\exp(tx)|_{t=0} = x$, so $\exp'(0) = \mathrm{Id}$. By the inverse function theorem this implies that exp is a diffeomorphism near the origin.

(ii) Holds since $\exp(tx) = \gamma_x(t)$.

(iii) Both $\phi(\exp(tx))$ and $\exp(\phi_*(tx))$ satisfy the equation $\gamma'(t) = \gamma(t)\phi_*(x)$ with the same initial conditions.

(iv) is a special case of (iii) with $\phi : G \to G$, $\phi(h) = ghg^{-1}$. $\square$

Thus exp has an inverse $\log : U \to \mathfrak{g}$ defined on a neighborhood U of $1 \in G$ with $\log(1) = 0$. This map is called the **logarithm**. For $GL_n(\mathbb{K})$ and its Lie subgroups it coincides with the matrix logarithm. The logarithm map defines a canonical coordinate chart on G near 1, so a choice of a basis of $\mathfrak{g}$ gives a local coordinate system.

Proposition 7.6. *Let G be a connected Lie group and $\phi : G \to K$ a morphism of Lie groups. Then ϕ is completely determined by the linear map $\phi_* : T_1G \to T_1K$.*

Proof. We have $\phi(\exp(x)) = \exp(\phi_*(x))$, so since exp is a diffeomorphism near 0, ϕ is determined by ϕ_* on a neighborhood of $1 \in G$. This completely determines ϕ since this neighborhood generates G by Proposition 3.15. □

Exercise 7.7. (i) Show that a connected compact complex Lie group is abelian. (**Hint:** consider the adjoint representation and use that a holomorphic function on a compact complex manifold is constant, by the maximum principle.)

(ii) Classify such Lie groups of dimension n up to isomorphism (Show that they are compact complex tori whose isomorphism classes are bijectively labeled by elements of the set $GL_n(\mathbb{C})\backslash GL_{2n}(\mathbb{R})/GL_{2n}(\mathbb{Z})$.)

(iii) Work out the classification explicitly in the 1-dimensional case (this is the classification of complex elliptic curves). Namely, show that isomorphism classes are labeled by points of $\mathbb{H}/\Gamma$, where $\mathbb{H}$ is the upper half-plane and $\Gamma = SL_2(\mathbb{Z})$ acting on $\mathbb{H}$ by Möbius transformations $\tau \mapsto \frac{a\tau+b}{c\tau+d}$ (where $\mathrm{Im}(\tau) > 0$).

7.2. **The commutator.** In general (say, for $G = GL_n(\mathbb{K})$, $n \geq 2$), $\exp(x + y) \neq \exp(x)\exp(y)$. So let us consider the map

$$(x, y) \mapsto \mu(x, y) = \log(\exp(x)\exp(y))$$

which maps $U \times U \to \mathfrak{g}$, where $U \subset \mathfrak{g}$ is a neighborhood of 0. This map expresses the product in G in the coordinate chart coming from the logarithm map. We have $\mu(x, 0) = \mu(0, x) = x$ and $\mu_*(x, y) = x+y$, so

$$\mu(x, y) = x + y + \tfrac{1}{2}\mu_2(x, y) + ...$$

where $\mu_2 = d^2\mu_{(0,0)}$ is the quadratic part and ... are higher terms. Moreover, $\mu_2(x, 0) = \mu_2(0, y) = 0$, hence μ_2 is a bilinear map $\mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$. It is easy to see that $\mu(x, -x) = 0$, hence μ_2 is skew-symmetric.

Definition 7.8. The map μ_2 is called the **commutator** and denoted by $x, y \mapsto [x, y]$.

Thus we have

$$\exp(x)\exp(y) = \exp(x + y + \tfrac{1}{2}[x, y] + ...).$$

Example 7.9. Let $G = GL_n(\mathbb{K})$. Then

$$\exp(x)\exp(y) = (1+x+\tfrac{x^2}{2}+...)(1+y+\tfrac{y^2}{2}+...) = 1+x+y+\tfrac{x^2}{2}+xy+\tfrac{y^2}{2}+... =$$

$$1 + (x + y) + \frac{(x+y)^2}{2} + \frac{xy-yx}{2} + ... = \exp(x + y + \frac{xy-yx}{2} + ...)$$

Thus

$$[x, y] = xy - yx.$$

This justifies the term "commutator": it measures the failure of x and y to commute.

Corollary 7.10. *If $G \subset GL_n(\mathbb{K})$ is a Lie subgroup then $\mathfrak{g} = T_1G \subset \mathfrak{gl}_n(\mathbb{K})$ is closed under the commutator $[x, y] = xy - yx$, which coincides with the commutator of G.*

For $x \in \mathfrak{g}$ define the linear map $\mathrm{ad}x : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}x(y) = [x, y].$$

Proposition 7.11. *(i) Let G, K be Lie groups and $\phi : G \to K$ a morphism of Lie groups. Then $\phi_* : T_1G \to T_1K$ preserves the commutator:*

$$\phi_*([x, y]) = [\phi_*(x), \phi_*(y)].$$

(ii) The adjoint action preserves the commutator.

(iii) We have

$$\exp(x)\exp(y)\exp(x)^{-1}\exp(y)^{-1} = \exp([x, y] + ...)$$

where ... denotes cubic and higher terms.

(iv) Let $X(t), Y(s)$ be parametrized curves on G such that $X(0) = Y(0) = 1$, $X'(0) = x, Y'(0) = y$. Then we have

$$[x, y] = \lim_{s,t\to 0} \frac{\log(X(t)Y(s)X(t)^{-1}Y(s)^{-1})}{ts}.$$

In particular,

$$[x, y] = \lim_{s,t\to 0} \frac{\log(\exp(tx)\exp(sy)\exp(tx)^{-1}\exp(sy)^{-1})}{ts}$$

and

$$[x, y] = \frac{d}{dt}|_{t=0}\mathrm{Ad}_{X(t)}(y).$$

Thus $\mathrm{ad} = \mathrm{Ad}_*$*, the differential of* Ad *at* $1 \in G$*.*

(v) If G is commutative (=abelian) then $[x, y] = 0$ for all x, y.

Proof. (i) Follows since ϕ commutes with the exponential map.

(ii) Follows from (i) by setting $\phi = \mathrm{Ad}_g$.

(iii) Modulo cubic and higher terms we have

$$\log(\exp(x)\exp(y)) = \log(\exp(y)\exp(x)) + [x, y] + ...,$$

which implies the statement by exponentiation.

(iv) Let $\log X(t) = x(t)$, $\log Y(s) = y(s)$. Then by (iii) we have

$$\log(X(t)Y(s)X(t)^{-1}Y(s)^{-1}) =$$

$$\log(\exp(x(t))\exp(y(s))\exp(x(t))^{-1}\exp(y(s))^{-1}) = ts([x,y]+o(1)),\ t,s\to 0.$$

This implies the first two statements. The last statement follows by taking the limit in s first, then in t.

(v) follows from (iii). $\square$

8. **Lie algebras**

8.1. **The Jacobi identity.** The matrix commutator $[x, y] = xy - yx$ obviously satisfies the identity

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$$

called the **Jacobi identity**. Thus it is satisfied for any Lie subgroup of $GL_n(\mathbb{K})$.

Proposition 8.1. *The Jacobi identity holds for any Lie group G.*

Proof. Let $\mathfrak{g} = T_1G$. The Jacobi identity is equivalent to $\mathrm{ad}x$ being a derivation of the commutator:

$$\mathrm{ad}x([y, z]) = [\mathrm{ad}x(y), z] + [y, \mathrm{ad}x(z)],\ \ x, y, z \in \mathfrak{g}.$$

To show that it is indeed a derivation, let $g(t) = \exp(tx)$, then

$$\mathrm{Ad}_{g(t)}([y, z]) = [\mathrm{Ad}_{g(t)}(y), \mathrm{Ad}_{g(t)}(z)].$$

The desired identity is then obtained by differentiating this equality by t at $t = 0$ and using the Leibniz rule and Proposition 7.11(iv). □

Corollary 8.2. *We have* $\mathrm{ad}[x, y] = [\mathrm{ad}x, \mathrm{ad}y]$.

Proof. This is also equivalent to the Jacobi identity. □

Proposition 8.3. *For $x \in \mathfrak{g}$ one has* $\exp(\mathrm{ad}x) = \mathrm{Ad}_{\exp(x)} \in GL(\mathfrak{g})$.

Proof. We will show that $\exp(t\mathrm{ad}x) = \mathrm{Ad}_{\exp(tx)}$ for $t \in \mathbb{R}$. Let $\gamma_1(t) = \exp(t\mathrm{ad}x)$ and $\gamma_2(t) = \mathrm{Ad}_{\exp(tx)}$. Then γ_1, γ_2 both satisfy the differential equation $\gamma'(t) = \gamma(t)\mathrm{ad}x$ and equal 1 at $t = 0$. Thus $\gamma_1 = \gamma_2$. □

8.2. **Lie algebras.**

Definition 8.4. A **Lie algebra** over a field $\mathbf{k}$ is a vector space $\mathfrak{g}$ over $\mathbf{k}$ equipped with bilinear operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **commutator** or **(Lie) bracket** which satisfies the following identities:

(i) $[x, x] = 0$ for all $x \in \mathfrak{g}$;

(ii) the Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

A **(homo)morphism of Lie algebras** is a linear map between Lie algebras that preserves the commutator.

Remark 8.5. If $\mathbf{k}$ has characteristic $\neq 2$ then the condition $[x, x] = 0$ is equivalent to skew-symmetry $[x, y] = -[y, x]$, but in characteristic 2 it is stronger.

Example 8.6. Any subspace of $\mathfrak{gl}_n(\mathbf{k})$ closed under $[x, y] := xy - yx$ is a Lie algebra.

Example 8.7. The map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is a morphism of Lie algebras.

Thus we have

Theorem 8.8. *If G is a $\mathbb{K}$-Lie group (for $\mathbb{K} = \mathbb{R}, \mathbb{C}$) then $\mathfrak{g} := T_1G$ has a natural structure of a Lie algebra over $\mathbb{K}$. Moreover, if $\phi : G \to K$ is a morphism of Lie groups then $\phi_* : T_1G \to T_1K$ is a morphism of Lie algebras.*

We will denote the Lie algebra $\mathfrak{g} = T_1G$ by $\mathrm{Lie}G$ or $\mathrm{Lie}(G)$ and call it the **Lie algebra of** G. We see that the assignment $G \mapsto \mathrm{Lie}G$ is a functor from the category of Lie groups to the category of Lie algebras. Thus we have a map $\mathrm{Hom}(G, K) \to \mathrm{Hom}(\mathrm{Lie}G, \mathrm{Lie}K)$, which is injective if G is connected.

Motivated by Proposition 7.11(v), a Lie algebra $\mathfrak{g}$ is said to be **commutative** or **abelian** if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$.

8.3. **Lie subalgebras and ideals.** A **Lie subalgebra** of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ closed under the commutator. It is called a **Lie ideal** if moreover $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.

Proposition 8.9. *Let $H \subset G$ be a Lie subgroup. Then:*
(i) $\mathrm{Lie}H \subset \mathrm{Lie}G$ *is a Lie subalgebra;*
(ii) If H is normal then $\mathrm{Lie}H$ *is a Lie ideal in* $\mathrm{Lie}G$*;*
(iii) If G, H are connected and $\mathrm{Lie}H \subset \mathrm{LieG}$ *is a Lie ideal then H is normal in G.*

Proof. (i) If $x, y \in \mathfrak{h}$ then $\exp(tx), \exp(sy) \in H$, so by Proposition 7.11(iv)

$$[x, y] = \lim_{t,s\to 0} \frac{\log(\exp(tx)\exp(sy)\exp(-tx)\exp(-sy))}{ts} \in \mathfrak{h}.$$

(ii) We have $ghg^{-1} \in H$ for $g \in G$ and $h \in H$. Thus, taking $h = \exp(sy)$, $y \in \mathfrak{h}$ and taking the derivative in s at zero, we get $\mathrm{Ad}_g(y) \in \mathfrak{h}$. Now taking $g = \exp(tx)$, $x \in \mathfrak{g}$ and taking the derivative in t at zero, by Proposition 7.11(iv) we get $[x, y] \in \mathfrak{h}$, i.e., $\mathfrak{h}$ is a Lie ideal.

(iii) If $x \in \mathfrak{g}$, $y \in \mathfrak{h}$ are small then

$$\exp(x)\exp(y)\exp(x)^{-1} =$$

$$\exp(\mathrm{Ad}_{\exp(x)}y) = \exp(\exp(\mathrm{ad}x)y) = \exp(\sum_{n=0}^{\infty} \tfrac{(\mathrm{ad}x)^n y}{n!}) \in H$$

since $\sum_{n=0}^{\infty} \frac{(\mathrm{ad}x)^n y}{n!} \in \mathfrak{h}$. So G acting on itself by conjugation maps a small neighborhood of 1 in H into H (as G is generated by its neighborhood of 1 by Proposition 3.15, since it is connected). But H is also connected, so is generated by its neighborhood of 1, again by Proposition 3.15. Hence H is normal. $\square$

8.4. **The Lie algebra of vector fields.** Recall that a vector field on a manifold X is a compatible family of derivations $\mathbf{v} : O(U) \to O(U)$ for open subsets $U \subset X$.

Proposition 8.10. *If* $\mathbf{v}, \mathbf{w}$ *are derivations of an algebra* A *then so is* $[\mathbf{v}, \mathbf{w}] := \mathbf{v}\mathbf{w} - \mathbf{w}\mathbf{v}$.

Proof. We have

$$(\mathbf{v}\mathbf{w} - \mathbf{w}\mathbf{v})(ab) = \mathbf{v}(\mathbf{w}(a)b + a\mathbf{w}(b)) - \mathbf{w}(\mathbf{v}(a)b + a\mathbf{v}(b)) =$$

$$\mathbf{v}\mathbf{w}(a)b + \mathbf{w}(a)\mathbf{v}(b) + \mathbf{v}(a)\mathbf{w}(b) + a\mathbf{v}\mathbf{w}(b)$$

$$-\mathbf{w}\mathbf{v}(a)b - \mathbf{v}(a)\mathbf{w}(b) - \mathbf{w}(a)\mathbf{v}(b) - a\mathbf{w}\mathbf{v}(b) =$$

$$(\mathbf{v}\mathbf{w} - \mathbf{w}\mathbf{v})(a)b + a(\mathbf{v}\mathbf{w} - \mathbf{w}\mathbf{v})(b).$$

$\square$

Thus, the space $\mathrm{Vect}(X)$ of vector fields on X is a Lie algebra under the operation

$$\mathbf{v}, \mathbf{w} \mapsto [\mathbf{v}, \mathbf{w}],$$

called the **Lie bracket of vector fields**.[6]

In local coordinates we have

$$\mathbf{v} = \sum_i v_i \frac{\partial}{\partial x_i}, \ \ \mathbf{w} = \sum w_j \frac{\partial}{\partial x_j},$$

so

$$[\mathbf{v}, \mathbf{w}] = \sum_i \left(\sum_j (v_j \tfrac{\partial w_i}{\partial x_j} - w_j \tfrac{\partial v_i}{\partial x_j}) \right) \tfrac{\partial}{\partial x_i}.$$

This implies that if vector fields $\mathbf{v}, \mathbf{w}$ are tangent to a k-dimensional submanifold $Y \subset X$ then so is their Lie bracket $[\mathbf{v}, \mathbf{w}]$. Indeed, in local coordinates Y is given by equations $x_{k+1} = ... = x_n = 0$, and in such coordinates a vector field is tangent to Y iff it does not contain terms with $\frac{\partial}{\partial x_j}$ for $j > k$.

[6]Note that this Lie algebra is infinite dimensional for all real manifolds and many (but not all) complex manifolds of positive dimension.

Exercise 8.11. Let $U \subset \mathbb{R}^n$ be an open subset, $\mathbf{v}, \mathbf{w} \in \mathrm{Vect}(U)$ and g_t, h_t be the associated flows, defined in a neighborhood of every point of U for small t. Show that for any $\mathbf{x} \in U$

$$\lim_{t,s \to 0} \frac{g_t h_s g_t^{-1} h_s^{-1}(\mathbf{x}) - \mathbf{x}}{ts} = [\mathbf{v}, \mathbf{w}](\mathbf{x}).$$

Now let G be a Lie group and $\mathrm{Vect}_L(G), \mathrm{Vect}_R(G) \subset \mathrm{Vect}(G)$ be the subspaces of left and right invariant vector fields.

Proposition 8.12. $\mathrm{Vect}_L(G), \mathrm{Vect}_R(G) \subset \mathrm{Vect}(G)$ *are Lie subalgebras which are both canonically isomorphic to* $\mathfrak{g} = \mathrm{Lie}G$.

Proof. The first statement is obvious, so we prove only the second statement. Let $\mathbf{x}, \mathbf{y} \in \mathrm{Vect}_L(G)$. Then $\mathbf{x} = \mathbf{L}_x$, $\mathbf{y} = \mathbf{L}_y$ for $x = \mathbf{x}(1), y = \mathbf{y}(1) \in \mathfrak{g}$, where $\mathbf{L}_z$ denotes the vector field on G obtained by right translations of $z \in \mathfrak{g}$. Then $[\mathbf{L}_x, \mathbf{L}_y] = \mathbf{L}_z$, where $z = [\mathbf{L}_x, \mathbf{L}_y](1)$. So let us compute z.

Let f be a regular function on a neighborhood of $1 \in G$. We have shown that for $u \in \mathfrak{g}$

$$(\mathbf{L}_u f)(g) = \tfrac{d}{dt}|_{t=0} f(g \exp(tu)).$$

Thus,

$$z(f) = x(\mathbf{L}_y f) - y(\mathbf{L}_x f) = x(\tfrac{\partial}{\partial s}|_{s=0} f(\bullet \exp(sy))) - y(\tfrac{\partial}{\partial t}|_{t=0} f(\bullet \exp(tx))) =$$
$$\tfrac{\partial}{\partial t}|_{t=0} \tfrac{\partial}{\partial s}|_{s=0} f(\exp(tx)\exp(sy)) - \tfrac{\partial}{\partial s}|_{s=0} \tfrac{\partial}{\partial t}|_{t=0} f(\exp(sy)\exp(tx)) =$$
$$\tfrac{\partial^2}{\partial t \partial s}|_{t=s=0} (F(tx + sy + \tfrac{1}{2} ts[x,y] + ...) - F(tx + sy - \tfrac{1}{2} ts[x,y] + ...)),$$

where $F(u) := f(\exp(u))$. It is easy to see by using Taylor expansion that this expression equals to $[x,y](f)$. Thus $z = [x,y]$, i.e., the map $\mathfrak{g} \to \mathrm{Vect}_L(G)$ given by $x \mapsto \mathbf{L}_x$ is a Lie algebra isomorphism. Similarly, the map $\mathfrak{g} \to \mathrm{Vect}_R(G)$ given by $x \mapsto -\mathbf{R}_x$ is a Lie algebra isomorphism, as claimed. $\square$

9. **Fundamental theorems of Lie theory**

9.1. **Proofs of Theorem 3.13, Proposition 4.12, Proposition 4.7.** Let G be a Lie group with Lie algebra $\mathfrak{g}$ and X be a manifold with an action $a : G \times X \to X$. Then for any $z \in \mathfrak{g}$ we have a vector field $a_*(z)$ on X given by

$$(a_*(z)f)(x) = \tfrac{d}{dt}|_{t=0} f(\exp(-tz)x),$$

where $t \in \mathbb{R}$, $f \in O(U)$ for some open set $U \subset X$ and $x \in U$.

Proposition 9.1. *The map a_* is linear and we have*

$$a_*([z,w]) = [a_*(z), a_*(w)].$$

In other words, the map $a_* : \mathfrak{g} \to \mathrm{Vect}(X)$ is a homomorphism of Lie algebras.

Exercise 9.2. Prove Proposition 9.1.

This motivates the following definition.

Definition 9.3. An action of a Lie algebra $\mathfrak{g}$ on a manifold X is a homomorphism of Lie algebras $\mathfrak{g} \to \mathrm{Vect}(X)$.

Thus an action of a Lie group G on X induces an action of the Lie algebra $\mathfrak{g} = \mathrm{Lie}G$ on X.

Now let $x \in X$. Then we have a linear map $a_{*x} : \mathfrak{g} \to T_x X$ given by $a_{*x}(z) := a_*(z)(x)$.

Theorem 9.4. *(i) The stabilizer G_x is a closed subgroup of G with Lie algebra*

$$\mathfrak{g}_x := \mathrm{Ker}(a_{*x}).$$

(ii) The map $G/G_x \to X$ given by $g \mapsto gx$ is an immersion. So the orbit Gx is an immersed submanifold of X, and

$$T_x(Gx) \cong \mathrm{Im}(a_{*x}) \cong \mathfrak{g}/\mathfrak{g}_x.$$

Part (i) of Theorem 9.4 is the promised weaker version of Theorem 3.13 sufficient for our purposes. Also, part (ii) implies Proposition 4.12.

Proof. (i) It is clear that G_x is closed in G, but we need to show it is a Lie subgroup and compute its Lie algebra.[7] It suffices to show that for some neighborhood U of 1 in G, $U \cap G_x$ is a (closed) submanifold of U such that $T_1(U \cap G_x) = \mathfrak{g}_x$.

Note that $\mathfrak{g}_x \subset \mathfrak{g}$ is a Lie subalgebra, since the commutator of vector fields vanishing at x also vanishes at x (by the formula for commutator

[7] Although we claimed in Theorem 3.13 that a closed subgroup of a Lie group is always a Lie subgroup, we did not prove it, so we need to prove it in this case.

in local coordinates). Also, for any $z \in \mathfrak{g}_x$, $\exp(tz)x$ is a solution of the ODE $\gamma'(t) = a_{*\gamma(t)}(z)$ with initial condition $\gamma(0) = x$, and $\gamma(t) = x$ is such a solution, so by uniqueness of ODE solutions $\exp(tz)x = x$, thus $\exp(tz) \in G_x$.

Now choose a complement $\mathfrak{u}$ of $\mathfrak{g}_x$ in $\mathfrak{g}$, so that $\mathfrak{g} = \mathfrak{g}_x \oplus \mathfrak{u}$. Then $a_{*x} : \mathfrak{u} \to T_xX$ is injective. By the implicit function theorem, the map $\mathfrak{u} \to X$ given by $u \mapsto \exp(u)x$ is injective for small u, so $\exp(u) \in G_x$ for small $u \in \mathfrak{u}$ if and only if $u = 0$.

But in a small neighborhood U of 1 in G, any element g can be uniquely written as $g = \exp(u)\exp(z)$, where $u \in \mathfrak{u}$ and $z \in \mathfrak{g}_x$. So we see that $g \in G_x$ iff $u = 0$, i.e., $\log(g) \in \mathfrak{g}_x$. This shows that $U \cap G_x$ coincides with $U \cap \exp(\mathfrak{g}_x)$, as desired.

(ii) The same proof shows that we have an isomorphism $T_1(G/G_x) \cong \mathfrak{g}/\mathfrak{g}_x = \mathfrak{u}$, so the injectivity of $a_{*x} : \mathfrak{u} \to T_xX$ implies that the map $G/G_x \to X$ given by $g \mapsto gx$ is an immersion, as claimed. $\square$

Corollary 9.5. *(Proposition 4.7) Let $\phi : G \to K$ be a morphism of Lie groups and $\phi_* : \mathrm{Lie}G \to \mathrm{Lie}K$ be the corresponding morphism of Lie algebras. Then $H := \mathrm{Ker}(\phi)$ is a closed normal Lie subgroup with Lie algebra $\mathfrak{h} := \mathrm{Ker}(\phi_*)$, and the map $\overline{\phi} : G/H \to K$ is an immersion. Moreover, if $\mathrm{Im}\overline{\phi}$ is a submanifold of K then it is a closed Lie subgroup, and we have an isomorphism of Lie groups $\overline{\phi} : G/H \cong \mathrm{Im}\overline{\phi}$.*

Proof. Apply Theorem 9.4 to the action of G on $X = K$ via $g \circ k = \phi(g)k$, and take $x = 1$. $\square$

Corollary 9.6. *Let V be a finite dimensional representation of a Lie group G, and $v \in V$. Then the stabilizer G_v is a closed Lie subgroup of G with Lie algebra $\mathfrak{g}_v := \{z \in \mathfrak{g} : zv = 0\}$.*

Example 9.7. Let A be a finite dimensional algebra (not necessarily associative, e.g. a Lie algebra). Then the group $G = \mathrm{Aut}(A) \subset GL(A)$ is a closed Lie subgroup with Lie algebra $\mathrm{Der}(A) \subset \mathrm{End}(A)$ of derivations of A, i.e., linear maps $d : A \to A$ such that

$$d(ab) = d(a) \cdot b + a \cdot d(b).$$

Indeed, consider the action of $GL(A)$ on $\mathrm{Hom}(A \otimes A, A)$. Then $G = G_\mu$ where $\mu : A \otimes A \to A$ is the multiplication map. Also, if $g_t(ab) = g_t(a)g_t(b)$ and $d = \frac{d}{dt}|_{t=0} g_t$ then $d(ab) = d(a) \cdot b + a \cdot d(b)$ and conversely, if d is a derivation then $g_t := \exp(td)$ is an automorphism.

9.2. **The center of G and $\mathfrak{g}$.** Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $Z = Z(G)$ the center of G, i.e. the set of $z \in G$ such that $zg = gz$ for all $g \in G$. Also let $\mathfrak{z} = \mathfrak{z}(\mathfrak{g})$ be the set of $x \in \mathfrak{g}$ such that $[x, y] = 0$ for all $y \in \mathfrak{g}$; it is called the **center** of $\mathfrak{g}$.

Proposition 9.8. *If G is connected then Z is a closed (normal, commutative) Lie subgroup of G with Lie algebra $\mathfrak{z}$.*

Proof. Since G is connected, an element $g \in G$ belongs to Z iff it commutes with $\exp(tu)$ for all $u \in \mathfrak{g}$, i.e., iff $\mathrm{Ad}_g(u) = u$. Thus $Z = \mathrm{Ker}(\mathrm{Ad})$, where $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is the adjoint representation. Thus by Proposition 4.7, $Z \subset G$ is a closed Lie subgroup with Lie algebra $\mathrm{Ker}(\mathrm{ad})$, as claimed. □

Remark 9.9. In general (when G is not necessarily connected), it is easy to show that G/G° acts on $\mathfrak{z}$, and Z is a closed Lie subgroup of G with Lie algebra $\mathfrak{z}^{G/G^\circ}$ (the subspace of invariant vectors).

Definition 9.10. For a connected Lie group G, the group $G/Z(G)$ is called the **adjoint group** of G.

It is clear that $G/Z(G)$ is naturally isomorphic to the image of the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$, which motivates the terminology.

9.3. **The statements of the fundamental theorems of Lie theory.**

Theorem 9.11. *(First fundamental theorem of Lie theory) For a Lie group G, there is a bijection between connected Lie subgroups $H \subset G$ and Lie subalgebras $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}G$, given by $\mathfrak{h} = \mathrm{Lie}H$.*

Theorem 9.12. *(Second fundamental theorem of Lie theory) If G and K are Lie groups with G simply connected then the map*

$$\mathrm{Hom}(G, K) \to \mathrm{Hom}(\mathrm{Lie}G, \mathrm{Lie}K)$$

given by $\phi \mapsto \phi_*$ is a bijection.

Theorem 9.13. *(Third fundamental theorem of Lie theory) Any finite dimensional Lie algebra is the Lie algebra of a Lie group.*

These theorems hold for real as well as complex Lie groups. Thus we have

Corollary 9.14. *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the assignment $G \mapsto \mathrm{Lie}G$ is an equivalence between the category of simply connected $\mathbb{K}$-Lie groups and the category of finite dimensional $\mathbb{K}$-Lie algebras. Moreover, any connected Lie group K has the form G/Γ where G 'is simply connected and $\Gamma \subset G$ is a discrete central subgroup.*

Proof. The second fundamental theorem says that the functor $G \mapsto \mathrm{Lie}G$ is fully faithful, and the third fundamental theorem says that it is essentially surjective. Thus it is an equivalence of categories. The last statement follows from Proposition 3.5 (G is the universal covering of K). $\square$

We will discuss proofs of the fundamental theorems of Lie theory in Subsection 10.2. The third theorem is the hardest one, and we will give its complete proof only in Section 49.

9.4. **Complexification of real Lie groups and real forms of complex Lie groups.** Let $\mathfrak{g}$ be a real Lie algebra. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex Lie algebra. We say that $\mathfrak{g}_{\mathbb{C}}$ is the **complexification** of $\mathfrak{g}$, and $\mathfrak{g}$ is a **real form** of $\mathfrak{g}_{\mathbb{C}}$.

Note that two non-isomorphic real Lie algebras can have isomorphic complexifications; in other words, the same complex Lie algebra can have different real forms. For example,

$$\mathfrak{u}(n)_{\mathbb{C}} \cong \mathfrak{gl}_n(\mathbb{R})_{\mathbb{C}} \cong \mathfrak{gl}_n(\mathbb{C})$$

while for $n > 1$,

$$\mathfrak{u}(n) \ncong \mathfrak{gl}_n(\mathbb{R}),$$

since in the first algebra any element x with nilpotent $\mathrm{ad}x$ must be zero, while in the second one it must not.

Definition 9.15. Let G be a connected complex Lie group and $K \subset G$ a real Lie subgroup such that Lie$K$ is a real form of Lie$G$ (i.e., the natural map Lie$K \otimes_{\mathbb{R}} \mathbb{C} \to$ Lie$G$ is an isomorphism). Then K is called a **real form** of G.

Example 9.16. Both $U(n)$ and $GL_n(\mathbb{R})$ are real forms of $GL_n(\mathbb{C})$ (but $GL_n(\mathbb{R})$ is not connected). Also $GL_n(\mathbb{R})^\circ \subset GL_n(\mathbb{R})$, the subgroup of matrices with positive determinant, is a connected real form of $GL_n(\mathbb{C})$. So with this definition two different real forms (at least one of which is disconnected) may have the same Lie algebra.[8]

Let K be a simply connected real Lie group, and $\mathfrak{g} =$ Lie$K \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of Lie$K$. By the third fundamental theorem of Lie theory, there exists a unique simply connected complex Lie group G such that Lie$G = \mathfrak{g}$. It is called the **complexification** of K. We have a natural homomoprhism $K \to G$ coming from the homomorphism Lie$K \to$ LieG (from the second fundamental theorem), but it need not be injective; e.g. it is not for K being the universal covering of $SL_2(\mathbb{R})$.

[8]Later, when we discuss real forms of semisimple Lie groups, we will need a more restrictive notion of a real form of a complex group, namely the group of fixed points of an antiholomorphic involution. With this more restrictive definition, the disconnected group $GL_n(\mathbb{R})$ is a real form of $GL_n(\mathbb{C})$, but the connected group $GL_n(\mathbb{R})^\circ$ is not.

Exercise 9.17. (i) Classify complex Lie algebras of dimension at most 3, up to isomorphism.

(ii) Classify real Lie algebras of dimension at most 3.

(iii) Classify connected complex and real Lie groups of dimension at most 3.

10. **Proofs of the fundamental theorems of Lie theory**

10.1. **Distributions and the Frobenius theorem.** The proofs of the fundamental theorems of Lie theory are based on the notion of an integrable distribution in differential geometry, and the Frobenius theorem about such distributions.

Definition 10.1. A k-dimensional **distribution** on a manifold X is a rank k subbundle $D \subset TX$.

This means that in every tangent space T_xX we fix a k-dimensional subspace D_x which varies regularly with x. In other words, on some neighborhood $U \subset X$ of every $x \in X$, D is spanned by vector fields $\mathbf{v}_1, ..., \mathbf{v}_k$ linearly independent at every point of U.

Definition 10.2. A distribution D is **integrable** if every point $x \in X$ has a neighborhood U and local coordinates $x_1, ..., x_n$ on U such that D is defined at every point of U by the equations $dx_{k+1} = ... = dx_n = 0$, i.e., it is spanned by vector fields $\partial_i = \frac{\partial}{\partial x_i}$, $i = 1, ..., k$.

This is equivalent to saying that every point x of X is contained in an **integral submanifold** for D, i.e., an immersed submanifold $S = S_x \subset X$ such that for any $y \in S$ the tangent space $T_yS \subset T_yX$ coincides with D_y. Namely, S_x is the set of all points of $y \in X$ that can be connected to x by a smooth curve $\gamma : [0,1] \to X$ with $\gamma(0) = x, \gamma(1) = y$ and $\gamma'(t) \in D_{\gamma(t)}$ for all $t \in [0,1]$ (show it!).

For this reason an integrable distribution is also called a **foliation** and the integral submanifolds S_x are called the **sheets of the foliation**. The manifold X falls into a disjoint union of such sheets. But note that the sheets need not be closed (i.e., think of the irrational torus winding!)

Example 10.3. A 1-dimensional distribution is the same thing as a **direction field.** It is always integrable, as follows from the existence theorem for ODE, and its integral submanifolds are called **integral curves**. They are geometric realizations of solutions of the corresponding ODE.

However, for $k \geq 2$ a distribution is not always integrable.

Theorem 10.4. *(The Frobenius theorem) A distribution D is integrable if and only if for every two vector fields $\mathbf{v}, \mathbf{w}$ contained in D, their commutator $[\mathbf{v}, \mathbf{w}]$ is also contained in D.*

Example 10.5. Let $\mathbf{v} = \partial_x$, $\mathbf{w} = x\partial_y + \partial_z$ in $\mathbb{R}^3$, and D be the 2-dimensional distribution spanned by $\mathbf{v}, \mathbf{w}$. Then $[\mathbf{v}, \mathbf{w}] = \partial_y \notin D$. So D is not integrable.

Proof. If D is integrable, a vector field is contained in D iff it is tangent to integral submanifolds of D. But the commutator of two vector fields tangent to a submanifold is itself tangent to this submanifold. This establishes the "only if" part.

It remains to prove the "if " part. The proof is by induction in the rank k of D. The base case $k = 0$ is trivial, so it suffices to establish the inductive step. The question is local, so we may work in a neighborhood U of $P \in X$. Suppose that $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathrm{Vect}(U)$ is a basis of D in U (on every tangent space). By local existence and uniqueness of solutions of ODE, in some local coordinates $x_1, ..., x_n = z$, the vector field $\mathbf{v}_k$ equals ∂_z. By subtracting from $\mathbf{v}_i, i < k$ a multiple of $\mathbf{v}_k$ we can make sure that $\mathbf{v}_i$ has no ∂_z-component. Then

$$\mathbf{v}_i = \sum_{j=1}^{n-1} a_{ij}(x_1, ..., x_{n-1}, z)\partial_{x_j}.$$

Thus we have

$$[\partial_z, \mathbf{v}_i] = \sum_{m=1}^{n-1} b_{im}(x_1, ..., x_{n-1}, z)\mathbf{v}_m.$$

Thus, setting $A = (a_{ij}(x_1, ..., x_{n-1}, z)), B = (b_{im}(x_1, ..., x_{n-1}, z))$, we have

$$\partial_z A = BA.$$

Let A_0 be the solution of this linear ODE with $A_0(x_1, ..., x_{n-1}, 0) = 1$. Then $A = A_0 C$, where $C = C(x_1, ..., x_{n-1})$ does not depend on z. So we have a basis of D given by

$$\mathbf{w}_i = \sum_j c_{ij}(x_1, ..., x_{n-1})\partial_{x_j}.$$

Thus there is a neighborhood U of P which can be represented as $U = (-a, a) \times U'$, where $\dim U' = n - 1$, so that $D = \mathbb{R} \oplus D'$, where D' is a $k - 1$-dimensional distribution on U'. It is clear that for any two vector fields $\mathbf{v}, \mathbf{w}$ on U' contained in D', so is $[\mathbf{v}, \mathbf{w}]$. Hence D' is integrable by the induction assumption. Therefore, so is D, justifying the inductive step.

□

10.2. **Proofs of the fundamental theorems of Lie theory.**

10.2.1. *Proof of Theorem 9.11.* Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra. We need to show that there is a unique (not necessarily closed) connected Lie subgroup $H \subset G$ with

Lie algebra $\mathfrak{h}$. The proof of existence of H is based on the Frobenius theorem.

Define the distribution D on G by left-translating $\mathfrak{h} \subset \mathfrak{g} = T_1G$, i.e., $D_g = L_g\mathfrak{h}$. So any vector field contained in D is of the form

$$\mathbf{v} = \sum f_i \mathbf{L}_{a_i},$$

where a_i is a basis of $\mathfrak{h}$ and f_i are regular functions. Now if

$$\mathbf{w} = \sum g_j \mathbf{L}_{a_j}$$

is another such field then

$$[\mathbf{v}, \mathbf{w}] = \sum_{i,j} (f_i \mathbf{L}_{a_i}(g_j)\mathbf{L}_{a_j} - g_j L_{a_j}(f_i)\mathbf{L}_{a_i} + f_i g_j [\mathbf{L}_{a_i}, \mathbf{L}_{a_j}]).$$

But $[a_i, a_j] = \sum_k c_{ij}^k a_k$, so

$$[\mathbf{L}_{a_i}, \mathbf{L}_{a_j}] = \sum_k c_{ij}^k \mathbf{L}_{a_k}.$$

Thus if $\mathbf{v}, \mathbf{w}$ are contained in D then so is $[\mathbf{v}, \mathbf{w}]$. Hence by the Frobenius theorem, D is integrable.

Now consider the integral (embedded) submanifold H of D going through $1 \in G$. We claim that H is a Lie subgroup of G with Lie algebra $\mathfrak{h}$. Indeed, it suffices to show that H is a subgroup of G. But this is clear since H is the collection of elements of G of the form

$$g = \exp(a_1) ... \exp(a_m),$$

where $a_i \in \mathfrak{h}$.

Moreover, H is unique since it has to be generated by the image of the exponential map $\exp : \mathfrak{h} \to G$.

10.3. **Proof of Theorem 9.12.** We need to show that the natural map $\mathrm{Hom}(G, K) \to \mathrm{Hom}(\mathrm{Lie}G, \mathrm{Lie}K)$ is a bijection if G is simply connected.

We know this map is injective so we only need to establish surjectivity. For any morphism $\psi : \mathrm{Lie}G \to \mathrm{Lie}K$, consider the morphism

$$\theta = (\mathrm{id}, \psi) : \mathrm{Lie}G \to \mathrm{Lie}(G \times K) = \mathrm{Lie}G \oplus \mathrm{Lie}K$$

The previous proposition implies that there is a connected Lie subgroup $H \subset G \times K$ whose Lie algebra is $\mathrm{Im}\theta$. We have projection homomorphisms $p_1 : H \to G$, $p_2 : H \to K$, and $(p_1)_* = \mathrm{id}$, so p_1 is a covering. Since G is simply connected, p_1 is an isomorphism, so we can define $\phi := p_2 \circ p_1^{-1} : G \to K$, and it is easy to see that $\psi = \phi_*$.

10.4. **Proof of Theorem 9.13.** Finally, let us discuss a proof of Theorem 9.13, stating that any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is the Lie algebra of a Lie group. We will deduce it from the following purely algebraic *Ado's theorem*.

Theorem 10.6. *Any finite dimensional Lie algebra over* $\mathbb{K}$ *is a Lie subalgebra of* $\mathfrak{gl}_n(\mathbb{K})$.

Ado's theorem in fact holds over any ground field, but it is rather nontrivial and we won't prove it now. A proof can be found, for example, in [J]. But Ado's theorem immediately implies Theorem 9.13. Indeed, using Theorem 9.11, Ado's theorem implies the following even stronger statement:

Theorem 10.7. *Any finite dimensional* $\mathbb{K}$*-Lie algebra is the Lie algebra of a Lie subgroup of* $\mathfrak{gl}_n(\mathbb{K})$ *for some* n.

This implies

Corollary 10.8. *Any simply connected Lie group is the universal covering of a linear Lie group, i.e., of a Lie subgroup of* $GL_n(\mathbb{K})$.

However, it is not true that any Lie group is isomorphic to a Lie subgroup of $GL_n(\mathbb{K})$, see Exercise 11.20.

One can also prove Theorem 9.13 directly and then deduce Ado's theorem as a corollary. We will do this in Sections 49 and 50. We note that Theorem 9.13 will not be used in proofs of other results until that point.

11. **Representations of Lie groups and Lie algebras**

11.1. **Representations.** We have previously defined (finite dimensional) representations of Lie groups and (iso)morphisms between them. We can do the same for Lie algebras:

Definition 11.1. A **representation of a Lie algebra** $\mathfrak{g}$ over a field $\mathbf{k}$ (or a $\mathfrak{g}$-module) is a vector space V over $\mathbf{k}$ equipped with a homomorphism of Lie algebras $\rho = \rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$. A **(homo)morphism of representations** $A : V \to W$ (also called an **intertwining operator**) is a linear map which commutes with the $\mathfrak{g}$-action: $A\rho_V(b) = \rho_W(b)A$ for $b \in \mathfrak{g}$. Such A is an isomorphism if it is an isomorphism of vector spaces.

The first and second fundamental theorems of Lie theory imply:

Corollary 11.2. *Let G be a Lie group and $\mathfrak{g} = \mathrm{Lie}G$.*

(i) Any finite dimensional representation $\rho : G \to GL(V)$ gives rise to a Lie algebra representation $\rho_* : \mathfrak{g} \to \mathfrak{gl}(V)$, and any morphism of G-representations is also a morphism of $\mathfrak{g}$-representations.

(ii) If G is connected then any morphism of $\mathfrak{g}$-representations is a morphism of G-representations.

(iii) If G is simply connected then the assignment $\rho \mapsto \rho_*$ is an equivalence of categories $\mathrm{Rep}\, G \to \mathrm{Rep}\, \mathfrak{g}$ between the corresponding categories of finite dimensional representations. In particular, any finite dimensional representation of the Lie algebra $\mathfrak{g}$ can be uniquely exponentiated to the group G.

Example 11.3. 1. The trivial representation: $\rho(g) = 1, g \in G$, $\rho_*(x) = 0$, $x \in \mathfrak{g}$.

2. The adjoint representation: $\rho(g) = \mathrm{Ad}_g, \rho_*(x) = \mathrm{ad}x$.

Exercise 11.4. Let $\mathfrak{g}$ be a complex Lie algebra. Show that $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus \mathfrak{g}$. Deduce that if G is a simply connected complex Lie group then $\mathrm{Rep}_{\mathbb{R}}\, G \cong \mathrm{Rep}(\mathfrak{g} \oplus \mathfrak{g})$, where $\mathrm{Rep}_{\mathbb{R}}\, G$ is the category of finite dimensional representations of G regarded as a real Lie group.

As usual, a **subrepresentation** of a representation V is a subspace $W \subset V$ invariant under the G-action (resp. $\mathfrak{g}$-action). In this case the quotient space V/W has a natural structure of a representation, called the **quotient representation**. The notion of **direct sum** of representations is defined in an obvious way:

$$\rho_{V\oplus W}(x) = \rho_V(x) \oplus \rho_W(x).$$

Also we have the notion of **dual representation**:

$$\rho_{V^*}(g) = \rho_V(g^{-1})^*, g \in G;\ \rho_{V^*}(x) = -\rho_V(x)^*, x \in \mathfrak{g},$$

and **tensor product**:

$$\rho_{V\otimes W}(g) = \rho_V(g)\otimes\rho_W(g),\ \rho_{V\otimes W}(x) = \rho_V(x)\otimes 1_W + 1_V\otimes\rho_W(x).$$

Thus we have the notion of **symmetric and exterior powers** $S^mV, \wedge^m V$ of a representation V, which can be defined either as quotients or (over a field of characteristic zero) as subrepresentations of $V^{\otimes n}$. Also for representations V, W, $\mathrm{Hom}(V, W)$ is a representation via

$$g\circ A = \rho_W(g)A\rho_V(g^{-1}),\ x\circ A = \rho_W(x)A - A\rho_V(x),$$

so if V is finite dimensional then $\mathrm{Hom}(V, W)\cong V^*\otimes W$. Finally, for every representation V we have the notion of invariants:

$$V^G = \{v\in V : gv = v\ \forall g\in G\},\ V^{\mathfrak{g}} = \{v\in V : xv = 0\ \forall x\in\mathfrak{g}\}.$$

Thus $V^G\subset V^{\mathfrak{g}}$ and $V^G = V^{\mathfrak{g}}$ for connected G (in general, $V^G = (V^{\mathfrak{g}})^{G/G^\circ}$). Also $\mathrm{Hom}(V, W)^G\cong\mathrm{Hom}_G(V, W)$ and $\mathrm{Hom}(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$, the spaces of intertwining operators. Note that in all cases the formula for Lie algebras is determined by the formula for groups by the requirement that these definitions should be consistent with the assignment $\rho\mapsto\rho_*$.

Definition 11.5. A representation $V\neq 0$ of G or $\mathfrak{g}$ is **irreducible** if any subrepresentation $W\subset V$ is either 0 or V and is **indecomposable** if for any decomposition $V\cong V_1\oplus V_2$, we have $V_1 = 0$ or $V_2 = 0$.

It is clear that any finite dimensional representation is isomorphic to a direct sum of indecomposable representations (in fact, uniquely so up to order of summands by the *Krull-Schmidt theorem*). However, not any V is a direct sum of irreducible representations, e.g.

$$\rho:\mathbb{C}\to GL_2(\mathbb{C}),\ \rho(x) = \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}.$$

Definition 11.6. A representation V is called **completely reducible** if it is isomorphic to a direct sum of irreducible representations.

Some of the main problems of representation theory are:
1) Classify irreducible representations;
2) If V is a completely reducible representation, find its decomposition into irreducibles.
3) For which G are all representations completely reducible?

Example 11.7. Let V be a finite dimensional $\mathbb{C}$-representation of $\mathfrak{g}$ or G and $A : V\to V$ be a homomorphism of representations (e.g., defined by a central element). Then we have a decomposition of representations $V = \oplus_\lambda V(\lambda)$, where $V(\lambda)$ is the generalized eigenspace of A with eigenvalue λ.

Example 11.8. Let V be the vector representation of $GL(V)$. Then V is irreducible, and more generally so are $S^mV, \wedge^nV$ (show it!). Thus $V \otimes V$ is completely reducible: $V \otimes V \cong S^2V \oplus \wedge^2V$.

11.2. **Schur's lemma.**

Lemma 11.9. *(Schur's lemma) Let V, W be irreducible finite dimensional complex representations of G or $\mathfrak{g}$. Then $\mathrm{Hom}_{G,\mathfrak{g}}(V, W) = 0$ if V, W are not isomorphic, and every endomorphism of the representation V is a scalar.*

Proof. Let $A : V \to W$ be a nonzero morphism of representations. Then $\mathrm{Im}(A) \subset W$ is a nonzero subrepresentation, hence $\mathrm{Im}(A) = W$. Also $\mathrm{Ker}(A) \subset V$ is a proper subrepresentation, so $\mathrm{Ker}(A) = 0$. Thus A is an isomorphism, i.e., we may assume that $W = V$. In this case, let λ be an eigenvalue of A. Then $A - \lambda \cdot \mathrm{Id} : V \to V$ is a morphism of representations but not an isomorphism, hence it must be zero, so $A = \lambda \cdot \mathrm{Id}$. $\square$

Note that the second statement of Schur's lemma (unlike the first one) does not hold over $\mathbb{R}$. For example, consider the rotation group $SO(2)$ (or its finite subgroup of order > 2) acting on $V = \mathbb{R}^2$ by rotations. Then $\mathrm{End}(V) = \mathbb{C} \neq \mathbb{R}$. Similarly, if V is the representation of $SU(2)$ on $\mathbb{H}$ defined by right multiplication by unit quaternions then V is an irreducible real representation but $\mathrm{End}(V) = \mathbb{H} \neq \mathbb{R}$. For this reason, in representation theory of Lie groups and Lie algebras one usually considers complex representations. Thus from now on all representations we consider will be assumed complex unless specified otherwise.[9]

Corollary 11.10. *The center of $G, \mathfrak{g}$ acts on an irreducible representation by a scalar. In particular, if G or $\mathfrak{g}$ is abelian then its every irreducible representation is 1-dimensional.*

Example 11.11. Irreducible representations of $\mathbb{R}$ are χ_s given by $\chi_s(a) = \exp(sa)$, $s \in \mathbb{C}$. Irreducible representations of $\mathbb{R}^\times = \mathbb{R}_{>0} \times \mathbb{Z}/2$ are $\chi_{s,+}(a) = |a|^s$, $\chi_{s,-}(a) = |a|^s\mathrm{sign}(a)$. Irreducible representations of S^1 are $\chi_n(z) = z^n$, $n \in \mathbb{Z}$. Irreducible representations of the real group $\mathbb{C}^\times = \mathbb{R}_{>0} \times S^1$ are $\chi_{s,n}(z) = |z|^s(z/|z|)^n$, $s \in \mathbb{C}$, $n \in \mathbb{Z}$.

Corollary 11.12. *Let V_i be irreducible and $V = \oplus_i n_iV_i, W = \oplus_i m_iV_i$ be completely reducible complex representations of G or $\mathfrak{g}$. Then we have a natural linear isomorphism*

$$\mathrm{Hom}_{G,\mathfrak{g}}(V, W) \cong \oplus_i \mathrm{Mat}_{m_i,n_i}(\mathbb{C}).$$

[9]An exception is the adjoint representation of a real Lie group and associated tensor representations, which are real.

Moreover, if $V = W$ then this is an isomorphism of algebras.

11.3. **Unitary representations.** A finite dimensional representation V of G is said to be **unitary** if it is equipped with a positive definite Hermitian inner product $B(,)$ invariant under G, i.e., $B(gv, gw) = B(v, w)$ for $v, w \in V$, $g \in G$.

Proposition 11.13. *Any unitary representation can be written as an orthogonal direct sum of irreducible unitary representations. In particular, it is completely reducible.*

Proof. If $W \subset V$ is a subrepresentation of a unitary representation V then let $W^\perp$ be its orthogonal complement under B. Then $W^\perp$ is also a subrepresentation since B is invariant, and $V = W \oplus W^\perp$ since B is positive definite.

Now we can prove that V is an orthogonal direct sum of irreducible unitary representations by induction in $\dim V$. The base $\dim V = 1$ is clear so let us make the inductive step. Pick an irreducible $W \subset V$. Then $V = W \oplus W^\perp$, and $W^\perp$ is a unitary representation of dimension smaller than $\dim V$, so is an orthogonal direct sum of irreducible unitary representations by the induction assumption. □

Proposition 11.14. *Any finite dimensional representation V of a finite group G is unitary. Moreover, if V is irreducible, the unitary structure is unique up to a positive factor.*

Proof. Let B be any positive definite inner product on V. Let

$$\widehat{B}(v, w) := \sum_{g \in G} B(gv, gw).$$

Then $\widehat{B}$ is positive definite and invariant, so V is unitary.

If V is irreducible and B_1, B_2 are two unitary structures on V then $B_1(v, w) = B_2(Av, w)$ for some homomorphism $A : V \to V$. Thus by Schur's lemma $A = \lambda \cdot \mathrm{Id}$, and $\lambda > 0$ since B_1, B_2 are positive definite. □

Corollary 11.15. *Every finite dimensional complex representation of a finite group G is completely reducible.*

11.4. **Representations of $\mathfrak{sl}_2$.** The Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ has basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

with commutator

$$[e, f] = h, \ [h, e] = 2e, \ [h, f] = -2f.$$

Since 2-by-2 matrices act on variables x, y, they also act on the space $V = \mathbb{C}[x, y]$ of polynomials in x, y. Namely, this action is given by the formulas

$$e = x\partial_y,\ f = y\partial_x,\ h = x\partial_x - y\partial_y.$$

This infinite-dimensional representation has the form $V = \oplus_{n\geq 0} V_n$, where V_n is the space of polynomials of degree n. The space V_n is invariant under e, f, h, so it is an $n+1$-dimensional representation of $\mathfrak{sl}_2$. It has basis $v_{pq} = x^p y^q$, such that

$$hv_{pq} = (p-q)v_{pq},\ ev_{pq} = qv_{p+1,q-1},\ fv_{pq} = pv_{p-1,q+1}.$$

Thus V_0 is the trivial representation, and V_1 is the tautological representation by 2-by-2 matrices. Also it is easy to see that V_2 is the adjoint representation.

Theorem 11.16. *(i) V_n is irreducible.*

(ii) If $V \neq 0$ is a finite dimensional representation of $\mathfrak{sl}_2$ then $e|_V$ and $f|_V$ are nilpotent, so $U := \mathrm{Ker}(e) \neq 0$. Moreover, h preserves U and acts diagonalizably on it, with nonnegative integer eigenvalues.

(iii) Any irreducible finite dimensional representation V of $\mathfrak{sl}_2$ is isomoprhic to V_n for some n.

(iv) Any finite dimensional representation V of $\mathfrak{sl}_2$ is completely reducible.

Proof. (i) Let $W \subset V_n$ be a nonzero subrepresentation. Since it is h-invariant, it must be spanned by vectors $v_{p,n-p}$ for p from a nonempty subset $S \subset [0, n]$. Since W is e-invariant and f-invariant, if $m \in S$ then so are $m+1, m-1$ (if they are in $[0, n]$). Thus $S = [0, n]$ and $W = V_n$.

(ii) Let V be a finite dimensional representation of $\mathfrak{sl}_2$. We can write V as a direct sum of generalized eigenspaces of h: $V = \oplus_\lambda V(\lambda)$. Since $he = e(h+2)$, $hf = f(h-2)$, we have $e : V(\lambda) \to V(\lambda+2)$, $f : V(\lambda) \to V(\lambda-2)$. Thus $e|_V$, $f|_V$ are nilpotent, so $U \neq 0$.

If $v \in U$ then $e(hv) = (h-2)ev = 0$, so $hv \in U$, i.e., U is h-invariant.

Given $v \in U$, consider the vector $v_m := e^m f^m v$. We have

(11.1)
$$ef^m v = fef^{m-1}v + hf^{m-1}v = fef^{m-1}v + f^{m-1}(h - 2(m-1))v = ...$$
$$= f^{m-1}m(h-m+1)v.$$

Thus

$$v_m = e^{m-1}f^{m-1}m(h-m+1)v = m(h-m+1)v_{m-1}.$$

Hence

$$v_m = m!h(h-1)...(h-m+1)v.$$

But for large enough m, $v_m = 0$, since f is nilpotent, so

$$h(h-1)...(h-m+1)v = 0.$$

Thus h acts diagonalizably on U with nonnegative integer eigenvalues.

(iii) Let $v \in U$ be an eigenvector of h, i.e., $hv = \lambda v$. Let $w_m = f^m v$. Then

$$fw_m = w_{m+1}, hw_m = (\lambda - 2m)w_m.$$

Also, it follows from (11.1) that

$$ew_m = m(\lambda - m + 1)w_{m-1}.$$

Thus if $w_m \neq 0$ and $\lambda \neq m$ then $w_{m+1} \neq 0$. Also the nonzero vectors w_m are linearly independent since they have different eigenvalues of h. Thus $\lambda = n$ must be a nonnegative integer (as also follows from (ii)), and $w_{n+1} = 0$. So V, being irreducible, has a basis w_m, $m = 0, ..., n$. Now it is easy to see that $V \cong V_n$, via the assignment

$$w_m \mapsto n(n-1)...(n-m+1)x^m y^{n-m}.$$

(iv) Consider the **Casimir operator**

$$C = 2fe + \frac{h^2}{2} + h.$$

It is easy to check that $[C, e] = [C, f] = [C, h] = 0$, so $C : V \to V$ is a homomorphism. Thus $C|_{V_n} = \frac{n(n+2)}{2}$ (it is a scalar by Schur's lemma, and acts with such eigenvalue on $v_{0n} \in V_n$); note that these are different for different n. For a general representation, we have $V = \oplus_c V_c$, the direct sum of generalized eigenspaces of C.

Assume V is indecomposable. Then C has a single eigenvalue c on V. Fix a **Jordan-Hölder filtration** on V, i.e. a filtration

$$0 = F_0 V \subset F_1 V \subset ... \subset F_m V = V$$

such that $Y_i := F_i V / F_{i-1} V$ are irreducible for all i. By (iii), for each i we have $Y_i \cong V_n$ for some n, so $c = \frac{n(n+2)}{2}$ and thus this n is the same for all i. Thus $V(k)$ has dimension m, with h acting on it by $k \cdot \mathrm{Id}$ for $k = n, n-2, ..., -n$ and $V(k) = 0$ otherwise, by (ii); in particular, $\dim V = m(n+1)$. Let $u_1, ..., u_m$ be a basis of $V(n)$. As in (iii), we define subrepresentations $W_i \subset V$ generated by u_i. It is easy to see that $W_i \cong V_n$ and the natural morphism $W_1 \oplus ... \oplus W_m \to V$ is injective. Hence it is an isomorphism by dimension count, i.e., V is completely reducible. $\square$

Corollary 11.17. *(The Jacobson-Morozov lemma for $GL(V)$) Let V be a finite dimensional complex vector space and $N : V \to V$ be a* *nilpotent operator. Then there is a unique up to isomorphism action of $\mathfrak{sl}_2$ on V for which e acts by N.*

Proof. This follows from Theorem 11.16 and the Jordan normal form theorem for operators on V. $\square$

For a representation V define its **character** by

$$\chi_V(z) = \mathrm{Tr}_V(z^h) = \sum_m \dim V(m) z^m.$$

Thus

$$\chi_{V_n}(z) = z^n + z^{n-2} + ... + z^{-n} = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}.$$

It is easy to see that

$$\chi_{V \oplus W} = \chi_V + \chi_W, \chi_{V \otimes W} = \chi_V \chi_W.$$

Since the functions χ_{V_n} are linearly independent, we see that a finite dimensional representation of $\mathfrak{sl}_2$ is determined by its character.

Theorem 11.18. *(The Clebsch-Gordan rule) We have*

$$V_m \otimes V_n \cong \oplus_{i=0}^{\min(m,n)} V_{|m-n|+2i}.$$

Proof. It suffices to note that we have the corresponding character identity:

$$\chi_{V_m} \chi_{V_n} = \sum_{i=0}^{\min(m,n)} \chi_{V_{|m-n|+2i}}.$$

$\square$

Exercise 11.19. Show that V_n has an invariant nondegenerate inner product (i.e., such that $(av, w) + (v, aw) = 0$ for $a \in \mathfrak{sl}_2$, $v, w \in V_n$) which is symmetric for even n and skew-symmetric for odd n. In particular, $V_n^* \cong V_n$.

Exercise 11.20. Let G be the universal cover of $SL_2(\mathbb{R})$. Show that G is not isomorphic to a Lie subgroup of $GL_n(\mathbb{R})$ for any n and that moreover, the only quotients of G that are such subgroups are $SL_2(\mathbb{R})$ and $PSL_2(\mathbb{R})$.

12. The universal enveloping algebra of a Lie algebra

12.1. **The definition of the universal enveloping algebra.** Let V be a vector space over a field $\mathbf{k}$. Recall that the **tensor algebra** of V is the $\mathbb{Z}$-graded associative algebra $TV := \oplus_{n\geq 0} V^{\otimes n}$ (with $\deg(V^{\otimes n}) = n$), with multiplication given by $a \cdot b = a \otimes b$ for $a \in V^{\otimes m}$ and $b \in V^{\otimes n}$. If $\{x_i\}$ is a basis of V then TV is just the free algebra with generators x_i (i.e., without any relations). Its basis consists of various words in the letters x_i.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{k}$.

Definition 12.1. The **universal enveloping algebra** of $\mathfrak{g}$, denoted $U(\mathfrak{g})$, is the quotient of $T\mathfrak{g}$ by the ideal I generated by the elements $xy - yx - [x, y]$, $x, y \in \mathfrak{g}$.

Recall that any associative algebra A is also a Lie algebra with operation $[a, b] := ab - ba$. The following proposition follows immediately from the definition of $U(\mathfrak{g})$.

Proposition 12.2. *(i) Let $J \subset T\mathfrak{g}$ be an ideal, and $\rho : \mathfrak{g} \to T\mathfrak{g}/J$ the natural linear map. Then ρ is a homomorphism of Lie algebras if and only if $J \supset I$, so that $T\mathfrak{g}/J$ is a quotient of $T\mathfrak{g}/I = U(\mathfrak{g})$. In other words, $U(\mathfrak{g})$ is the largest quotient of $T\mathfrak{g}$ for which ρ is a homomorphism of Lie algebras.*

(ii) Let A be any associative algebra over $\mathbf{k}$. Then the map

$$\mathrm{Hom}_{\mathrm{associative}}(U(\mathfrak{g}), A) \to \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, A)$$

given by $\phi \mapsto \phi \circ \rho$ is a bijection.

Part (ii) of this proposition implies that any Lie algebra map $\psi : \mathfrak{g} \to A$ can be uniquely extended to an associative algebra map $\phi : U(\mathfrak{g}) \to A$ so that $\psi = \phi \circ \rho$. This is the universal property of $U(\mathfrak{g})$ which justifies the term "universal enveloping algebra".

In particular, it follows that a representation of $\mathfrak{g}$ on a vector space V is the same thing as an algebra map $U(\mathfrak{g}) \to \mathrm{End}(V)$ (i.e., a representation of $U(\mathfrak{g})$ on V). Thus, to understand the representation theory of $\mathfrak{g}$, it is helpful to understand the structure of $U(\mathfrak{g})$; for example, every central element $C \in U(\mathfrak{g})$ gives rise to a morphism of representations $V \to V$ (note that this has already come in handy in studying representations of $\mathfrak{sl}_2$).

In terms of the basis $\{x_i\}$ of $\mathfrak{g}$, we can write the bracket as

$$[x_i, x_j] = \sum_k c_{ij}^k x_k,$$

where $c_{ij}^k \in \mathbf{k}$ are the **structure constants**. Then the algebra $U(\mathfrak{g})$ can be described as the quotient of the free algebra $\mathbf{k}\langle\{x_i\}\rangle$ by the relations

$$x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k.$$

Example 12.3. 1. If $\mathfrak{g}$ is abelian (i.e., $c_{ij}^k = 0$) then $U(\mathfrak{g}) = S\mathfrak{g} = \mathbf{k}[\{x_i\}]$ is the symmetric algebra of $\mathfrak{g}$, $S\mathfrak{g} = \oplus_{n\geq 0} S^n\mathfrak{g}$, which in terms of the basis is the polynomial algebra in x_i.

2. $U(\mathfrak{sl}_2(\mathbf{k}))$ is generated by e, f, h with defining relations

$$he - eh = 2e,\ hf - fh = -2f,\ ef - fe = h.$$

Recall that $\mathfrak{g}$ acts on $T\mathfrak{g}$ by derivations via the adjoint action. Moreover, using the Jacobi identity, we have

$$\mathrm{ad}z(xy - yx - [x,y]) = [z,x]y + x[z,y] - [z,y]x - y[z,x] - [z,[x,y]] =$$
$$([z,x]y - y[z,x] - [[z,x],y]) + (x[z,y] - [z,y]x - [x,[z,y]]).$$

Thus $\mathrm{ad}z(I) \subset I$, and hence the action of $\mathfrak{g}$ on $T\mathfrak{g}$ descends to its action on $U(\mathfrak{g})$ by derivations (also called the adjoint action). It is easy to see that these derivations are in fact inner:

$$\mathrm{ad}z(a) = za - az$$

for $a \in U(\mathfrak{g})$ (although this is not so for $T\mathfrak{g}$). Indeed, it suffices to note that this holds for $a \in \mathfrak{g}$ by the definition of $U(\mathfrak{g})$.

Thus we get

Proposition 12.4. *The center $Z(U(\mathfrak{g}))$ of $U(\mathfrak{g})$ coincides with the subalgebra of invariants $U(\mathfrak{g})^{\mathrm{ad}\mathfrak{g}}$.*

Example 12.5. The Casimir operator $C = 2fe + \frac{h^2}{2} + h$ which we used to study representations of $\mathfrak{g} = \mathfrak{sl}_2$ is in fact a central element of $U(\mathfrak{g})$.

12.2. **Graded and filtered algebras.** Recall that a $\mathbb{Z}_{\geq 0}$-**filtered** algebra is an algebra A equipped with a filtration

$$0 = F_{-1}A \subset F_0 A \subset F_1 A \subset ... \subset F_n A \subset ...$$

such that $1 \in F_0 A$, $\cup_{n\geq 0} F_n A = A$ and $F_i A \cdot F_j A \subset F_{i+j}A$. In particular, if A is generated by $\{x_\alpha\}$ then a filtration on A can be obtained by declaring x_α to be of degree 1; i.e., $F_n A = (F_1 A)^n$ is the span of all words in x_α of degree $\leq n$.

If $A = \oplus_{i\geq 0} A_i$ is $\mathbb{Z}_{\geq 0}$-graded then we can define a filtration on A by setting $F_n A := \oplus_{i=0}^n A_i$; however, not any filtered algebra is obtained in this way, and having a filtration is a weaker condition than having a grading. Still, if A is a filtered algebra, we can define its **associated graded algebra** $\mathrm{gr}(A) := \oplus_{n\geq 0}\mathrm{gr}_n(A)$, where $\mathrm{gr}_n(A) := F_n A / F_{n-1}A$.

The multiplication in $\mathrm{gr}(A)$ is given by the "leading terms" of multiplication in A: for $a \in F_iA$, $b \in F_jA$, pick their representatives $\widetilde{a} \in F_iA$, $\widetilde{b} \in F_jA$ and let ab be the projection of $\widetilde{a}\widetilde{b}$ to $\mathrm{gr}_{i+j}(A)$.

Proposition 12.6. *If* $\mathrm{gr}(A)$ *is a domain (has no zero divisors) then so is* A.

Exercise 12.7. Prove Proposition 12.6.

12.3. **The coproduct of** $U(\mathfrak{g})$**.** For a vector space $\mathfrak{g}$ define the algebra homomorphism $\Delta : T\mathfrak{g} \to T\mathfrak{g} \otimes T\mathfrak{g}$ given for $x \in \mathfrak{g} \subset T\mathfrak{g}$ by $\Delta(x) = x \otimes 1 + 1 \otimes x$ (it exists and is unique since $T\mathfrak{g}$ is freely generated by $\mathfrak{g}$).

Lemma 12.8. *If* $\mathfrak{g}$ *is a Lie algebra then the kernel* I *of the map* $T\mathfrak{g} \to U(\mathfrak{g})$ *satisfies the property* $\Delta(I) \subset I \otimes T\mathfrak{g} + T\mathfrak{g} \otimes I \subset T\mathfrak{g} \otimes T\mathfrak{g}$. *Thus* Δ *descends to an algebra homomorphism* $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

Proof. For $x, y \in \mathfrak{g}$ and $a = a(x, y) := xy - yx - [x, y]$ we have $\Delta(a) = a \otimes 1 + 1 \otimes a$. The lemma follows since the ideal I is generated by elements of the form $a(x, y)$. $\square$

The homomorphism Δ is called the **coproduct** (of $T\mathfrak{g}$ or $U(\mathfrak{g})$).

Example 12.9. Let $\mathfrak{g} = V$ be abelian (a vector space). Then $U(\mathfrak{g}) = SV$, which for $\dim V < \infty$ can be viewed as the algebra of polynomial functions on V^*. Similarly, $SV \otimes SV$ is the algebra of polynomial functions on $V^* \times V^*$. In terms of this identification, we have $\Delta(f)(x, y) = f(x + y)$.

13. The Poincaré-Birkhoff-Witt theorem

13.1. **The statement of the Poincaré-Birkhoff-Witt theorem.** Let $\mathfrak{g}$ be a Lie algebra. Define a filtration[10] on $U(\mathfrak{g})$ by setting $\deg(\mathfrak{g}) = 1$. Thus $F_nU(\mathfrak{g})$ is the image of $\oplus_{i=0}^{n}\mathfrak{g}^{\otimes i} \subset T\mathfrak{g}$. Note that since

$$xy - yx = [x, y],\ x \in \mathfrak{g},$$

we have $[F_iU(\mathfrak{g}), F_jU(\mathfrak{g})] \subset F_{i+j-1}U(\mathfrak{g})$. Thus, $\mathrm{gr}U(\mathfrak{g})$ is commutative; in other words, we have a surjective algebra morpism $\phi : S\mathfrak{g} \to \mathrm{gr}U(\mathfrak{g})$.

Theorem 13.1. *(Poincaré-Birkhoff-Witt theorem) The homomorphism ϕ is an isomorphism.*

We will prove Theorem 13.1 in Subsection 13.2. Now let us discuss its reformulation in terms of a basis and corollaries.

Given a basis $\{x_i\}$ of $\mathfrak{g}$, fix an ordering on this basis and consider ordered monomials $\prod_i x_i^{n_i}$, where the product is ordered according to the ordering of the basis. The statement that ϕ is surjective is equivalent to saying that ordered monomials span $U(\mathfrak{g})$. This is also easy to see directly: any monomial can be ordered using the commutation relations at the cost of an error of lower degree, so proceeding recursively, we can write any monomial as a linear combination of ordered ones. Thus the PBW theorem can be formulated as follows:

Theorem 13.2. *The ordered monomials are linearly independent, hence form a basis of $U(\mathfrak{g})$.*

Corollary 13.3. *The map $\rho : \mathfrak{g} \to U(\mathfrak{g})$ is injective. Thus $\mathfrak{g} \subset U(\mathfrak{g})$.*

Remark 13.4. Let $\mathfrak{g}$ be a vector space equipped with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Then one can define the algebra $U(\mathfrak{g})$ as above. However, if the map $\rho : \mathfrak{g} \to U(\mathfrak{g})$ is injective then we clearly must have $[x, x] = 0$ for $x \in \mathfrak{g}$ and the Jacobi identity, i.e., $\mathfrak{g}$ has to be a Lie algebra. Thus the PBW theorem and even Corollary 13.3 fail without the axioms of a Lie algebra.

Corollary 13.5. *Let $\mathfrak{g}_i$, $1 \le i \le n$, be Lie subalgebras of $\mathfrak{g}$ such that $\mathfrak{g} = \oplus_i \mathfrak{g}_i$ as a vector space (but $[\mathfrak{g}_i, \mathfrak{g}_j]$ need not be zero). Then the multiplication map $\otimes_i U(\mathfrak{g}_i) \to U(\mathfrak{g})$ in any order is a linear isomorphism.*

Proof. The corollary follows immediately from the PBW theorem by choosing a basis of each $\mathfrak{g}_i$. $\square$

[10] The grading on $T\mathfrak{g}$ does not descend to $U(\mathfrak{g})$, in general, since the relation $xy - yx = [x, y]$ is not homogeneous: the right hand side has degree 1 while the left hand side has degree 2. So $U(\mathfrak{g})$ is not graded but is only filtered.

Remark 13.6. 1. Corollary 13.5 applies to the case of infinitely many $\mathfrak{g}_i$ if we understand the tensor product accordingly: the span of tensor products of elements of $U(\mathfrak{g}_i)$ where almost all of these elements are equal to 1.

2. Note that if $\dim \mathfrak{g}_i = 1$, this recovers the PBW theorem itself, so Corollary 13.5 is in fact a generalization of the PBW theorem.

Let $\mathrm{char}(\mathbf{k}) = 0$. Define the **symmetrization map** $\sigma : S\mathfrak{g} \to U(\mathfrak{g})$ given by

$$\sigma(y_1 \otimes ... \otimes y_n) = \frac{1}{n!} \sum_{s \in S_n} y_{s(1)} ... y_{s(n)}.$$

It is easy to see that this map commutes with the adjoint action of $\mathfrak{g}$.

Corollary 13.7. *σ is an isomorphism.*

Proof. It is easy to see that $\mathrm{gr}\sigma$ (the induced map on the associated graded algebra) coincides with ϕ, so the result follows from the PBW theorem. $\square$

Let $Z(U(\mathfrak{g}))$ denote the center of $U(\mathfrak{g})$.

Corollary 13.8. *The map σ defines a filtered vector space isomorphism $\sigma_0 : Z(U(\mathfrak{g})) \to (S\mathfrak{g})^{\mathrm{ad}\mathfrak{g}}$ whose associated graded is the algebra isomorphism $\phi|_{Z(U(\mathfrak{g}))} : Z(U(\mathfrak{g})) \to (S\mathfrak{g})^{\mathrm{ad}\mathfrak{g}}$.*

In the case when $\mathfrak{g} = \mathrm{Lie}G$ for a connected Lie group G, we thus obtain a filtered vector space isomorphism of the center of $U(\mathfrak{g})$ with $(S\mathfrak{g})^{\mathrm{Ad}G}$.

Remark 13.9. The map σ_0 is not, in general, an algebra homomorphism; however, a nontrivial theorem of M. Duflo says that if $\mathfrak{g}$ is finite dimensional then there exists a canonical filtered *algebra isomorphism* $\eta : Z(U(\mathfrak{g})) \to (S\mathfrak{g})^{\mathrm{ad}\mathfrak{g}}$ (a certain twisted version of σ_0) whose associated graded is $\phi|_{Z(U(\mathfrak{g}))}$. A construction of the Duflo isomorphism can be found in [CR].

Example 13.10. Let $\mathfrak{g} = \mathfrak{sl}_2 = \mathfrak{so}_3$. Then $\mathfrak{g}$ has a basis x, y, z with $[x, y] = z$, $[y, z] = x$, $[z, x] = y$, and $G = SO(3)$ acts on these elements by ordinary rotations of the 3-dimensional space. So the only G-invariant polynomials of x, y, z are polynomials of $r^2 = x^2 + y^2 + z^2$. Thus we get that $Z(U(\mathfrak{g})) = \mathbb{C}[x^2 + y^2 + z^2]$. In terms of e, f, h, we have

$$x^2 + y^2 + z^2 = -4fe - h^2 - 2h = -2C,$$

where C is the Casimir element.

13.2. **Proof of the PBW theorem.** The proof of Theorem 13.1 is based on the following key lemma.

Lemma 13.11. *There exists a unique linear map $\varphi : T\mathfrak{g} \to S\mathfrak{g}$ such that*

(i) for an **ordered** *monomial $X := x_{i_1}...x_{i_m} \in \mathfrak{g}^{\otimes m}$ one has $\varphi(X) = X$;*

(ii) one has $\varphi(I) = 0$; in other words, φ descends to a linear map $\overline{\varphi} : U(\mathfrak{g}) \to S\mathfrak{g}$.

Remark 13.12. The map φ is not canonical and depends on the choice of the ordered basis x_i of $\mathfrak{g}$.

Note that Lemma 13.11 immediately implies the PBW theorem, since by this lemma the images of ordered monomials under φ are linearly independent in $S\mathfrak{g}$, implying that these monomials themselves are linearly independent in $U(\mathfrak{g})$.

Proof. It is clear that φ is unique if exists since ordered monomials span $U(\mathfrak{g})$. We will construct φ by defining it inductively on $F_nT\mathfrak{g}$ for $n \geq 0$.

Suppose φ is already defined on $F_{n-1}T\mathfrak{g}$ and let us extend it to $F_nT\mathfrak{g} = F_{n-1}T\mathfrak{g} \oplus \mathfrak{g}^{\otimes n}$. So we should define φ on $\mathfrak{g}^{\otimes n}$. Since φ is already defined on ordered monomials X (by $\varphi(X) = X$), we need to extend this definition to all monomials.

Namely, let X be an ordered monomial of degree n, and let us define φ on monomials of the form $s(X)$ for $s \in S_n$, where

$$s(y_1...y_n) := y_{s(1)}...y_{s(n)}.$$

To this end, fix a decomposition D of s into a product of transpositions of neighbors:

$$s = s_{j_r}...s_{j_1},$$

and define $\varphi(s(X))$ by the formula

$$\varphi(s(X)) := X + \Phi_D(s, X),$$

where

$$\Phi_D(s, X) := \sum_{m=0}^{r-1} \varphi([,]_{j_{m+1}}(s_{j_m}...s_{j_1}(X))),$$

and

$$[,]_j(y_1...y_jy_{j+1}...y_n) := y_1...[y_j, y_{j+1}]...y_n.$$

We need to show that $\varphi(s(X))$ is well defined, i.e., $\Phi_D(s, X)$ does not really depend on the choice of D and s but only on $s(X)$. We first show that $\Phi_D(s, X)$ is independent on D.

To this end, recall that the symmetric group S_n is generated by $s_j, 1 \leq j \leq n-1$ with defining relations

$$s_j^2 = 1;\ s_j s_k = s_k s_j, |j-k| \geq 2;\ s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}.$$

Thus any two decompositions of s into a product of transpositions of neighbors can be related by a sequence of applications of these relations somewhere inside the decomposition.

Now, the first relation does not change the outcome by the identity $[x,y] = -[y,x]$.

For the second relation, suppose that $j < k$ and we have two decompositions D_1, D_2 of s given by $s = ps_j s_k q$ and $s = ps_k s_j q$, where q is a product of m transpositions of neighbors. Let $q(X) = YabZcdT$ where $a, b, c, d \in \mathfrak{g}$ stand in positions $j, j+1, k, k+1$. Let $\Phi_1 := \Phi_{D_1}(s, X)$, $\Phi_2 := \Phi_{D_2}(s, X)$. Then the sums defining Φ_1 and Φ_2 differ only in the m-th and $m+1$-th term, so we get

$$\Phi_1 - \Phi_2 =$$

$$\varphi(YabZ[c,d]T) + \varphi(Y[a,b]ZdcT) - \varphi(Y[a,b]ZcdT) - \varphi(YbaZ[c,d]T),$$

which equals zero by the induction assumption.

For the third relation, suppose that we have two decompositions D_1, D_2 of s given by $s = ps_j s_{j+1} s_j q$ and $s = ps_{j+1} s_j s_{j+1} q$, where q is a product of k transpositions of neighbors. Let $q(X) = YabcZ$ where $a, b, c \in \mathfrak{g}$ stand in positions $j, j+1, j+2$. Let $\Phi_1 := \Phi_{D_1}(s, X)$, $\Phi_2 := \Phi_{D_2}(s, X)$. Then the sums defining Φ_1 and Φ_2 differ only in the k-th, $k+1$-th, and $k+2$-th terms, so we get

$$\Phi_1 - \Phi_2 =$$

$$(\varphi(Y[a,b]cZ) + \varphi(Yb[a,c]Z) + \varphi(Y[b,c]aZ)) -$$
$$(\varphi(Ya[b,c]Z) + \varphi(Y[a,c]bZ) + \varphi(Yc[a,b]Z)).$$

So the Jacobi identity

$$[[b,c],a] + [b,[a,c]] + [[a,b],c] = 0$$

combined with property (ii) in degree $n-1$ implies that $\Phi_1 - \Phi_2 = 0$, i.e., $\Phi_1 = \Phi_2$, as claimed. Thus we will denote $\Phi_D(s, X)$ just by $\Phi(s, X)$.

It remains to show that $\Phi(s, X)$ does not depend on the choice of s and only depends on $s(X)$. Let $X = x_{i_1} ... x_{i_n}$; then $s(X) = s'(X)$ if and only if $s = s't$, where t is the product of transpositions s_k for which $i_k = i_{k+1}$. Thus, it suffices to show that $\Phi(s, X) = \Phi(ss_k, X)$ for such k. But this follows from the the fact that $[x,x] = 0$.

Now, it follows from the construction of φ that for any monomial X of degree n (not necessarily ordered), $\varphi(s_j(X)) = \varphi(X) + \varphi([,]_j(X))$.

Thus φ satisfies property (ii) in degree n. This concludes the proof of Lemma 13.11 and hence Theorem 13.1. $\square$

14. **Free Lie algebras, the Baker-Campbell-Hausdorff formula**

14.1. **Primitive elements.** Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbf{k}$. Let us say that $x \in U(\mathfrak{g})$ is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$. It is clear that if $x \in \mathfrak{g} \subset U(\mathfrak{g})$ then x is primitive.

Lemma 14.1. *If the ground field* $\mathbf{k}$ *has characteristic zero then every primitive element of* $U(\mathfrak{g})$ *is contained in* $\mathfrak{g}$.

Proof. Let $0 \neq f \in U(\mathfrak{g})$ be a primitive element. Suppose that the filtration degree of f is n. Let $f_0 \in S^n\mathfrak{g}$ be the leading term of f (it is well defined by the PBW Theorem). Then f_0 is primitive in $S\mathfrak{g}$, and in fact in SV for some finite dimensional subspace $V \subset \mathfrak{g}$. So $f_0(x+y) = f_0(x) + f_0(y)$, $x, y \in V^*$. In particular, $2^n f_0(x) = f_0(2x) = 2f_0(x)$, so $2^n - 2 = 0$, which implies that $n = 1$ as $\mathrm{char}(\mathbf{k}) = 0$. Thus $f = c + f_0$ where $f_0 \in \mathfrak{g}$, $c \in \mathbf{k}$ and $c = 0$ since f is primitive. $\square$

Remark 14.2. Note that the assumption of characteristic zero is essential. Indeed, if the characterictic of $\mathbf{k}$ is $p > 0$ and $x \in \mathfrak{g}$ then $x^{p^i} \in U(\mathfrak{g})$ is primitive for all i.

14.2. **Free Lie algebras.** Let V be a vector space over a field $\mathbf{k}$. The **free Lie algebra** $L(V)$ generated by V is the Lie subalgebra of TV generated by V. Note that $L(V)$ is a $\mathbb{Z}_{>0}$-graded Lie algebra: $L(V) = \oplus_{m \geq 1} L_m(V)$, with grading defined by $\deg V = 1$; thus $L_m(V)$ is spanned by commutators of m-tuples of elements of V inside TV.

Example 14.3. The free Lie algebra $FL_2 = L(\mathbf{k}^2)$ in two generators x, y is generated by x, y with $FL_2[1]$ having basis x, y, $FL_2[2]$ having basis $[x, y]$, $FL_2[3]$ having basis $[x, [x, y]]$, $[y, [x, y]]$, etc. Similarly, $FL_3 = L(\mathbf{k}^3)$ is generated by x, y, z with $FL_3[1]$ having basis x, y, z, $FL_3[2]$ having basis $[x, y], [x, z], [y, z]$, $FL_3[3]$ having basis $[x, [x, y]]$, $[y, [x, y]]$, $[y, [y, z]]$, $[z, [y, z]]$, $[x, [x, z]$, $[z, [x, z]]$, $[x, [y, z]]$, $[y, [z, x]]$ (note that $[z, [x, y]]$ expresses in terms of the last two using the Jacobi identity).

The Lie algebra embedding $L(V) \hookrightarrow TV$ gives rise to an associative algebra homomorphism $\psi : U(L(V)) \to TV$.

Proposition 14.4. *(i)* ψ *is an isomorphism, so* $U(L(V)) \cong TV$.

(ii) ψ *preserves the coproduct.*

(iii) (The universal property of free Lie algebras) If $\mathfrak{g}$ *is any Lie algebra over* $\mathbf{k}$ *then restriction to* V *defines an isomorphism*

$$\mathbf{res} : \mathrm{Hom}_{\mathrm{Lie}}(L(V), \mathfrak{g}) \cong \mathrm{Hom}_{\mathbf{k}}(V, \mathfrak{g}).$$

Proof. (i) By definition, $U(L(V))$ is generated by V as an associative algebra, so $U(L(V)) = TV/J$ for some 2-sided ideal J. Moreover, the map $\psi : TV/J \to TV$ restricts to the identity on the space V of generators. Thus $J = 0$ and $\psi = \mathrm{Id}$.

(ii) is clear since the two coproducts agree on generators.

(iii) Let $a : V \to \mathfrak{g}$ be a linear map. Then a can be viewed as a linear map $V \to U(\mathfrak{g})$. So it extends to a map of associative algebras $\widetilde{a} : TV \to U(\mathfrak{g})$ which restricts to a Lie algebra map $\widehat{a} : L(V) \to U(\mathfrak{g})$. Moreover, since $\widehat{a}(V) \subset \mathfrak{g} \subset U(\mathfrak{g})$ and $L(V)$ is generated by V as a Lie algebra, we obtain that $\widehat{a} : L(V) \to \mathfrak{g}$. It is easy to see that the assignment $a \mapsto \widehat{a}$ is inverse to **res**, implying that **res** is an isomorphism. $\square$

Exercise 14.5. Let $\dim V = n$ and $d_m(n) = \dim L_m(V)$. Use the PBW theorem to show that $d_m(n)$ are uniquely determined from the identity

$$\prod_{m=1}^{\infty} (1 - q^m)^{d_m(n)} = 1 - nq.$$

14.3. **The Baker-Campbell-Hausdorff formula.** We have defined the commutator $[x, y]$ on $\mathfrak{g} = \mathrm{Lie}G$ as the quadratic part of $\mu(x, y) = \log(\exp(x)\exp(y))$. So one may wonder if taking higher order terms in the Taylor explansion of $\mu(x, y)$,

$$\mu(x, y) = \sum_{n=1}^{\infty} \frac{\mu_n(x, y)}{n!}$$

would yield new operations on $\mathfrak{g}$. It turns out, however, that all these operations express via the commutator. Namely, we have

Theorem 14.6. *For each $n \geq 1$, $\mu_n(x, y)$ may be written as a $\mathbb{Q}$-Lie polynomial of x, y (i.e., a $\mathbb{Q}$-linear combination of Lie monomials, obtained by taking successive commutators of x, y), which is universal (i.e., independent on G).*

Proof. Let $T\mathbb{C}^2 = \mathbb{C}\langle x, y\rangle$ be the free noncommutative algebra in the letters x, y. The series $X = \exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ can be viewed as as an element of the degree completion $\widehat{\mathbb{C}\langle x, y\rangle}$ (i.e., the space of noncommutative formal series in x and y), and similarly for $Y := \exp(y)$. Thus we may define

$$\mu := \log(XY) \in \widehat{\mathbb{C}\langle x, y\rangle},$$

where

$$\log A := -\sum_{n=1}^{\infty} \frac{(1 - A)^n}{n}.$$

Then $\mu = \sum_{n=1}^{\infty} \frac{\mu_n}{n!}$ where $\mu_n \in \mathbb{C}\langle x, y\rangle$ is homogeneous of degree n. These μ_n are the desired universal expressions, and it remains to show that they are Lie polynomials, i.e., can be expressed solely in terms of commutators.

To this end, note that since $\Delta(x) = x \otimes 1 + 1 \otimes x$, the element X is **grouplike**, i.e., $\Delta(X) = X \otimes X$ (where we extend the coproduct to the completion by continuity). The same property is shared by Y and hence by $Z := XY$, i.e., we have $\Delta(Z) = Z \otimes Z$. Thus

$$\begin{aligned}\Delta(\log Z) = \log \Delta(Z) = \log(Z \otimes Z) = \log((Z \otimes 1)(1 \otimes Z)) \\ = \log Z \otimes 1 + 1 \otimes \log Z.\end{aligned}$$

Thus $\mu = \log Z$ is primitive, hence so is μ_n for each n. Thus by Lemma 14.1, $\mu_n \in FL_2 = L(\mathbb{C}^2)$, where $FL_2 \subset \mathbb{C}\langle x, y\rangle$ is the free Lie algebra generated by x, y. This implies the statement. $\square$

Example 14.7.

$$\mu_3(x, y) = \tfrac{1}{2}([x, [x, y]] + [y, [y, x]]).$$

Thus

$$\mu(x, y) = x + y + \tfrac{1}{2}[x, y] + \tfrac{1}{12}([x, [x, y]] + [y, [y, x]]) + ...$$

Remark 14.8. E. Dynkin derived an explicit formula for $\mu(x, y)$ making it apparent that it expresses solely in terms of commutators. Several proofs of this formula may be found in the expository paper [Mu].

15. **Solvable and nilpotent Lie algebras, theorems of Lie and Engel**

15.1. **Ideals and commutant.** Let $\mathfrak{g}$ be a Lie algebra. Recall that an ideal in $\mathfrak{g}$ is a subspace $\mathfrak{h}$ such that $[\mathfrak{g}, \mathfrak{h}] = \mathfrak{h}$. If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal then $\mathfrak{g}/\mathfrak{h}$ has a natural structure of a Lie algebra. Moreover, if $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomoprphism of Lie algebras then $\mathrm{Ker}\phi$ is an ideal in $\mathfrak{g}_1$, $\mathrm{Im}\phi$ is a Lie subalgebra in $\mathfrak{g}_2$, and ϕ induces an isomorphism $\mathfrak{g}_1/\mathrm{Ker}\phi \cong \mathrm{Im}\phi$ (check it!).

Lemma 15.1. *If $I_1, I_2 \subset \mathfrak{g}$ are ideals then so are $I_1 \cap I_2, I_1 + I_2, [I_1, I_2]$.*

Exercise 15.2. Prove Lemma 15.1.

Definition 15.3. The commutant of $\mathfrak{g}$ is the ideal $[\mathfrak{g}, \mathfrak{g}]$.

Lemma 15.4. *The quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian; moreover, if $I \subset \mathfrak{g}$ is an ideal such that $\mathfrak{g}/I$ is abelian then $I \supset [\mathfrak{g}, \mathfrak{g}]$.*

Exercise 15.5. Prove Lemma 15.4.

Example 15.6. The commutant of $\mathfrak{gl}_n(\mathbf{k})$ is $\mathfrak{sl}_n(\mathbf{k})$ (check it!).

Exercise 15.7. (i) Prove that if G is a connected Lie group with Lie algebra $\mathfrak{g}$ then the group commutant $[G, G]$ (the subgroup of G generated by elements $ghg^{-1}h^{-1}$, $g, h \in G$) is a Lie subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$.

(ii) Let $\widetilde{G} = \mathbb{R} \times H$, where H is the **Heisenberg group** of real matrices of the form

$$M(a, b, c) := \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \ a, b, c \in \mathbb{R}.$$

Let $\Gamma \cong \mathbb{Z}^2 \subset \widetilde{G}$ be the (closed) central subgroup generated by the pairs $(1, M(0, 0, 0) = \mathrm{Id})$ and $(\sqrt{2}, M(0, 0, 1))$. Let $G = \widetilde{G}/\Gamma$. Show that $[G, G]$ is not closed in G (although by (i) it is a Lie subgroup).

(iii) Does $[G, G]$ have to be closed in G if G is simply connected? (Consider $\mathrm{Hom}(G, \mathbb{R})$ and apply the second fundamental theorem of Lie theory).

15.2. **Solvable Lie algebras.** For a Lie algebra $\mathfrak{g}$ define its **derived series** recursively by the formulas $D^0(\mathfrak{g}) = \mathfrak{g}$, $D^{n+1}(\mathfrak{g}) = [D^n(\mathfrak{g}), D^n(\mathfrak{g})]$. This is a descending sequence of ideals in $\mathfrak{g}$.

Definition 15.8. A Lie algebra $\mathfrak{g}$ is said to be **solvable** if $D^n(\mathfrak{g}) = 0$ for some n.

Proposition 15.9. *The following conditions on* $\mathfrak{g}$ *are equivalent:*

(i) $\mathfrak{g}$ *is solvable;*

(ii) There exists a sequence of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset ... \supset \mathfrak{g}_m = 0$ *such that* $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ *is abelian.*

Proof. It is clear that (i) implies (ii), since we can take $\mathfrak{g}_i = D^i\mathfrak{g}$. Conversely, by induction we see that $D^i\mathfrak{g} \subset \mathfrak{g}_i$, as desired. □

Proposition 15.10. *(i) Any Lie subalgebra or quotient of a solvable Lie algebra is solvable.*

(ii) If $I \subset \mathfrak{g}$ *is an ideal and* $I, \mathfrak{g}/I$ *are solvable then* $\mathfrak{g}$ *is solvable.*

Exercise 15.11. Prove Proposition 15.10.

15.3. **Nilpotent Lie algebras.** For a Lie algebra $\mathfrak{g}$ define its **lower central series** recursively by the formulas $D_0(\mathfrak{g}) = \mathfrak{g}$, $D_{n+1}(\mathfrak{g}) = [\mathfrak{g}, D_n(\mathfrak{g})]$. This is a descending sequence of ideals in $\mathfrak{g}$.

Definition 15.12. A Lie algebra $\mathfrak{g}$ is said to be **nilpotent** if $D_n(\mathfrak{g}) = 0$ for some n.

Proposition 15.13. *The following conditions on* $\mathfrak{g}$ *are equivalent:*

(i) $\mathfrak{g}$ *is nilpotent;*

(ii) There exists a sequence of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset ... \supset \mathfrak{g}_m = 0$ *such that* $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$.

Proof. It is clear that (i) implies (ii), since we can take $\mathfrak{g}_i = D_i\mathfrak{g}$. Conversely, by induction we see that $D_i\mathfrak{g} \subset \mathfrak{g}_i$, as desired. □

Remark 15.14. Any nilpotent Lie algebra is solvable since $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ implies $[\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$, hence $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian.

Proposition 15.15. *Any Lie subalgebra or quotient of a nilpotent Lie algebra is nilpotent.*

Exercise 15.16. Prove Proposition 15.15.

Example 15.17. (i) The Lie algebra of upper triangular matrices of size n is solvable, but it is not nilpotent for $n \geq 2$.

(ii) The Lie algebra of strictly upper triangular matrices is nilpotent.

(iii) The Lie algebra of all matrices of size $n \geq 2$ is not solvable.

15.4. **Lie's theorem.** One of the main technical tools of the structure theory of finite dimensional Lie algebras is **Lie's theorem** for solvable Lie algebras. Before stating and proving this theorem, we will prove the following auxiliary lemma, which will be used several times.

Lemma 15.18. *Let $\mathfrak{g} = \mathbf{k}x \oplus \mathfrak{h}$ be a Lie algebra over a field $\mathbf{k}$ in which $\mathfrak{h}$ is an ideal. Let V be a finite dimensional $\mathfrak{g}$-module and $v \in V$ a common eigenvector of $\mathfrak{h}$:*

$$av = \lambda(a)v, \ a \in \mathfrak{h}$$

where $\lambda : \mathfrak{h} \to \mathbf{k}$ is a character. Then:

(i) $W := \mathbf{k}[x]v$ is a $\mathfrak{g}$-submodule on V on which $a - \lambda(a)$ is nilpotent for all $a \in \mathfrak{h}$.

(ii) If in addition λ vanishes on $[\mathfrak{g}, \mathfrak{g}]$ (i.e., $\lambda([a, x]) = 0$ for all $a \in \mathfrak{h}$) then every $a \in \mathfrak{h}$ acts on W by the scalar $\lambda(a)$. Thus the common eigenspace $V_\lambda \subset V$ of $\mathfrak{h}$ is a $\mathfrak{g}$-submodule.

(iii) The assumption (hence the conclusion) of (ii) always holds if $\mathrm{char}(\mathbf{k}) = 0$.

Proof. (i) For $a \in \mathfrak{h}$ we have

$$ax^n v = xax^{n-1}v + [a, x]x^{n-1}v. \tag{15.1}$$

Therefore, it follows by induction in n that $ax^n v$ is a linear combination of $v, xv, ..., x^n v$, hence $W \subset V$ is a submodule.

Let n be the smallest integer such that $x^n v$ is a linear combination of $x^i v$ with $i < n$. Then $v_i := x^{i-1}v$ for $i = 1, ..., n$ is a basis of W and $\dim W = n$. It follows from (15.1) that the element a acts in this basis by an upper triangular matrix with all diagonal entries equal $\lambda(a)$, as claimed.

(ii) It follows from (15.1) by induction in n that for every $a \in \mathfrak{h}$, $ax^n v = \lambda(a)x^n v$, as desired.

(iii) By (i), $\mathrm{Tr}(a|_W) = n\lambda(a)$ for all $a \in \mathfrak{h}$. On the other hand, if $a \in [\mathfrak{g}, \mathfrak{g}]$ then $\mathrm{Tr}(a|_W) = 0$, thus $n\lambda(a) = 0$ in $\mathbf{k}$. Since $\mathrm{char}(\mathbf{k}) = 0$, this implies that $\lambda(a) = 0$. $\square$

Theorem 15.19. *(Lie's theorem) Let $\mathbf{k}$ be an algebraically closed field of characteristic zero, and $\mathfrak{g}$ a finite dimensional solvable Lie algebra over $\mathbf{k}$. Then any irreducible finite dimensional representation of $\mathfrak{g}$ is 1-dimensional.*

Proof. Let V be a finite dimensional representation of $\mathfrak{g}$. It suffices to show that V contains a common eigenvector of $\mathfrak{g}$. The proof is by induction in $\dim \mathfrak{g}$. The base is trivial so let us justify the induction step. Since $\mathfrak{g}$ is solvable, $\mathfrak{g} \neq [\mathfrak{g}, \mathfrak{g}]$, so fix a subspace $\mathfrak{h} \subset \mathfrak{g}$ of codimension 1 containing $[\mathfrak{g}, \mathfrak{g}]$. Since $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, hence solvable. Thus by the induction assumption, there is a nonzero common eigenvector $v \in V$ for $\mathfrak{h}$, i.e., there is a linear functional $\lambda : \mathfrak{h} \to \mathbf{k}$ such that $av = \lambda(a)v$ for all $a \in \mathfrak{h}$.

Let $x \in \mathfrak{g}$ be an element not belonging to $\mathfrak{h}$ and W be the subspace of V spanned by $v, xv, x^2v, \dots$. By Lemma 15.18(i), W is a $\mathfrak{g}$-submodule of V and $a - \lambda(a)$ is nilpotent on W. Thus by Lemma 15.18(ii),(iii) every $a \in \mathfrak{h}$ acts on W by $\lambda(a)$, in particular $[\mathfrak{g}, \mathfrak{g}]$ acts by zero. Hence W is a representation of the abelian Lie algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. Now the statement follows since every finite dimensional representation of an abelian Lie algebra has a common eigenvector. $\square$

Remark 15.20. Lemma 15.18(iii) and Lie's theorem do not hold in characteristic $p > 0$. Indeed, let $\mathfrak{g}$ be the Lie algebra with basis x, y and $[x, y] = y$, and let V be the space with basis $v_0, \dots, v_{p-1}$ and action of $\mathfrak{g}$ given by

$$xv_i = iv_i, \ yv_i = v_{i+1},$$

where $i + 1$ is taken modulo p. It is easy to see that V is irreducible.

Here is another formulation of Lie's theorem:

Corollary 15.21. *Every finite dimensional representation V of a finite dimensional solvable Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbf{k}$ of characteristic zero has a basis in which all elements of $\mathfrak{g}$ act by upper triangular matrices. In other words, there is a sequence of subrepresentations $0 = V_0 \subset V_1 \subset \dots \subset V_n = V$ such that $\dim(V_{k+1}/V_k) = 1$.*

In the case $\dim \mathfrak{g} = 1$, this recovers the well known theorem in linear algebra that any linear operator on a finite dimensional $\mathbf{k}$-vector space is upper triangular in some basis (which is actually true in any characteristic).

Proof. The proof is by induction in $\dim V$ (where the base is obvious). By Lie's theorem, there is a common eigenvector $v_0 \in V$ for $\mathfrak{g}$. Let $V' := V/\mathbf{k}v_0$. Then by the induction assumption V' has a basis $v_1', \dots, v_n'$ in which $\mathfrak{g}$ acts by upper triangular matrices. Let $v_1, \dots, v_n$ be any lifts of $v_1', \dots, v_n'$ to V. Then $v_0, v_1, \dots, v_n$ is a basis of V in which $\mathfrak{g}$ acts by upper triangular matrices. $\square$

Corollary 15.22. *Over an algebraically closed field of characteristic zero, the following hold.*

(i) A solvable finite dimensional Lie algebra $\mathfrak{g}$ admits a sequence of ideals $0 = I_0 \subset I_1 \subset \dots \subset I_n = \mathfrak{g}$ such that $\dim(I_{k+1}/I_k) = 1$.

(ii) A finite dimensional Lie algebra $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Proof. (i) Apply Corollary 15.21 to the adjoint representation of $\mathfrak{g}$.

(ii) If $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent then it is solvable and $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian, so $\mathfrak{g}$ is solvable. Conversely, if $\mathfrak{g}$ is solvable then by Corollary 15.21 elements of $[\mathfrak{g},\mathfrak{g}]$ act on $\mathfrak{g}$, hence on $[\mathfrak{g},\mathfrak{g}]$ by strictly upper triangular matrices, which implies the statement. $\square$

Example 15.23. Let $\mathfrak{g}, V$ be as in Remark 15.20 and $\mathfrak{h} = \mathfrak{g} \ltimes V$ be the semidirect product, i.e. $\mathfrak{h} = \mathfrak{g} \oplus V$ as a space with

$$[(g_1, v_1), (g_2, v_2)] = ([g_1, g_2], g_1 v_2 - g_2 v_1).$$

Then $\mathfrak{h}$ is a counterexample to Corollary 15.22 both (i) and (ii) in characteristic $p > 0$.

15.5. **Engel's theorem.** Another key tool of the structure theory of finite dimensional Lie algebras is **Engel's theorem**. Before stating and proving this theorem, we prove an auxiliary result.

Theorem 15.24. *Let $V \neq 0$ be a finite dimensional vector space over any field $\mathbf{k}$, and $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie algebra consisting of nilpotent operators. Then there exists a nonzero vector $v \in V$ such that $\mathfrak{g}v = 0$.*

Proof. The proof is by induction on the dimension of $\mathfrak{g}$. The base case $\mathfrak{g} = 0$ is trivial and we assume the dimension of $\mathfrak{g}$ is positive.

First we find an ideal $\mathfrak{h}$ of codimension one in $\mathfrak{g}$. Let $\mathfrak{h}$ be a maximal (proper) subalgebra of $\mathfrak{g}$, which exists by finite-dimensionality of $\mathfrak{g}$. We claim that $\mathfrak{h} \subset \mathfrak{g}$ is an ideal and has codimension one.

Indeed, for each $a \in \mathfrak{h}$, the operator $\mathrm{ad}a$ induces a linear map $\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$, and this induced map is nilpotent (in fact, $\mathrm{ad}a : \mathfrak{g} \to \mathfrak{g}$ is nilpotent). Thus, by the inductive hypothesis, there exists a nonzero element $\bar{x}$ in $\mathfrak{g}/\mathfrak{h}$ such that $\mathrm{ad}a \cdot \bar{x} = 0$ for each $a \in \mathfrak{h}$. Let x be a lift of $\bar{x}$ to $\mathfrak{g}$. Then $[a, x] \in \mathfrak{h}$ for all $a \in \mathfrak{g}$. Let $\mathfrak{h}'$ be the span of $\mathfrak{h}$ and x. Then $\mathfrak{h}' \subset \mathfrak{g}$ is a Lie subalgebra in which $\mathfrak{h}$ is an ideal. Hence, by maximality, $\mathfrak{h}' = \mathfrak{g}$. This proves the claim.

Now let $W = V^{\mathfrak{h}} \subset V$. By the inductive hypothesis, $W \neq 0$. Also by Lemma 15.18(ii) (with $\lambda = 0$), W is a $\mathfrak{g}$-subrepresentation of V.

Now take $w \neq 0$ in W. Let k be the smallest positive integer such that $x^k w = 0$; it exists since x acts nilpotently on V. Let $v = x^{k-1} w \in W$. Then $v \neq 0$ but $\mathfrak{h}v = xv = 0$, so $\mathfrak{g}v = 0$, as desired. $\square$

Definition 15.25. An element $x \in \mathfrak{g}$ is said to be **nilpotent** if the operator $\mathrm{ad}x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent.

Corollary 15.26. *(Engel's theorem) A finite dimensional Lie algebra $\mathfrak{g}$ is nilpotent if and only if every element $x \in \mathfrak{g}$ is nilpotent.*

Proof. The "only if" direction is easy. To prove the "if" direction, note that by Theorem 15.24, in some basis v_i of $\mathfrak{g}$ all elements $\mathrm{ad}x$ act by strictly upper triangular matrices. Let I_m be the subspace of $\mathfrak{g}$ spanned by the vectors $v_1, ..., v_m$. Then $I_m \subset I_{m+1}$ and $[\mathfrak{g}, I_{m+1}] \subset I_m$, hence $\mathfrak{g}$ is nilpotent. $\square$

16. **Semisimple and reductive Lie algebras, the Cartan criteria**

16.1. **Semisimple and reductive Lie algebras, the radical.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $\mathbf{k}$.

Proposition 16.1. *$\mathfrak{g}$ contains the largest solvable ideal which contains all solvable ideals of $\mathfrak{g}$.*

Definition 16.2. This ideal is called **the radical** of $\mathfrak{g}$ and denoted $\mathrm{rad}(\mathfrak{g})$.

Proof. Let I, J be solvable ideals of $\mathfrak{g}$. Then $I + J \subset \mathfrak{g}$ is an ideal, and $(I+J)/I = J/(I\cap J)$ is solvable, so $I+J$ is solvable. Thus the sum of finitely many solvable ideals is solvable. Hence the sum of all solvable ideals in $\mathfrak{g}$ is a solvable ideal, as desired. □

Definition 16.3. (i) $\mathfrak{g}$ is called **semisimple** if $\mathrm{rad}(\mathfrak{g}) = 0$, i.e., $\mathfrak{g}$ does not contain nonzero solvable ideals.

(ii) A non-abelian $\mathfrak{g}$ is called **simple** if it contains no ideals other than $0, \mathfrak{g}$. In other words, a non-abelian $\mathfrak{g}$ is simple if its adjoint representation is irreducible (=simple).

Thus if $\mathfrak{g}$ is both solvable and semisimple then $\mathfrak{g} = 0$.

Proposition 16.4. *(i) We have $\mathrm{rad}(\mathfrak{g}\oplus\mathfrak{h}) = \mathrm{rad}(\mathfrak{g})\oplus\mathrm{rad}(\mathfrak{h})$. In particular, the direct sum of semisimple Lie algebras is semisimple.*

(ii) A simple Lie algebra is semisimple. Thus a direct sum of simple Lie algebras is semisimple.

Proof. (i) The images of $\mathrm{rad}(\mathfrak{g}\oplus\mathfrak{h})$ in $\mathfrak{g}$ and in $\mathfrak{h}$ are solvable, hence contained in $\mathrm{rad}(\mathfrak{g})$, respectively $\mathrm{rad}(\mathfrak{h})$. Thus

$$\mathrm{rad}(\mathfrak{g}\oplus\mathfrak{h}) \subset \mathrm{rad}(\mathfrak{g})\oplus\mathrm{rad}(\mathfrak{h}).$$

But $\mathrm{rad}(\mathfrak{g})\oplus\mathrm{rad}(\mathfrak{h})$ is a solvable ideal in $\mathfrak{g}\oplus\mathfrak{h}$, so

$$\mathrm{rad}(\mathfrak{g}\oplus\mathfrak{h}) = \mathrm{rad}(\mathfrak{g})\oplus\mathrm{rad}(\mathfrak{h}).$$

(ii) The only nonzero ideal in $\mathfrak{g}$ is $\mathfrak{g}$, and $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ since $\mathfrak{g}$ is not abelian. Hence $\mathfrak{g}$ is not solvable. Thus $\mathfrak{g}$ is semisimple. □

Example 16.5. The Lie algebra $\mathfrak{sl}_2(\mathbf{k})$ is simple if $\mathrm{char}(\mathbf{k}) \neq 2$. Likewise, $\mathfrak{so}_3(\mathbf{k})$ is simple.

Theorem 16.6. *(weak Levi decomposition) The Lie algebra $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple. Thus any $\mathfrak{g}$ can be included in an exact sequence*

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}_{\mathrm{ss}} \to 0,$$

where $\mathrm{rad}(\mathfrak{g})$ *is a solvable ideal and* $\mathfrak{g}_{\mathrm{ss}}$ *is semisimple. Moreover, if* $\mathfrak{h} \subset \mathfrak{g}$ *is a solvable ideal such that* $\mathfrak{g}/\mathfrak{h}$ *is semisimple then* $\mathfrak{h} = \mathrm{rad}(\mathfrak{g})$.

Proof. Let $I \subset \mathfrak{g}_{\mathrm{ss}}$ be a solvable ideal, and let $\widetilde{I}$ be its preimage in $\mathfrak{g}$. Then $\widetilde{I}$ is a solvable ideal in $\mathfrak{g}$. Thus $\widetilde{I} = \mathrm{rad}(\mathfrak{g})$ and $I = 0$. □

In fact, in characteristic zero there is a stronger statement, which says that the extension in Theorem 16.6 splits. Namely, given a Lie algebra $\mathfrak{h}$ and another Lie algebra $\mathfrak{a}$ acting on $\mathfrak{h}$ by derivations, we may form the **semidirect product** Lie algebra $\mathfrak{a} \ltimes \mathfrak{h}$ which is $\mathfrak{a} \oplus \mathfrak{h}$ as a vector space with commutator defined by

$$[(a_1, h_1), (a_2, h_2)] = ([a_1, a_2], a_1 \circ h_2 - a_2 \circ h_1 + [h_1, h_2]).$$

Note that a special case of this construction has already appeared in Example 15.23.

Theorem 16.7. *(Levi decomposition) If* $\mathrm{char}(\mathbf{k}) = 0$ *then we have* $\mathfrak{g} \cong \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{\mathrm{ss}}$*, where* $\mathfrak{g}_{\mathrm{ss}} \subset \mathfrak{g}$ *is a semisimple subalgebra (but not necessarily an ideal); i.e.,* $\mathfrak{g}$ *is isomorphic to the semidirect product* $\mathfrak{g}_{\mathrm{ss}} \ltimes \mathrm{rad}(\mathfrak{g})$*. In other words, the projection* $p : \mathfrak{g} \to \mathfrak{g}_{\mathrm{ss}}$ *admits an (in general, non-unique) splitting* $q : \mathfrak{g}_{\mathrm{ss}} \to \mathfrak{g}$*, i.e., a Lie algebra map such that* $p \circ q = \mathrm{Id}$.

Theorem 16.7 will be proved in Subsection 48.2.

Example 16.8. Let G be the group of motions of the Euclidean space $\mathbb{R}^3$ (generated by rotations and translations). Then $G = SO_3(\mathbb{R}) \ltimes \mathbb{R}^3$, so $\mathfrak{g} = \mathrm{Lie}G = \mathfrak{so}_3(\mathbb{R}) \ltimes \mathbb{R}^3$, hence $\mathrm{rad}(\mathfrak{g}) = \mathbb{R}^3$ (abelian Lie algebra) and $\mathfrak{g}_{\mathrm{ss}} = \mathfrak{so}_3(\mathbb{R})$.

Proposition 16.9. *Let* $\mathrm{char}(\mathbf{k}) = 0$*,* $\mathbf{k}$ *algebraically closed, and* V *be an irreducible representation of* $\mathfrak{g}$*. Then* $\mathrm{rad}(\mathfrak{g})$ *acts on* V *by scalars, and* $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ *by zero.*

Proof. By Lie's theorem, there is a nonzero $v \in V$ and $\lambda \in \mathrm{rad}(\mathfrak{g})^*$ such that $av = \lambda(a)v$ for $a \in \mathrm{rad}(\mathfrak{g})$. Let $x \in \mathfrak{g}$ and $\mathfrak{g}_x \subset \mathfrak{g}$ be the Lie subalgebra spanned by $\mathrm{rad}(\mathfrak{g})$ and x. Let W be the span of $x^n v$ for $n \geq 0$. By Lemma 15.18(i), W is a $\mathfrak{g}_x$-subrepresentation of V on which $a \in \mathrm{rad}(\mathfrak{g})$ has the only eigenvalue $\lambda(a)$. Thus by Lemma 15.18(iii), for $a \in \mathrm{rad}(\mathfrak{g})$ we have $\lambda([x, a]) = 0$, so the λ-eigenspace V_λ of $\mathrm{rad}(\mathfrak{g})$ in V is a $\mathfrak{g}$-subrepresentation of V, which implies that $V_\lambda = V$ since V is irreducible. □

Definition 16.10. $\mathfrak{g}$ is called **reductive** if $\mathrm{rad}(\mathfrak{g})$ coincides with the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$.

In other words, $\mathfrak{g}$ is reductive if $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = 0$.

The Levi decomposition theorem implies that a reductive Lie algebra in characteristic zero is a direct sum of a semisimple Lie algebra and an abelian Lie algebra (its center). We will also prove this in Corollary 18.8.

16.2. **Invariant inner products.** Let B be a bilinear form on a Lie algebra $\mathfrak{g}$. Recall that B is invariant if $B([x,y],z) = B(x,[y,z])$ for any $x,y,z \in \mathfrak{g}$.

Example 16.11. If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a finite dimensional representation of $\mathfrak{g}$ then the form

$$B_V(x,y) := \mathrm{Tr}(\rho(x)\rho(y))$$

is an invariant symmetric bilinear form on $\mathfrak{g}$. Indeed, the symmetry is obvious and

$$B_V([x,y],z) = B_V(x,[y,z]) = \mathrm{Tr}|_V(\rho(x)\rho(y)\rho(z) - \rho(x)\rho(z)\rho(y)).$$

Proposition 16.12. *If B is a symmetric invariant bilinear form on $\mathfrak{g}$ and $I \subset \mathfrak{g}$ is an ideal then the orthogonal complement $I^\perp \subset \mathfrak{g}$ is also an ideal. In particular, $\mathfrak{g}^\perp = \mathrm{Ker}(B)$ is an ideal in $\mathfrak{g}$.*

Exercise 16.13. Prove Proposition 16.12.

Proposition 16.14. *If B_V is nondegenerate for some V then $\mathfrak{g}$ is reductive.*

Proof. Let $V_1, ..., V_n$ be the simple composition factors of V; i.e., V has a filtration by subrepresentations such that $F_iV/F_{i-1}V = V_i$, $F_0V = 0$ and $F_nV = V$. Then $B_V(x,y) = \sum_i B_{V_i}(x,y)$. Now, if $x \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$ then $x|_{V_i} = 0$, so $B_{V_i}(x,y) = 0$ for all $y \in \mathfrak{g}$, hence $B_V(x,y) = 0$. $\square$

Example 16.15. It is clear that if $\mathfrak{g} = \mathfrak{gl}_n(\mathbf{k})$ and $V = \mathbf{k}^n$ then the form B_V is nondegenerate, as $B_V(E_{ij}, E_{kl}) = \delta_{il}\delta_{jk}$. Thus $\mathfrak{g}$ is reductive. Also if n is not divisible by the characteristic of $\mathbf{k}$ then $\mathfrak{sl}_n(\mathbf{k})$ is semisimple, since it is orthogonal to scalars under B_V (hence reductive), and has trivial center. In fact, it is easy to show that in this case $\mathfrak{sl}_n(\mathbf{k})$ is a simple Lie algebra (another way to see that it is semisimple).

In fact, we have the following proposition.

Proposition 16.16. *All classical Lie algebras over $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$ are reductive.*

Proof. Let $\mathfrak{g}$ be a classical Lie algebra and V its standard matrix representation. It is easy to check that the form B_V on $\mathfrak{g}$ is nondegenerate, which implies that $\mathfrak{g}$ is reductive. $\square$

For example, the Lie algebras $\mathfrak{so}_n(\mathbb{K})$, $\mathfrak{sp}_{2n}(\mathbb{K})$, $\mathfrak{su}(p,q)$ have trivial center and therefore are semisimple.

16.3. **The Killing form and the Cartan criteria.**

Definition 16.17. The **Killing form** of a Lie algebra $\mathfrak{g}$ is the form $B_{\mathfrak{g}}(x,y) = \mathrm{Tr}(\mathrm{ad}x \cdot \mathrm{ad}y)$.

The Killing form is denoted by $K_{\mathfrak{g}}(x,y)$ or shortly by $K(x,y)$.

Theorem 16.18. *(Cartan criterion of solvability) A Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ of characteristic zero is solvable if and only if $[\mathfrak{g},\mathfrak{g}] \subset \mathrm{Ker}(K)$.*

Theorem 16.19. *(Cartan criterion of semisimplicity) A Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ of characteristic zero is semisimple if and only if its Killing form is nondegenerate.*

16.4. **Jordan decomposition.** To prove the Cartan criteria, we will use the Jordan decomposition of a square matrix. Let us recall it.

Proposition 16.20. *A square matrix $A \in \mathfrak{gl}_N(\mathbf{k})$ over a field $\mathbf{k}$ of characteristic zero can be uniquely written as $A_s + A_n$, where $A_s \in \mathfrak{gl}_N(\mathbf{k})$ is semisimple (i.e. diagonalizes over the algebraic closure of $\mathbf{k}$) and $A_n \in \mathfrak{gl}_N(\mathbf{k})$ is nilpotent in such a way that $A_sA_n = A_nA_s$. Moreover, $A_s = P(A)$ for some $P \in \mathbf{k}[x]$.*

Proof. By the Chinese remainder theorem, there exists a polynomial $P \in \overline{\mathbf{k}}[x]$ such that for every eigenvalue λ of A we have $P(x) = \lambda$ modulo $(x-\lambda)^N$, i.e.,

$$P(x) - \lambda = (x-\lambda)^N Q_\lambda(x)$$

for some polynomial Q_λ. Then on the generalized eigenspace $V(\lambda)$ for A, we have

$$P(A) - \lambda = (A-\lambda)^N Q_\lambda(A) = 0,$$

so $A_s := P(A)$ is semisimple and $A_n = A - P(A)$ is nilpotent, with $A_nA_s = A_sA_n$. If $A = A_s' + A_n'$ is another such decomposition then A_s', A_n' commute with A, hence with A_s and A_n. Also we have

$$A_s - A_s' = A_n' - A_n.$$

Thus this matrix is both semisimple and nilpotent, so it is zero. Finally, since A_s, A_n are unique, they are invariant under the Galois group of $\overline{\mathbf{k}}$ over $\mathbf{k}$ and therefore have entries in $\mathbf{k}$. $\square$

Remark 16.21. 1. If $\mathbf{k}$ is algebraically closed, then A admits a basis in which it is upper triangular, and A_s is the diagonal part while A_n is the off-diagonal part of A.

2. Proposition 16.20 holds with the same proof in characteristic p if the field $\mathbf{k}$ is perfect, i.e., the Frobenius map $x \to x^p$ is surjective on $\mathbf{k}$. However, if $\mathbf{k}$ is not perfect, the proof fails: the fact that A_s, A_n are Galois invariant does not imply that their entries are in $\mathbf{k}$. Also the statement fails: if $\mathbf{k} = \mathbb{F}_p(t)$ and $Ae_i = e_{i+1}$ for $i = 1, .., p-1$ while $Ae_p = te_1$ then A has only one eigenvalue $t^{1/p}$, so $A_s = t^{1/p} \cdot \mathrm{Id}$, i.e., does not have entries in $\mathbf{k}$.

17. **Proofs of the Cartan criteria, properties of semisimple Lie algebras**

17.1. **Proof of the Cartan solvability criterion.** It is clear that $\mathfrak{g}$ is solvable if and only if so is $\mathfrak{g} \otimes_{\mathbf{k}} \overline{\mathbf{k}}$, so we may assume that $\mathbf{k}$ is algebraically closed.

For the "only if" part, note that by Lie's theorem, $\mathfrak{g}$ has a basis in which the operators $\mathrm{ad}x$, $x \in \mathfrak{g}$, are upper triangular. Then $[\mathfrak{g}, \mathfrak{g}]$ acts in this basis by strictly upper triangular matrices, so $K(x, y) = 0$ for $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$.

To prove the "if" part, let us prove the following lemma.

Lemma 17.1. *Let $\mathfrak{g} \subset \mathfrak{gl}(V)$ be a Lie subalgebra such that for any $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$ we have $\mathrm{Tr}(xy) = 0$. Then $\mathfrak{g}$ is solvable.*

Proof. Let $x \in [\mathfrak{g}, \mathfrak{g}]$. Let $\lambda_i, i = 1, ..., m$, be the distinct eigenvalues of x. Let $E \subset \mathbf{k}$ be a $\mathbb{Q}$-span of λ_i. Let $b : E \to \mathbb{Q}$ be a linear functional. There exists an interpolation polynomial $Q \in \mathbf{k}[t]$ such that $Q(\lambda_i - \lambda_j) = b(\lambda_i - \lambda_j) = b(\lambda_i) - b(\lambda_j)$ for all i, j.

By Proposition 16.20, we can write x as $x = x_s + x_n$. Then the operator $\mathrm{ad}x_s$ is diagonalizable with eigenvalues $\lambda_i - \lambda_j$. So

$$Q(\mathrm{ad}x_s) = \mathrm{ad}b,$$

where $b : V \to V$ is the operator acting by $b(\lambda_j)$ on the generalized λ_j-eigenspace of x.

Also we have

$$\mathrm{ad}x = \mathrm{ad}x_s + \mathrm{ad}x_n$$

a sum of commuting semisimple and nilpotent operators. Thus

$$\mathrm{ad}x_s = (\mathrm{ad}x)_s = P(\mathrm{ad}x),$$

and $P(0) = 0$ since 0 is an eigenvalue of $\mathrm{ad}x$. Thus

$$\mathrm{ad}b = R(\mathrm{ad}x),$$

where $R(t) = Q(P(t))$ and $R(0) = 0$.

Let $x = \sum_j [y_j, z_j]$, $y_j, z_j \in \mathfrak{g}$, and d_j be the dimension of the generalized λ_j-eigenspace of x. Then

$$\sum_j d_j b(\lambda_j)\lambda_j = \mathrm{Tr}(bx) =$$

$$\mathrm{Tr}(\sum_j b[y_j, z_j]) = \mathrm{Tr}(\sum_j [b, y_j]z_j) = \mathrm{Tr}(\sum_j R(\mathrm{ad}x)(y_j)z_j).$$

Since $R(0) = 0$, we have $R(\mathrm{ad}x)(y_j) \in [\mathfrak{g}, \mathfrak{g}]$, so by assumption we get

$$\sum_j d_j b(\lambda_j)\lambda_j = 0.$$

Applying b, we get $\sum_j d_j b(\lambda_j)^2 = 0$. Thus $b(\lambda_j) = 0$ for all j. Hence $b = 0$, so $E = 0$.

Thus, the only eigenvalue of x is 0, i.e., x is nilpotent. But then by Engel's theorem, $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Thus $\mathfrak{g}$ is solvable. Thus proves the lemma. □

Now the "if" part of the Cartan solvability criterion follows easily by applying Lemma 17.1 to $V = \mathfrak{g}$ and replacing $\mathfrak{g}$ by the quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$.

17.2. **Proof of the Cartan semisimplicity criterion.** Assume that $\mathfrak{g}$ is semisimple, and let $I = \mathrm{Ker}(K_\mathfrak{g})$, an ideal in $\mathfrak{g}$. Then $K_I = (K_\mathfrak{g})|_I = 0$. Thus by Cartan's solvability criterion I is solvable. Hence $I = 0$.

Conversely, suppose $K_\mathfrak{g}$ is nondegenerate. Then $\mathfrak{g}$ is reductive. Moreover, the center of $\mathfrak{g}$ is contained in the kernel of $K_\mathfrak{g}$, so it must be trivial. Thus $\mathfrak{g}$ is semisimple.

17.3. **Properties of semisimple Lie algebras.**

Proposition 17.2. *Let* $\mathrm{char}(\mathbf{k}) = 0$ *and* $\mathfrak{g}$ *be a finite dimensional Lie algebra over* $\mathbf{k}$. *Then* $\mathfrak{g}$ *is semisimple iff* $\mathfrak{g} \otimes_\mathbf{k} \overline{\mathbf{k}}$ *is semisimple.*

Proof. Immediately follows from Cartan's criterion of semisimplicity. Here is another proof (of the nontrivial direction): if $\mathfrak{g}$ is semisimple and I is a nonzero solvable ideal in $\mathfrak{g} \otimes_\mathbf{k} \overline{\mathbf{k}}$ then it has a finite Galois orbit $I_1, ..., I_n$ and $I_1 + ... + I_n$ is a Galois invariant solvable ideal, so it comes from a solvable ideal in $\mathfrak{g}$. □

Remark 17.3. This theorem fails if we replace the word "semisimple" by "simple": e.g., if $\mathfrak{g}$ is a simple complex Lie algebra regarded as a real Lie algebra then $\mathfrak{g}_\mathbb{C} \cong \mathfrak{g} \oplus \mathfrak{g}$ is semisimple but not simple.

Theorem 17.4. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra and* $I \subset \mathfrak{g}$ *an ideal. Then there is an ideal* $J \subset \mathfrak{g}$ *such that* $\mathfrak{g} = I \oplus J$.

Proof. Let $I^\perp$ be the orthogonal complement of I with respect to the Killing form, an ideal in $\mathfrak{g}$. Consider the intersection $I \cap I^\perp$. It is an ideal in $\mathfrak{g}$ with the zero Killing form. Thus, by the Cartan solvability criterion, it is solvable. By definition of a semisimple Lie algebra, this means that $I \cap I^\perp = 0$, so we may take $J = I^\perp$. □

We will see below (in Proposition 17.7) that J is in fact unique and must equal $I^\perp$.

Corollary 17.5. *A Lie algebra* $\mathfrak{g}$ *is semisimple iff it is a direct sum of simple Lie algebras.*

Proof. We have already shown that a direct sum of simple Lie algebras is semisimple. The opposite direction easily follows by induction from Theorem 17.4. □

Corollary 17.6. *If* $\mathfrak{g}$ *is a semisimple Lie algebra, then* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Proof. For a simple Lie algebra it is clear because $[\mathfrak{g}, \mathfrak{g}]$ is an ideal in $\mathfrak{g}$ which cannot be zero (otherwise, $\mathfrak{g}$ would be abelian). So the result follows from Corollary 17.5. □

Proposition 17.7. *Let* $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_k$ *be a semisimple Lie algebra, with* $\mathfrak{g}_i$ *being simple. Then any ideal* I *in* $\mathfrak{g}$ *is of the form* $I = \oplus_{i\in S}\mathfrak{g}_i$ *for some subset* $S \subset \{1, ..., k\}$.

Proof. The proof goes by induction in k. Let $p_k : \mathfrak{g} \to \mathfrak{g}_k$ be the projection. Consider $p_k(I) \subset \mathfrak{g}_k$. Since $\mathfrak{g}_k$ is simple, either $p_k(I) = 0$, in which case $I \subset \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_{k-1}$ and we can use the induction assumption, or $p_k(I) = \mathfrak{g}_k$. Then $[\mathfrak{g}_k, I] = [\mathfrak{g}_k, p_k(I)] = \mathfrak{g}_k$. Since I is an ideal, $I \supset \mathfrak{g}_k$, so $I = I' \oplus \mathfrak{g}_k$ for some subspace $I' \subset \mathfrak{g}_1 \oplus \ \oplus \mathfrak{g}_{k-1}$. It is immediate that then I' is an ideal in $\mathfrak{g}_1 \oplus \ \oplus \mathfrak{g}_{k-1}$ and the result again follows from the induction assumption. □

Corollary 17.8. *Any ideal in a semisimple Lie algebra is semisimple. Also, any quotient of a semisimple Lie algebra is semisimple.*

Let $\mathrm{Der}\mathfrak{g}$ be the Lie algebra of derivations of a Lie algebra $\mathfrak{g}$. We have a homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathrm{Der}\mathfrak{g}$ whose kernel is the center $\mathfrak{z}(\mathfrak{g})$. Thus if $\mathfrak{g}$ has trivial center (e.g., is semisimple) then the map ad is injective and identifies $\mathfrak{g}$ with a Lie subalgebra of $\mathrm{Der}\mathfrak{g}$. Moreover, for $d \in \mathrm{Der}\mathfrak{g}$ and $x \in \mathfrak{g}$, we have

$$[d, \mathrm{ad}x](y) = d[x, y] - [x, dy] = [dx, y] = \mathrm{ad}(dx)(y).$$

Thus $\mathfrak{g} \subset \mathrm{Der}\mathfrak{g}$ is an ideal.

Proposition 17.9. *If* $\mathfrak{g}$ *is semisimple then* $\mathfrak{g} = \mathrm{Der}\mathfrak{g}$.

Proof. Consider the invariant symmetric bilinear form

$$K(a, b) = \mathrm{Tr}|_{\mathfrak{g}}(ab)$$

on $\mathrm{Der}\mathfrak{g}$. This is an extension of the Killing form of $\mathfrak{g}$ to $\mathrm{Der}\mathfrak{g}$, so its restriction to $\mathfrak{g}$ is nondegenerate. Let $I = \mathfrak{g}^\perp$ be the orthogonal complement of $\mathfrak{g}$ in $\mathrm{Der}\mathfrak{g}$ under K. It follows that I is an ideal, $I \cap \mathfrak{g} = 0$, and $I \oplus \mathfrak{g} = \mathrm{Der}\mathfrak{g}$. Since both I and $\mathfrak{g}$ are ideals, we have $[\mathfrak{g}, I] = 0$. Thus for $d \in I$ and $x \in \mathfrak{g}$, $[d, \mathrm{ad}x] = \mathrm{ad}(dx) = 0$, so dx belongs to the center of $\mathfrak{g}$. Thus $dx = 0$, i.e., $d = 0$. It follows that $I = 0$, as claimed. $\square$

Corollary 17.10. *Let $\mathfrak{g}$ be a real or complex semisimple Lie algebra, and $G = \mathrm{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$. Then G is a Lie group with $\mathrm{Lie}G = \mathfrak{g}$. Thus G acts on $\mathfrak{g}$ by the adjoint action.*

Proof. It is easy to show that for any finite dimensional real or complex Lie algebra $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathrm{Der}(\mathfrak{g})$, so the statement follows from Proposition 17.9. $\square$

18. **Extensions of representations, Whitehead's theorem, compete reducibility**

18.1. **Extensions.** Let $\mathfrak{g}$ be a Lie algebra and U, W be representations of $\mathfrak{g}$. We would like to classify all representations V which fit into a short exact sequence

$$0 \to U \to V \to W \to 0, \tag{18.1}$$

i.e., $U \subset V$ is a subrepresentation such that the surjection $p : V \to W$ has kernel U and thus defines an isomorphism $V/U \cong W$. In other words, V is endowed with a 2-step filtration with $F_0V = U$ and $F_1V = V$ such that $F_1V/F_0V = W$, so $\mathrm{gr}(V) = U \oplus W$. To do so, pick a splitting of this sequence as a sequence of vector spaces, i.e. an injection $i : W \to V$ (not a homomorphism of representations, in general) such that $p \circ i = \mathrm{Id}_W$. This defines a linear isomorphism $\widetilde{i} : U \oplus W \to V$ given by $(u, w) \mapsto u + i(w)$, which allows us to rewrite the action of $\mathfrak{g}$ on V as an action on $U \oplus W$. Since $\widetilde{i}$ is not in general a morphism of representations, this action is given by

$$\rho(x)(u, w) = (xu + a(x)w, xw)$$

where $a : \mathfrak{g} \to \mathrm{Hom}_{\mathbf{k}}(W, U)$ is a linear map, and $\widetilde{i}$ is a morphism of representations iff $a = 0$.

What are the conditions on a to give rise to a representation? We compute:

$$\rho([x, y])(u, w) = ([x, y]u + a([x, y])w, [x, y]w),$$

$$[\rho(x), \rho(y)](u, w) = ([x, y]u + ([x, a(y)] + [a(x), y])w, [x, y]w).$$

Thus the condition to give a representation is the Leibniz rule

$$a([x, y]) = [x, a(y)] + [a(x), y] = [x, a(y)] - [y, a(x)].$$

In general, if E is a representation of $\mathfrak{g}$ then a linear function $a : \mathfrak{g} \to E$ such that

$$a([x, y]) = x \circ a(y) - y \circ a(x)$$

is called a $\mathbf{1-}$**cocycle** of $\mathfrak{g}$ with values in E. The space of 1-cocycles is denoted by $Z^1(\mathfrak{g}, E)$.

Example 18.1. We have $Z^1(\mathfrak{g}, \mathbf{k}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ and $Z^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{Der}\mathfrak{g}$.

Thus we see that in our setting $a : \mathfrak{g} \to \mathrm{Hom}_{\mathbf{k}}(W, U)$ defines a representation if and only if $a \in Z^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U))$. Denote the representation V attached to such a by V_a. Then we have a natural short exact sequence

$$0 \to U \to V_a \to W \to 0.$$

It may, however, happen that some $a \neq 0$ defines a trivial extension $V \cong U \oplus W$, i.e., $V_a \cong V_0$, and more generally $V_a \cong V_b$ for $a \neq b$. Let us determine when this happens. More precisely, let us look for isomorphisms $f : V_a \to V_b$ preserving the structure of the short exact sequences, i.e., such that $\mathrm{gr}(f) = \mathrm{Id}$. Then

$$f(u, w) = (u + Aw, w)$$

where $A : W \to U$ is a linear map. Then we have

$$xf(u, w) = x(u + Aw, w) = (xu + xAw + b(x)w, xw)$$

and

$$fx(u, w) = f(xu + a(x)w, xw) = (xu + a(x)w + Axw, xw),$$

so we get that $xf = fx$ iff

$$[x, A] = a(x) - b(x).$$

In particular, setting $b = 0$, we see that V is a trivial extension if and only if $a(x) = [x, A]$ for some A.

More generally, if E is a $\mathfrak{g}$-module, the linear function $a : \mathfrak{g} \to E$ given by $a(x) = xv$ for some $v \in E$ is called the **1-coboundary** of v, and one writes $a = dv$. The space of 1-coboundaries is denoted by $B^1(\mathfrak{g}, E)$; it is easy to see that it is a subspace of $Z^1(\mathfrak{g}, E)$, i.e., a 1-coboundary is always a 1-cocycle. Thus in our setting $f : V_a \to V_b$ is an isomorphism of representations iff

$$a - b = dA,$$

i.e., there is an isomorphism $f : V_a \cong V_b$ with $\mathrm{gr}(f) = \mathrm{Id}$ if and only if $a = b$ in the quotient space

$$\mathrm{Ext}^1(W, U) := Z^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U))/B^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U)).$$

The notation is justified by the fact that this space parametrizes extensions of W by U. More precisely, every short exact sequence (18.1) gives rise to a class $[V] \in \mathrm{Ext}^1(W, U)$, and the extension defined by this sequence is trivial iff $[V] = 0$.

More generally, for a $\mathfrak{g}$-module E the space

$$H^1(\mathfrak{g}, E) := Z^1(\mathfrak{g}, E)/B^1(\mathfrak{g}, E)$$

is called the **first cohomology** of $\mathfrak{g}$ with coefficients in E. Thus,

$$\mathrm{Ext}^1(W, U) = H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(W, U)).$$

Lemma 18.2. *A short exact sequence* $0 \to U \to V \to W \to 0$ *gives rise to an exact sequence*

$$H^1(\mathfrak{g}, U) \to H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, W).$$

Exercise 18.3. Prove Lemma 18.2.

18.2. **Whitehead's theorem.** We have shown in Corollary 17.6 and Proposition 17.9 that for a semisimple $\mathfrak{g}$ over a field of characteristic zero, $H^1(\mathfrak{g}, \mathbf{k}) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* = 0$, and $H^1(\mathfrak{g}, \mathfrak{g}) = \mathrm{Der}\mathfrak{g}/\mathfrak{g} = 0$. In fact, these are special cases of a more general theorem.

Theorem 18.4. *(Whitehead) If $\mathfrak{g}$ is semisimple in characteristic zero then for every finite dimensional representation V of $\mathfrak{g}$, $H^1(\mathfrak{g}, V) = 0$.*

18.3. **Proof of Theorem 18.4.** We will use the following lemma, which holds over any field.

Lemma 18.5. *Let E be a representation of a Lie algebra $\mathfrak{g}$ and $C \in U(\mathfrak{g})$ be a central element which acts by 0 on the trivial representation of $\mathfrak{g}$ and by some scalar $\lambda \neq 0$ on E. Then $H^1(\mathfrak{g}, E) = 0$.*

Proof. We have seen that $H^1(\mathfrak{g}, E) = \mathrm{Ext}^1(\mathbf{k}, E)$, so our job is to show that any extension
$$0 \to E \to V \to \mathbf{k} \to 0$$
splits. Let $p : V \to \mathbf{k}$ be the projection. We claim that there exists a unique vector $v \in V$ such that $p(v) = 1$ and $Cv = 0$. Indeed, pick some $w \in V$ with $p(w) = 1$. Then $Cw \in E$, so set $v = w - \lambda^{-1}Cw$. Since $C^2w = \lambda Cw$, we have $Cv = 0$. Also if v' is another such vector then $v - v' \in E$ so $C(v - v') = \lambda(v - v') = 0$, hence $v = v'$.

Thus $\mathbf{k}v \subset V$ is a $\mathfrak{g}$-invariant complement to E (as C is central), which implies the statement. $\square$

It remains to construct a central element of $U(\mathfrak{g})$ for a semisimple Lie algebra $\mathfrak{g}$ to which we can apply Lemma 18.5. This can be done as follows. Let a_i be a basis of $\mathfrak{g}$ and a^i the dual basis under an invariant inner product on $\mathfrak{g}$ (for example, the Killing form). Define the **(quadratic) Casimir element**
$$C := \sum_i a_i a^i.$$
It is easy to show that C is independent on the choice of the basis (although it depends on the choice of the inner product). Also C is central: for $y \in \mathfrak{g}$,
$$[y, C] = \sum_i ([y, a_i]a^i + a_i[y, a^i]) = 0$$
since
$$\sum_i ([y, a_i] \otimes a^i + a_i \otimes [y, a^i]) = 0.$$

Finally, note that for $\mathfrak{g} = \mathfrak{sl}_2$, C is proportional to the Casimir element $2fe + \frac{h^2}{2} + h = ef + fe + \frac{h^2}{2}$ considered previously, as the basis $f, e, \frac{h}{\sqrt{2}}$ is dual to the basis $e, f, \frac{h}{\sqrt{2}}$ under an invariant inner product of $\mathfrak{g}$.

The key lemma used in the proof of Theorem 18.4 is the following.

Lemma 18.6. *Let $\mathfrak{g}$ be semisimple in characteristic zero and V be a nontrivial finite dimensional irreducible $\mathfrak{g}$-module. Then there is a central element $C \in U(\mathfrak{g})$ such that $C|_{\mathbf{k}} = 0$ and $C|_V \neq 0$.*

Proof. Consider the invariant symmetric bilinear form on $\mathfrak{g}$

$$B_V(x, y) = \mathrm{Tr}|_V(xy).$$

We claim that $B_V \neq 0$. Indeed, let $\bar{\mathfrak{g}} \subset \mathfrak{gl}(V)$ be the image of $\mathfrak{g}$. By Lemma 17.1, if $B_V = 0$ then $\bar{\mathfrak{g}}$ is solvable, so, being the quotient of a semisimple Lie algebra $\mathfrak{g}$, it must be zero, hence V is trivial, a contradiction.

Let $I = \mathrm{Ker}(B_V)$. Then $I \subset \mathfrak{g}$ is an ideal, so by Proposition 17.7, $\mathfrak{g} = I \oplus \mathfrak{g}'$ for some semisimple Lie algebra $\mathfrak{g}'$, and B_V is nondegenerate on $\mathfrak{g}'$. Let C be the Casimir element of $U(\mathfrak{g}')$ corresponding to the inner product B_V. Then $\mathrm{Tr}_V(C) = \sum_i B_V(a_i, a^i) = \dim \mathfrak{g}'$, so $C|_V = \frac{\dim \mathfrak{g}'}{\dim V} \neq 0$. Also it is clear that $C|_{\mathbf{k}} = 0$, so the lemma follows. $\square$

Corollary 18.7. *For any irreducible finite dimensional representation V of a semisimple Lie algebra $\mathfrak{g}$ over a field $\mathbf{k}$ of characteristic zero, we have $H^1(\mathfrak{g}, V) = 0$.*

Proof. If V is nontrivial, this follows from Lemmas 18.5 and 18.6. On the other hand, if $V = \mathbf{k}$ then $H^1(\mathfrak{g}, V) = (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* = 0$. $\square$

Now we can prove Theorem 18.4. By Lemma 18.2, it suffices to prove the theorem for irreducible V, which is guaranteed by Corollary 18.7.

Corollary 18.8. *A reductive Lie algebra $\mathfrak{g}$ in characteristic zero is uniquely a direct sum of a semisimple and abelian Lie algebra.*

Proof. Consider the adjoint representation of $\mathfrak{g}$. It is a representation of $\mathfrak{g}' = \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, which fits into a short exact sequence

$$0 \to \mathfrak{z}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}' \to 0.$$

By complete reducibility, this sequence splits, i.e. we have a decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}(\mathfrak{g})$ as a direct sum of ideals, and it is clearly unique. $\square$

18.4. **Complete reducibility of representations of semisimple Lie algebras.**

Theorem 18.9. *Every finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ over a field of characteristic zero is completely reducible, i.e., isomorphic to a direct sum of irreducible representations.*

Proof. Theorem 18.4 implies that for any finite dimensional representations W, U of $\mathfrak{g}$ one has $\mathrm{Ext}^1(W, U) = 0$. Thus any short exact sequence

$$0 \to U \to V \to W \to 0$$

splits, which implies the statement. $\square$

19. **Structure of semisimple Lie algebras, I**

19.1. **Semisimple elements.** Let x be an element of a Lie algebra $\mathfrak{g}$ over an algebraically closed field $\mathbf{k}$. Let $\mathfrak{g}_\lambda \subset \mathfrak{g}$ be the generalized eigenspace of $\mathrm{ad}x$ with eigenvalue λ. Then $\mathfrak{g} = \oplus_\lambda \mathfrak{g}_\lambda$.

Lemma 19.1. *We have* $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.

Proof. Let $y \in \mathfrak{g}_\lambda, z \in \mathfrak{g}_\mu$. We have

$$(\mathrm{ad}x - \lambda - \mu)^N([y,z]) =$$

$$\sum_{p+q+r+s=N} (-1)^{r+s} \frac{N!}{p!q!r!s!} \lambda^r \mu^s [(\mathrm{ad}x)^p(y), (\mathrm{ad}x)^q(z)] =$$

$$\sum_{k+\ell=N} \frac{N!}{k!\ell!} [(\mathrm{ad}x - \lambda)^k(y), (\mathrm{ad}x - \mu)^\ell(z)].$$

Thus if $(\mathrm{ad}x - \lambda)^n(y) = 0$ and $(\mathrm{ad}x - \mu)^m(z) = 0$ then

$$(\mathrm{ad}x - \lambda - \mu)^{m+n}([y,z]) = 0,$$

so $[y,z] \in \mathfrak{g}_{\lambda+\mu}$. □

Definition 19.2. An element x of a Lie algebra $\mathfrak{g}$ is called **semisimple** if the operator $\mathrm{ad}x$ is semisimple and **nilpotent** if this operator is nilpotent.

It is clear that any element which is both semisimple and nilpotent is central, so for a semisimple Lie algebra it must be zero. Note also that for $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{k})$ this coincides with the usual definition.

Proposition 19.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic zero. Then every element $x \in \mathfrak{g}$ has a unique decomposition as $x = x_s + x_n$, where x_s is semisimple, x_n is nilpotent and $[x_s, x_n] = 0$. Moreover, if $y \in \mathfrak{g}$ and $[x,y] = 0$ then $[x_s, y] = [x_n, y] = 0$.*

Proof. Recall that $\mathfrak{g} \subset \mathfrak{gl}(\mathfrak{g})$ via the adjoint representation. So we can consider the Jordan decomposition $x = x_s + x_n$, with $x_s, x_n \in \mathfrak{gl}(\mathfrak{g})$. We have $x_s(y) = \lambda y$ for $y \in \mathfrak{g}_\lambda$. Thus $y \mapsto x_s(y)$ is a derivation of $\mathfrak{g}$ by Lemma 19.1. But by Proposition 17.9 every derivation of $\mathfrak{g}$ is inner, which implies that $x_s \in \mathfrak{g}$, hence $x_n \in \mathfrak{g}$. It is clear that x_s is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$. Also if $[x,y] = 0$ then $\mathrm{ad}y$ preserves $\mathfrak{g}_\lambda$ for all λ, hence $[x_s, y] = 0$ as linear operators on $\mathfrak{g}$ and thus as elements of $\mathfrak{g}$. This also implies that the decomposition is unique since if $x = x_s' + x_n'$ then $[x_s, x_s'] = [x_n, x_n'] = 0$, so $x_s - x_s' = x_n' - x_n$ is both semisimple and nilpotent, hence zero. □

Corollary 19.4. *Any semisimple Lie algebra $\mathfrak{g} \neq 0$ over a field of characteristic zero contains nonzero semisimple elements.*

Proof. Otherwise, by Proposition 19.3, every element $x \in \mathfrak{g}$ is nilpotent, which by Engel's theorem would imply that $\mathfrak{g}$ is nilpotent, hence solvable, hence zero. $\square$

19.2. **Toral subalgebras.** From now on we assume that $\mathrm{char}(\mathbf{k}) = 0$ unless specified otherwise.

Definition 19.5. An abelian Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a **toral subalgebra** if it consists of semisimple elements.

Proposition 19.6. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a toral subalgebra, and B a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ (e.g., the Killing form).*

(i) We have a decomposition $\mathfrak{g} = \oplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the subspace of $x \in \mathfrak{g}$ such that for $h \in \mathfrak{h}$ we have $[h, x] = \alpha(h)x$, and $\mathfrak{g}_0 \supset \mathfrak{h}$.

(ii) We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

(iii) If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal under B.

(iv) B restricts to a nondegenerate pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbf{k}$.

Proof. (i) is just the joint eigenspace decomposition for $\mathfrak{h}$ acting in $\mathfrak{g}$. (ii) is a very easy special case of Lemma 19.1. (iii) and (iv) follow from the fact that B is nondegenerate and invariant. $\square$

Corollary 19.7. *(i) The Lie subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ is reductive.*

(ii) if $x \in \mathfrak{g}_0$ then $x_s, x_n \in \mathfrak{g}_0$.

Proof. (i) This follows from Proposition 16.14 and the fact that the form $(x, y) \mapsto \mathrm{Tr}|_{\mathfrak{g}}(xy)$ on $\mathfrak{g}_0$ is nondegenerate (Proposition 19.6(iv) for the Killing form of $\mathfrak{g}$).

(ii) We have $[h, x] = 0$ for $h \in \mathfrak{h}$, so $[h, x_s] = 0$, hence $x_s \in \mathfrak{g}_0$. $\square$

19.3. **Cartan subalgebras.**

Definition 19.8. A **Cartan subalgebra** of a semisimple Lie algebra $\mathfrak{g}$ is a toral subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g}_0 = \mathfrak{h}$.

Example 19.9. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{k})$. Then the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of diagonal matrices is a Cartan subalgebra.

It is clear that any Cartan subalgebra is a maximal toral subalgebra of $\mathfrak{g}$. The following theorem, stating the converse, shows that Cartan subalgebras exist.

Theorem 19.10. *Let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra.*

Proof. Let $x \in \mathfrak{g}_0$, then by Corollary 19.7(ii) $x_s \in \mathfrak{g}_0$, so $x_s \in \mathfrak{h}$ by maximality of $\mathfrak{h}$. Thus $\mathrm{ad}x|_{\mathfrak{g}_0} = \mathrm{ad}x_n|_{\mathfrak{g}_0}$ is nilpotent. So by Engel's theorem $\mathfrak{g}_0$ is nilpotent. But it is also reductive, hence abelian.

Now let us show that every $x \in \mathfrak{g}_0$ which is nilpotent in $\mathfrak{g}$ must be zero. Indeed, in this case, for any $y \in \mathfrak{g}_0$, the operator $\mathrm{ad}x \cdot \mathrm{ad}y : \mathfrak{g} \to \mathfrak{g}$ is nilpotent (as $[x, y] = 0$), so $\mathrm{Tr}|_{\mathfrak{g}}(\mathrm{ad}x \cdot \mathrm{ad}y) = 0$. But this form is nondegenerate on $\mathfrak{g}_0$, which implies that $x = 0$.

Thus for any $x \in \mathfrak{g}_0$, $x_n = 0$, so $x = x_s$ is semisimple. Hence $\mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{h}$ is a Cartan subalgebra. □

We will show in Theorem 20.10 that all Cartan subalgebras of $\mathfrak{g}$ are conjugate under $\mathrm{Aut}(\mathfrak{g})$, in particular they all have the same dimension, which is called the **rank** of $\mathfrak{g}$.

19.4. **Root decomposition.**

Proposition 19.11. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and B a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$ (e.g., the Killing form).*

(i) We have a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the subspace of $x \in \mathfrak{g}$ such that for $h \in \mathfrak{h}$ we have $[h, x] = \alpha(h)x$, and R is the (finite) set of $\alpha \in \mathfrak{h}^*$, $\alpha \neq 0$, such that $\mathfrak{g}_\alpha \neq 0$.

(ii) We have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.

(iii) If $\alpha + \beta \neq 0$ then $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal under B.

(iv) B restricts to a nondegenerate pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbf{k}$.

Proof. This immediately follows from Theorem 19.6. □

Definition 19.12. The set R is called the **root system** of $\mathfrak{g}$ and its elements are called **roots**.

Proposition 19.13. *Let $\mathfrak{g}_1, ..., \mathfrak{g}_n$ be simple Lie algebras and let $\mathfrak{g} = \oplus_i \mathfrak{g}_i$.*

(i) Let $\mathfrak{h}_i \subset \mathfrak{g}_i$ be Cartan subalgebras of $\mathfrak{g}_i$ and $R_i \subset \mathfrak{h}_i^*$ the corresponding root systems of $\mathfrak{g}_i$. Then $\mathfrak{h} = \oplus_i \mathfrak{h}_i$ is a Cartan subalgebra in $\mathfrak{g}$ and the corresponding root system R is the disjoint union of R_i.

(ii) Each Cartan subalgebra in $\mathfrak{g}$ has the form $\mathfrak{h} = \oplus_i \mathfrak{h}_i$ where $\mathfrak{h}_i \subset \mathfrak{g}_i$ is a Cartan subalgebra in $\mathfrak{g}_i$.

Proof. (i) is obvious. To prove (ii), given a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, let $\mathfrak{h}_i$ be the projections of $\mathfrak{h}$ to $\mathfrak{g}_i$. It is easy to see that $\mathfrak{h}_i \subset \mathfrak{g}_i$ are Cartan subalgebras. Also $\mathfrak{h} \subset \oplus_i \mathfrak{h}_i$ and the latter is toral, which implies that $\mathfrak{h} = \oplus_i \mathfrak{h}_i$ since $\mathfrak{h}$ is a Cartan subalgebra. □

Example 19.14. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbf{k})$. Then the subspace of diagonal matrices $\mathfrak{h}$ is a Cartan subalgebra (cf. Example 19.9), and it can be naturally identified with the space of vectors $\mathbf{x} = (x_1, ..., x_n)$ such that $\sum_i x_i = 0$. Let $\mathbf{e}_i$ be the linear functionals on this space given by $\mathbf{e}_i(\mathbf{x}) = x_i$. We have $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{i \neq j} \mathbf{k} E_{ij}$ and $[\mathbf{x}, E_{ij}] = (x_i - x_j)E_{ij}$. Thus the root system R consists of vectors $\mathbf{e}_i - \mathbf{e}_j \in \mathfrak{h}^*$ for $i \neq j$ (so there are $n(n-1)$ roots).

Now let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Let $(,)$ be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, for example the Killing form. Since the restriction of $(,)$ to $\mathfrak{h}$ is nondegenerate, it defines an isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ given by $h \mapsto (h, ?)$. The inverse of this isomorphism will be denoted by $\alpha \mapsto H_\alpha$. We also have the inverse form on $\mathfrak{h}^*$ which we also will denote by $(,)$; it is given by $(\alpha, \beta) := \alpha(H_\beta) = (H_\alpha, H_\beta)$.

Lemma 19.15. *For any $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ we have*

$$[e, f] = (e, f)H_\alpha.$$

Proof. We have $[e, f] \in \mathfrak{h}$ so it is enough to show that the inner product of both sides with any $h \in \mathfrak{h}$ is the same. We have

$$([e, f], h) = (e, [f, h]) = \alpha(h)(e, f) = ((e, f)H_\alpha, h),$$

as desired. ☐

Lemma 19.16. *(i) If α is a root then $(\alpha, \alpha) \neq 0$.*

(ii) Let $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ be such that $(e, f) = \frac{2}{(\alpha,\alpha)}$, and let $h_\alpha := \frac{2H_\alpha}{(\alpha,\alpha)}$. Then e, f, h_α satisfy the commutation relations of the Lie algebra $\mathfrak{sl}_2$.

(iii) h_α is independent on the choice of $(,)$.

Proof. (i) Pick $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\alpha}$ with $(e, f) \neq 0$. Let $h := [e, f] = (e, f)H_\alpha$ (by Lemma 19.15) and consider the Lie algebra $\mathfrak{a}$ generated by e, f, h. Then we see that

$$[h, e] = \alpha(h)e = (\alpha, \alpha)(e, f)e, \ [h, f] = -\alpha(h)f = (\alpha, \alpha)(e, f)f.$$

Thus if $(\alpha, \alpha) = 0$ then $\mathfrak{a}$ is a solvable Lie algebra. By Lie's theorem, we can choose a basis in $\mathfrak{g}$ such that operators $\mathrm{ad}e$, $\mathrm{ad}f$, $\mathrm{ad}h$ are upper triangular. Since $h = [e, f]$, $\mathrm{ad}h$ will be strictly upper-triangular and thus nilpotent. But since $h \in \mathfrak{h}$, it is also semisimple. Thus, $\mathrm{ad}h = 0$, so $h = 0$ as $\mathfrak{g}$ is semisimple. On the other hand, $h = (e, f)H_\alpha \neq 0$. This contradiction proves the first part of the theorem.

(ii) This follows immediately from the formulas in the proof of (i).

(iii) It's enough to check the statement for a simple Lie algebra, and in this case this is easy since $(,)$ is unique up to scaling. ☐

The Lie subalgebra of $\mathfrak{g}$ spanned by e, f, h_α, which we've shown to be isomorphic to $\mathfrak{sl}_2(\mathbf{k})$, will be denoted by $\mathfrak{sl}_2(\mathbf{k})_\alpha$ (we will see that $\mathfrak{g}_\alpha$ are 1-dimensional so it is independent on the choices).

Proposition 19.17. *Let $\mathfrak{a}_\alpha = \mathbf{k}H_\alpha \oplus \bigoplus_{k\neq 0} \mathfrak{g}_{k\alpha} \subset \mathfrak{g}$. Then $\mathfrak{a}_\alpha$ is a Lie subalgebra of $\mathfrak{g}$.*

Proof. This follows from the fact that for $e \in \mathfrak{g}_{k\alpha}, f \in \mathfrak{g}_{-k\alpha}$ we have $[e, f] = (e, f)H_{k\alpha} = k(e, f)H_\alpha$. □

Corollary 19.18. *(i) The space $\mathfrak{g}_\alpha$ is 1-dimensional for each root α of $\mathfrak{g}$.*

(ii) If α is a root of $\mathfrak{g}$ and $k \geq 2$ is an integer then $k\alpha$ is not a root of $\mathfrak{g}$.

Proof. For a root α the Lie algebra $\mathfrak{a}_\alpha$ contains $\mathfrak{sl}_2(\mathbf{k})_\alpha$, so it is a finite dimensional representation of this Lie algebra. Also the kernel of h_α on this representation is spanned by h_α, hence 1-dimensional, and eigenvalues of h_α are even integers since $\alpha(h_\alpha) = 2$. Thus by the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4), this representation is irreducible, i.e., eigenspaces of h_α (which are $\mathfrak{g}_{k\alpha}$ and $\mathbf{k}H_\alpha$) are 1-dimensional. Therefore the map $[e, ?] : \mathfrak{g}_\alpha \to \mathfrak{g}_{2\alpha}$ is zero (as $\mathfrak{g}_\alpha$ is spanned by e). So again by representation theory of $\mathfrak{sl}_2$ we have $\mathfrak{g}_{k\alpha} = 0$ for $|k| \geq 2$. □

Theorem 19.19. *Let $\mathfrak{g}$ be a semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$. Let $(,)$ be a nondegenerate symmetric invariant bilinear form on $\mathfrak{g}$.*

(i) R spans $\mathfrak{h}^*$ as a vector space, and elements h_α, $\alpha \in R$ span $\mathfrak{h}$ as a vector space.

(ii) For any two roots α, β, the number $a_{\alpha,\beta} := \beta(h_\alpha) = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

(iii) For $\alpha \in R$, define the **reflection operator** *$s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by*

$$s_\alpha(\lambda) = \lambda - \lambda(h_\alpha)\alpha = \lambda - 2\frac{(\lambda, \alpha)}{(\alpha, \alpha)}\alpha.$$

Then for any roots α, β, $s_\alpha(\beta)$ is also a root.

(iv) For roots $\alpha, \beta \neq \pm\alpha$, the subspace $V_{\alpha,\beta} = \oplus_{k\in\mathbb{Z}}\mathfrak{g}_{\beta+k\alpha} \subset \mathfrak{g}$ is an is an irreducible representation of $\mathfrak{sl}_2(\mathbf{k})_\alpha$.

Proof. (i) Suppose $h \in \mathfrak{h}$ is such that $\alpha(h) = 0$ for all roots α. Then $\mathrm{ad}h = 0$, hence $h = 0$ as $\mathfrak{g}$ is semisimple. This implies both statements.

(ii) $a_{\alpha,\beta}$ is the eigenvalue of h_α on e_β, hence an integer by the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4).

(iii) Let $x \in \mathfrak{g}_\beta$ be nonzero. If $\beta(h_\alpha) \geq 0$ then let $y = f_\alpha^{\beta(h_\alpha)} x$. If $\beta(h_\alpha) \leq 0$ then let $y = e_\alpha^{-\beta(h_\alpha)} x$. Then by representation theory of $\mathfrak{sl}_2$, $y \neq 0$. We also have $[h, y] = s_\alpha(\beta)(h) y$. This implies the statement.

(iv) It is clear that $V_{\alpha,\beta}$ is a representation. Also all h_α-eigenspaces in $V_{\alpha,\beta}$ are 1-dimensional, and the eigenvalues are either all odd or all even. This implies that it is irreducible. $\square$

Corollary 19.20. *Let $\mathfrak{h}_\mathbb{R}$ be the $\mathbb{R}$-span of h_α. Then $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \oplus i\mathfrak{h}_\mathbb{R}$ and the restriction of the Killing form to $\mathfrak{h}_\mathbb{R}$ is real-valued and positive definite.*

Proof. It follows from the previous theorem that the eigenvalues of $\mathrm{ad}h$, $h \in \mathfrak{h}_\mathbb{R}$, are real. So $\mathfrak{h}_\mathbb{R} \cap i\mathfrak{h}_\mathbb{R} = 0$, which implies the first statement. Now, $K(h,h) = \sum_i \lambda_i^2$ where λ_i are the eigenvalues of $\mathrm{ad}h$ (which are not all zero if $h \neq 0$). Thus $K(h,h) > 0$ if $h \neq 0$. $\square$

20. **Structure of semisimple Lie algebras, II**

20.1. **Strongly regular (regular semisimple) elements.** In this section we will discuss another way of constructing Cartan subalgebras. First consider an example.

Example 20.1. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ and $x \in \mathfrak{g}$ be a diagonal matrix with distinct eigenvalues. Then the centralizer $\mathfrak{h} = C(x)$ is the space of all diagonal matrices of trace 0, which is a Cartan subalgebra. Thus the same applies to any diagonalizable matrix with distinct eigenvalues, i.e., a generic matrix (one for which the discriminant of the characteristic polynomial is nonzero).

So we may hope that if we take a generic element x in a semisimple Lie algebra then its centralizer is a Cartan subalgebra. But for that we have to define what we mean by generic.

Definition 20.2. The **nullity** $n(x)$ of an element $x \in \mathfrak{g}$ is the multiplicity of the eigenvalue 0 for the operator $\mathrm{ad}x$ (i.e., the dimension of the generalized 0-eigenspace). The **rank** $\mathrm{rank}(\mathfrak{g})$ of $\mathfrak{g}$ is the minimal value of $n(x)$. An element x is **strongly regular** if $n(x) = \mathrm{rank}(\mathfrak{g})$.

Example 20.3. It is easy to check that for $\mathfrak{g} = \mathfrak{sl}_n$, x is strongly regular if and only if its eigenvalues are distinct.

We will need the following auxiliary lemma.

Lemma 20.4. *Let $P(z_1, ..., z_n)$ be a nonzero complex polynomial, and $U \subset \mathbb{C}^n$ be the set of points $(z_1, ..., z_n) \in \mathbb{C}^n$ such that $P(z_1, ..., z_n) \neq 0$. Then U is path-connected, dense and open.*

Proof. It is clear that U is open, since it is the preimage of the open set $\mathbb{C}^\times \subset \mathbb{C}$ under a continuous map. It is also dense, as its complement, the hypersurface $P = 0$, cannot contain a ball. Finally, to see that it is path-connected, take $\mathbf{x}, \mathbf{y} \in U$, and consider the polynomial $Q(t) := P((1-t)\mathbf{x} + t\mathbf{y})$. It has only finitely many zeros, hence the entire complex line $\mathbf{z} = (1-t)\mathbf{x} + t\mathbf{y}$ except finitely many points is contained in U. Clearly, $\mathbf{x}$ and $\mathbf{y}$ can be connected by a path inside this line avoiding this finite set of points. $\square$

Lemma 20.5. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then the set $\mathfrak{g}^{\mathrm{sr}}$ of strongly regular elements is connected, dense and open in $\mathfrak{g}$.*

Proof. Consider the characteristic polynomial $P_x(t)$ of $\mathrm{ad}x$. We have

$$P_x(t) = t^{\mathrm{rank}(\mathfrak{g})}(t^m + a_{m-1}(x)t^{m-1} + ... + a_0(x)),$$

where $m = \dim \mathfrak{g} - \text{rank}\mathfrak{g}$ and a_i are some polynomials of x, with $a_0 \neq 0$. Then x is strongly regular if and only if $a_0(x) \neq 0$. This implies the statement by Lemma 20.4. $\square$

Proposition 20.6. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Then*

(i) $\dim \mathfrak{h} = \text{rank}(\mathfrak{g})$; and

(ii) the set $\mathfrak{h}^{\text{reg}} := \mathfrak{h} \cap \mathfrak{g}^{\text{sr}}$ coincides with the set

$$V := \{h \in \mathfrak{h} : \alpha(h) \neq 0 \ \forall \alpha \in R\}.$$

In particular, $\mathfrak{h}^{\text{reg}}$ is open and dense in $\mathfrak{h}$.

Proof. (i) Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ (we know it exists, e.g. we can take G to be the connected component of the identity in $\text{Aut}(\mathfrak{g})$).

Lemma 20.7. *Let $\phi : G \times V \to \mathfrak{g}$ be the map defined by $\phi(g, x) := \text{Ad}g \cdot x$. Then the set $U := \text{Im}\phi \subset \mathfrak{g}$ is open.*

Proof. Let us compute the differential $\phi_* : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g}$ at the point $(1, x)$ for $x \in \mathfrak{h}$. We obtain

$$\phi_*(y, h) = [y, x] + h.$$

The kernel of this map is identified with the set of $y \in \mathfrak{g}$ such that $[y, x] \in \mathfrak{h}$. But then $K([y, x], z) = K(y, [x, z]) = 0$ for all $z \in \mathfrak{h}$, so $[y, x] = 0$. Thus $\text{Ker}\phi_* = C(x)$.

Now let $x \in V$. Then $C(x) = \mathfrak{h}$. Thus ϕ_* is surjective by dimension count, hence ϕ is a submersion at $(1, x)$. This means that $U := \text{Im}\phi$ contains x together with its neighborhood in $\mathfrak{g}$. Hence the same holds for $\text{Ad}g \cdot x$, which implies that U is open. $\square$

Since $\mathfrak{g}^{\text{sr}}$ is open and dense and U is open by Lemma 20.7 and non-empty, we see that $U \cap \mathfrak{g}^{\text{sr}} \neq \emptyset$. But

$$n(\text{Ad}g \cdot x) = n(x) = \dim C(x) = \dim \mathfrak{h}.$$

for $x \in V$. This implies that $\text{rank}\mathfrak{g} = \dim \mathfrak{h}$, which yields (i).

(ii) It is clear that for $x \in \mathfrak{h}$, we have

$$n(x) = \dim \text{Ker}(\text{ad}x) = \dim \mathfrak{h} + \#\{\alpha \in R : \alpha(x) = 0\}.$$

This implies the statement. $\square$

20.2. **Conjugacy of Cartan subalgebras.**

Theorem 20.8. *(i) Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $x \in \mathfrak{g}$ be a strongly regular semisimple element (which exists by Proposition 20.6). Then the centralizer $C(x)$ of x in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$.*

(ii) Any Cartan subalgebra of $\mathfrak{g}$ is of this form.

Proof. Consider the eigenspace decomposition of $\mathrm{ad}x$: $\mathfrak{g} = \oplus_\lambda \mathfrak{g}_\lambda$. Since $\mathbb{C}x$ is a toral subalgebra, the Lie algebra $\mathfrak{g}_0 = C(x)$ is reductive, with $\dim(\mathfrak{g}_0) = \mathrm{rank}\mathfrak{g}$.

We claim that $\mathfrak{g}_0$ is also nilpotent. By Engel's theorem, to establish this, it suffices to show that the restriction of $\mathrm{ad}y$ to $\mathfrak{g}_0$ is nilpotent for $y \in \mathfrak{g}_0$. But $\mathrm{ad}(x+ty) = \mathrm{ad}x + t\mathrm{ad}y$ is invertible on $\mathfrak{g}/\mathfrak{g}_0$ for small t, since it is so for $t = 0$ and the set of invertible matrices is open. Thus $\mathrm{ad}(x+ty)$ must be nilpotent on $\mathfrak{g}_0$, as the multiplicity of the eigenvalue 0 for this operator must be (at least) $\mathrm{rank}\mathfrak{g} = \dim \mathfrak{g}_0$. But $\mathrm{ad}(x+ty) = t\mathrm{ad}y$ on $\mathfrak{g}_0$, which implies that $\mathrm{ad}y$ is nilpotent on $\mathfrak{g}_0$, as desired.

Thus $\mathfrak{g}_0$ is abelian. Moreover, for $y, z \in \mathfrak{g}_0$ the operator $\mathrm{ad}y_n \cdot \mathrm{ad}z$ is nilpotent on $\mathfrak{g}$ (as the product of two commuting operators one of which is nilpotent), so $K_\mathfrak{g}(y_n, z) = 0$, which implies that $y_n = 0$, as $K_\mathfrak{g}$ restricts to a nondegenerate form on $\mathfrak{g}_0$ and z is arbitrary. It follows that any $y \in \mathfrak{g}_0$ is semisimple, so $\mathfrak{g}_0$ is a toral subalgebra. Moreover, it is maximal since any element commuting with x is in $\mathfrak{g}_0$. Thus $\mathfrak{g}_0$ is a Cartan subalgebra.

(ii) Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra. By Proposition 20.6 it contains a strongly regular element x, which is automatically semisimple. Then $\mathfrak{h} = C(x)$. □

Corollary 20.9. *(i) Any strongly regular element $x \in \mathfrak{g}$ is semisimple.*

(ii) Such x is contained in a unique Cartan subalgebra, namely $\mathfrak{h}_x = C(x)$.

Proof. (i) It is clear that if x is strongly regular then so is x_s. Since $x \in C(x_s)$ and as shown above $C(x_s)$ is a Cartan subalgebra, it follows that x is semisimple.

(ii) Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra containing x. Then $\mathfrak{h} \supset \mathfrak{h}_x$, thus by dimension count $\mathfrak{h} = \mathfrak{h}_x$. □

We note that there is also a useful notion of a **regular element**, which is an $x \in \mathfrak{g}$ for which the **ordinary** (rather than generalized) 0-eigenspace of $\mathrm{ad}x$ (i.e., the centralizer $C(x)$ of x) has dimension $\mathrm{rank}\mathfrak{g}$. Such elements don't have to be semisimple, e.g. the nilpotent Jordan block in $\mathfrak{sl}_n$ is regular. It follows from Proposition 20.9(i) that an element is strongly regular if and only if it is both regular and semisimple. For this reason, from now on we will follow standard terminology and call strongly regular elements **regular semisimple**.

Theorem 20.10. *Any two Cartan subalgebras of a complex semisimple Lie algebra $\mathfrak{g}$ are conjugate. I.e., if $\mathfrak{h}_1, \mathfrak{h}_2 \subset \mathfrak{g}$ are two Cartan subalgebras and G a connected Lie group with Lie algebra $\mathfrak{g}$ then there exists an element $g \in G$ such that $\mathrm{Ad}g \cdot \mathfrak{h}_1 = \mathfrak{h}_2$.*

Proof. By Corollary 20.9(ii), every element $x \in \mathfrak{g}^{\mathrm{sr}}$ is contained in a unique Cartan subalgebra $\mathfrak{h}_x$. Introduce an equivalence relation on $\mathfrak{g}^{\mathrm{sr}}$ by setting $x \sim y$ if $\mathfrak{h}_x$ is conjugate to $\mathfrak{h}_y$. It is clear that if $x, y \in \mathfrak{h}$ are regular elements in a Cartan subalgebra $\mathfrak{h}$ then $\mathfrak{h}_x = \mathfrak{h}_y = \mathfrak{h}$, so for any $g \in G$, $\mathrm{Ad}g \cdot x \sim y$, and any element equivalent to y has this form. So by Lemma 20.7 the equivalence class U_y of y is open. However, by Lemma 20.5, $\mathfrak{g}^{\mathrm{sr}}$ is connected. Thus there is only one equivalence class. Hence any two Cartan subalgebras of the form $\mathfrak{h}_x$ for regular x are conjugate. This implies the result, since by Theorem 20.8 any Cartan subalgebra is of the form $\mathfrak{h}_x$. □

Remark 20.11. The same results and proofs apply over any algebraically closed field $\mathbf{k}$ of characteristic zero if we use the Zariski topology instead of the usual topology of $\mathbb{C}^n$ when working with the notions of a connected, open and dense set.

20.3. **Root systems of classical Lie algebras.**

Example 20.12. Let $\mathfrak{g}$ be the symplectic Lie algebra $\mathfrak{sp}_{2n}(\mathbf{k})$. Thus $\mathfrak{g}$ consists of square matrices A of size $2n$ such that

$$AJ + JA^T = 0$$

where $J = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix}$, with blocks being of size n. So we get $A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$, where b, c are symmetric. A Cartan subalgebra $\mathfrak{h}$ is then spanned by matrices A such that $a = \mathrm{diag}(x_1, ..., x_n)$ and $b = c = 0$. So $\mathfrak{h} \cong \mathbf{k}^n$. In this case we have roots coming from the a-part, which are simply the roots $\mathbf{e}_i - \mathbf{e}_j$ of $\mathfrak{gl}_n \subset \mathfrak{sp}_{2n}$ (defined by the condition that $b = c = 0$) and also the roots coming form the b-part, which are $\mathbf{e}_i + \mathbf{e}_j$ (including $i = j$, when we get $2\mathbf{e}_i$), and the c-part, which gives the negatives of these roots, $-\mathbf{e}_i - \mathbf{e}_j$, including $-2\mathbf{e}_i$.

This is the **root system of type C_n.**

Example 20.13. Let $\mathfrak{g}$ be the orthogonal Lie algebra $\mathfrak{so}_{2n}(\mathbf{k})$, preserving the quadratic form $Q = x_1x_{n+1} + ... + x_nx_{2n}$. Then the story is almost the same. The Lie algebra $\mathfrak{g}$ consists of square matrices A of size $2n$ such that

$$AJ + JA^T = 0$$

where $J = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$, with blocks being of size n. So we get $A = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix}$, where b, c are now skew-symmetric. A Cartan subalgebra $\mathfrak{h}$ is again spanned by matrices A such that $a = \mathrm{diag}(x_1, ..., x_n)$ and $b = c = 0$. So $\mathfrak{h} \cong \mathbf{k}^n$. In this case we again have roots coming from the a-part, which are simply the roots $\mathbf{e}_i - \mathbf{e}_j$ of $\mathfrak{gl}_n \subset \mathfrak{so}_{2n}$ (defined by the condition that $b = c = 0$) and also the roots coming form the b-part, which are $\mathbf{e}_i + \mathbf{e}_j$ (but now excluding $i = j$, so only for $i \neq j$), and the c-part, which gives the negatives of these roots, $-\mathbf{e}_i - \mathbf{e}_j$, $i \neq j$.

This is the **root system of type D_n.**

Example 20.14. Let $\mathfrak{g}$ be the orthogonal Lie algebra $\mathfrak{so}_{2n+1}(\mathbf{k})$, preserving the quadratic form $Q = x_0^2 + x_1 x_{n+1} + ... + x_n x_{2n}$. Then the Lie algebra $\mathfrak{g}$ consists of square matrices A of size $2n + 1$ such that

$$AJ + JA^T = 0$$

where

$$J = \begin{pmatrix} \mathbf{1}_1 & 0 & 0 \\ 0 & 0 & \mathbf{1}_n \\ 0 & \mathbf{1}_n & 0 \end{pmatrix},$$

So we get

$$A = \begin{pmatrix} 0 & u & -u \\ w & a & b \\ -w & c & -a^T \end{pmatrix},$$

where b, c are skew-symmetric. A Cartan subalgebra $\mathfrak{h}$ is spanned by matrices A such that $a = \mathrm{diag}(x_1, ..., x_n)$ and $b = c = 0$, $u = w = 0$. So $\mathfrak{h} \cong \mathbf{k}^n$. In this case we again have roots coming from the a-part, which are simply the roots $\mathbf{e}_i - \mathbf{e}_j$ of $\mathfrak{gl}_n \subset \mathfrak{so}_{2n+1}$ (defined by the condition that $b = c = 0$, $u = w = 0$) and also the roots coming form the b-part, which are $\mathbf{e}_i + \mathbf{e}_j$, $i \neq j$, and the c-part, which gives the negatives of these roots, $-\mathbf{e}_i - \mathbf{e}_j$, $i \neq j$. But we also have the roots coming from the w-part, which are $\mathbf{e}_i$, and from the u part, which are $-\mathbf{e}_i$.

This is the **root system of type B_n.**

21. **Root systems**

21.1. **Abstract root systems.** Let $E \cong \mathbb{R}^r$ be a Euclidean space with a positive definite inner product.

Definition 21.1. An **abstract root system** is a finite set $R \subset E \setminus 0$ satisfying the following axioms:

(R1) R spans E;

(R2) For all $\alpha, \beta \in R$ the number $n_{\alpha\beta} := \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer;

(R3) If $\beta \in R$ then $s_\alpha(\beta) := \beta - n_{\alpha\beta}\alpha \in R$.

Elements of R are called **roots**. The number $r = \dim E$ is called the **rank** of R.

In particular, taking $\beta = \alpha$ in R3 yields that R is centrally symmetric, i.e., $R = -R$. Also note that s_α is the reflection with respect to the hyperplane $(\alpha, x) = 0$, so R3 just says that R is invariant under such reflections.

Note also that if $R \subset E$ is a root system, $\bar{E} \subset E$ a subspace, and $R' = R \cap E'$ then R' is also a root system inside $E' = \mathrm{Span}(R') \subset \bar{E}$.

For a root α the corresponding **coroot** $\alpha^\vee \in E^*$ is defined by the formula $\alpha^\vee(x) = \frac{2(\alpha,x)}{(\alpha,\alpha)}$. Thus $\alpha^\vee(\alpha) = 2$, $n_{\alpha\beta} = \alpha^\vee(\beta)$ and $s_\alpha(\beta) = \beta - \alpha^\vee(\beta)\alpha$.

Definition 21.2. A root system R is **reduced** if for $\alpha, c\alpha \in R$, we have $c = \pm 1$.

Proposition 21.3. *If $\mathfrak{g}$ is a semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra then the corresponding set of roots R is a reduced root system, and $\alpha^\vee = h_\alpha$.*

Proof. This follows immediately from Theorem 19.19. $\square$

Example 21.4. 1. The root system of $\mathfrak{sl}_n$ is called A_{n-1}. In this case, as we have seen in Example 19.14, the roots are $\mathbf{e}_i - \mathbf{e}_j$, and $s_{\mathbf{e}_i - \mathbf{e}_j} = (ij)$, the transposition of the i-th and j-th coordinates.

2. The subset $\{1, 2, -1, -2\}$ of $\mathbb{R}$ is a root system which is not reduced.

Definition 21.5. Let $R_1 \subset E_1, R_2 \subset E_2$ be root systems. An **isomorphism of root systems** $\phi : R_1 \to R_2$ is an isomorphism $\phi : E_1 \to E_2$ which maps R_1 to R_2 and preserves the numbers $n_{\alpha\beta}$.

So an isomorphism does not have to preserve the inner product, e.g. it may rescale it.

21.2. **The Weyl group.**

Definition 21.6. The **Weyl group** of a root system R is the group of automorphisms of E generated by s_α.

Proposition 21.7. *W is a finite subgroup of $O(E)$ which preserves R.*

Proof. Since s_α are orthogonal reflections, $W \subset O(E)$. By R3, s_α preserves R. By R1 an element of W is determined by its action on R, hence W is finite. $\square$

Example 21.8. For the root system A_{n-1}, $W = S_n$, the symmetric group. Note that for $n \geq 3$, the automorphism $x \mapsto -x$ of R is not in W, so W is, in general, a proper subgroup of $\mathrm{Aut}(R)$.

21.3. **Root systems of rank 2.** If α, β are linearly independent roots in R and $E' \subset E$ is spanned by α, β then $R' = R \cap E'$ is a root system in E' of rank 2. So to classify reduced root systems, it is important to classify reduced root systems of rank 2 first.

Theorem 21.9. *Let R be a reduced root system and $\alpha, \beta \in R$ be two linearly independent roots with $|\alpha| \geq |\beta|$. Let ϕ be the angle between α and β. Then we have one of the following possibilities:*
(1) $\phi = \pi/2$, $n_{\alpha\beta} = n_{\beta\alpha} = 0$;
(2a) $\phi = 2\pi/3$, $|\alpha|^2 = |\beta|^2$, $n_{\alpha\beta} = n_{\beta\alpha} = -1$;
(2b) $\phi = \pi/3$, $|\alpha|^2 = |\beta|^2$, $n_{\alpha\beta} = n_{\beta\alpha} = 1$;
(3a) $\phi = 3\pi/4$, $|\alpha|^2 = 2|\beta|^2$, $n_{\alpha\beta} = -1$, $n_{\beta\alpha} = -2$;
(3b) $\phi = \pi/4$, $|\alpha|^2 = 2|\beta|^2$, $n_{\alpha\beta} = 1$, $n_{\beta\alpha} = 2$;
(4a) $\phi = 5\pi/6$, $|\alpha|^2 = 3|\beta|^2$, $n_{\alpha\beta} = -1$, $n_{\beta\alpha} = -3$;
(4b) $\phi = \pi/6$, $|\alpha|^2 = 3|\beta|^2$, $n_{\alpha\beta} = 1$, $n_{\beta\alpha} = 3$.

Proof. We have $(\alpha, \beta) = 2|\alpha| \cdot |\beta| \cos\phi$, so $n_{\alpha\beta} = 2\frac{|\beta|}{|\alpha|}\cos\phi$. Thus $n_{\alpha\beta}n_{\beta\alpha} = 4\cos^2\phi$. Hence this number can only take values $0, 1, 2, 3$ (as it is an integer by R2) and $\frac{n_{\alpha\beta}}{n_{\beta\alpha}} = \frac{|\alpha|^2}{|\beta|^2}$ if $n_{\alpha\beta} \neq 0$. The rest is obtained by analysis of each case. $\square$

In fact, all these possibilities are realized. Namely, we have root systems $A_1 \times A_1$, A_2, $B_2 = C_2$ (the root system of the Lie algebras $\mathfrak{sp}_4$ and $\mathfrak{so}_5$, which are in fact isomorphic, consisting of the vertices and midpoints of edges of a square), and G_2, generated by α, β with $(\alpha, \alpha) = 6$, $(\beta, \beta) = 2$, $(\alpha, \beta) = -3$, and roots being $\pm\alpha, \pm\beta$, $\pm(\alpha+\beta)$, $\pm(\alpha + 2\beta)$, $\pm(\alpha + 3\beta)$, $\pm(2\alpha + 3\beta)$.

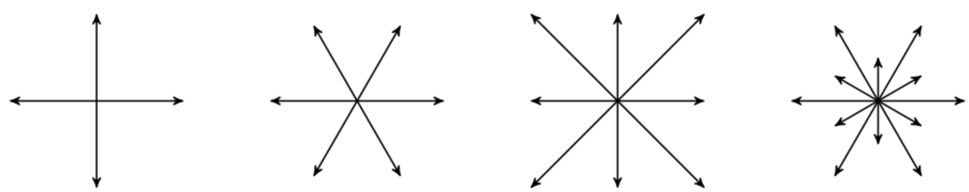
Theorem 21.10. *Any reduced rank 2 root system R is of the form $A_1 \times A_1$, A_2, B_2 or G_2.*

Proof. Pick independent roots $\alpha, \beta \in R$ such that the angle ϕ is as large as possible. Then $\phi \geq \pi/2$ (otherwise can replace α with $-\alpha$), so we are in one of the cases $1, 2a, 3a, 4a$. Now the statement follows by inspection of each case, giving $A_1 \times A_1$, A_2, B_2 and G_2 respectively. $\square$

Corollary 21.11. *If $\alpha, \beta \in R$ are independent roots with $(\alpha, \beta) < 0$ then $\alpha + \beta \in R$.*

Proof. This is easy to see from the classification of rank 2 root systems. $\square$

The root systems of rank 2 are shown in the following picture.



$A_1 \times A_1$ $\quad$ A_2 $\quad$ $B_2 \cong C_2$ $\quad$ G_2

21.4. **Positive and simple roots.** Let R be a reduced root system and $t \in E^*$ be such that $t(\alpha) \neq 0$ for any $\alpha \in R$. We say that a root is **positive** (with respect to t) if $t(\alpha) > 0$ and **negative** if $t(\alpha) < 0$. The set of positive roots is denoted by R_+ and of negative ones by R_-, so $R_+ = -R_-$ and $R = R_+ \cup R_-$ (disjoint union). This decomposition is called a **polarization** of R; it depends on the choice of t.

Example 21.12. Let R be of type A_{n-1}. Then for $t = (t_1, ..., t_n)$ we have $t(\alpha) \neq 0$ for all α iff $t_i \neq t_j$ for any i, j. E.g. suppose $t_1 > t_2 > ... > t_n$, then we have $\mathbf{e}_i - \mathbf{e}_j \in R_+$ iff $i < j$. We see that polarizations are in bijection with permutations in S_n, i.e., with elements of the Weyl group, which acts simply transitively on them. We will see that this is, in fact, the case for any reduced root system.

Definition 21.13. A root $\alpha \in R_+$ is **simple** if it is not a sum of two other positive roots.

Lemma 21.14. *Every positive root is a sum of simple roots.*

Proof. If α is not simple then $\alpha = \beta + \gamma$ where $\beta, \gamma \in R_+$. We have $t(\alpha) = t(\beta) + t(\gamma)$, so $t(\beta), t(\gamma) < t(\alpha)$. If β or γ is not simple, we can continue this process, and it will terminate since t has finitely many values on R. $\square$

Lemma 21.15. *If $\alpha, \beta \in R_+$ are simple roots then $(\alpha, \beta) \leq 0$.*

Proof. Assume $(\alpha, \beta) > 0$. Then $(-\alpha, \beta) < 0$ so by Lemma 21.11 $\gamma := \beta - \alpha$ is a root. If γ is positive then $\beta = \alpha + \gamma$ is not simple. If γ is negative then $-\gamma$ is positive so $\alpha = \beta + (-\gamma)$ is not simple. $\square$

Theorem 21.16. *The set $\Pi \subset R_+$ of simple roots is a basis of E.*

Proof. We will use the following linear algebra lemma:

Lemma 21.17. *Let v_i be vectors in a Euclidean space E such that $(v_i, v_j) \leq 0$ when $i \neq j$ and $t(v_i) > 0$ for some $t \in E^*$. Then v_i are linearly independent.*

Proof. Suppose we have a nontrivial relation

$$\sum_{i \in I} c_i v_i = \sum_{i \in J} c_i v_i$$

where I, J are disjoint and $c_i > 0$ (clearly, every nontrivial relation can be written in this form). Evaluating t on this relation, we deduce that both sides are nonzero. Now let us compute the square of the left hand side:

$$0 < |\sum_{i \in I} c_i v_i|^2 = (\sum_{i \in I} c_i v_i, \sum_{j \in J} c_j v_j) \leq 0.$$

This is a contradiction. $\square$

Now the result follows from Lemma 21.15 and Lemma 21.17. $\square$

Thus the set Π of simple roots has r elements: $\Pi = (\alpha_1, ..., \alpha_r)$.

Example 21.18. Let us describe simple roots for classical root systems. Suppose the polarization is given by $t = (t_1, ..., t_n)$ with decreasing coordinates. Then:

1. For type A_{n-1}, i.e., $\mathfrak{g} = \mathfrak{sl}_n$, the simple roots are $\alpha_i := \mathbf{e}_i - \mathbf{e}_{i+1}$, $1 \leq i \leq n-1$.

2. For type C_n, i.e., $\mathfrak{g} = \mathfrak{sp}_{2n}$, the simple roots are

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ..., \ \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n, \ \alpha_n = 2\mathbf{e}_n.$$

3. For type B_n, i.e., $\mathfrak{g} = \mathfrak{so}_{2n+1}$, we have the same story as for C_n except $\alpha_n = \mathbf{e}_n$ rather than $2\mathbf{e}_n$. Thus the simple roots are

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ..., \ \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n, \ \alpha_n = \mathbf{e}_n.$$

4. For type D_n, i.e., $\mathfrak{g} = \mathfrak{so}_{2n}$, the simple roots are

$$\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ..., \ \alpha_{n-2} = \mathbf{e}_{n-2} - \mathbf{e}_{n-1}, \ \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n, \ \alpha_n = \mathbf{e}_{n-1} + \mathbf{e}_n.$$

We thus obtain

Corollary 21.19. *Any root $\alpha \in R$ can be uniquely written as $\alpha = \sum_{i=1}^{r} n_i\alpha_i$, where $n_i \in \mathbb{Z}$. If α is positive then $n_i \geq 0$ for all i and if α is negative then $n_i \leq 0$ for all i.*

For a positive root α, its **height** $h(\alpha)$ is the number $\sum n_i$. So simple roots are the roots of height 1, and the height of $\mathbf{e}_i - \mathbf{e}_j$ in $R = A_{n-1}$ is $j - i$.

21.5. **Dual root system.** For a root system R, the set $R^\vee \subset E^*$ of $\alpha^\vee$ for all $\alpha \in R$ is also a root system, such that $(R^\vee)^\vee = R$. It is called the **dual root system** to R. For example, B_n is dual to C_n, while A_{n-1}, D_n and G_2 are selfdual.

Moreover, it is easy to see that any polarization of R gives rise to a polarization of $R^\vee$ (using the image $t^\vee$ of t under the isomorphism $E \to E^*$ induced by the inner product), and the corresponding system $\Pi^\vee$ of simple roots consists of $\alpha_i^\vee$ for $\alpha_i \in \Pi$.

21.6. **Root and weight lattices.** Recall that a **lattice** in a real vector space E is a subgroup $Q \subset E$ generated by a basis of E. Of course, every lattice is conjugate to $\mathbb{Z}^n \subset \mathbb{R}^n$ by an element of $GL_n(\mathbb{R})$. Also recall that for a lattice $Q \subset E$ the **dual lattice** $Q^* \subset E^*$ is the set of $f \in E^*$ such that $f(v) \in \mathbb{Z}$ for all $v \in Q$. If Q is generated by a basis $\mathbf{e}_i$ of E then Q^* is generated by the dual basis $\mathbf{e}_i^*$.

In particular, for a root system R we can define the **root lattice** $Q \subset E$, which is generated by the simple roots α_i with respect to some polarization of R. Since Q is also generated by all roots in R, it is independent on the choice of the polarization. Similarly, we can define the **coroot lattice** $Q^\vee \subset E^*$ generated by $\alpha^\vee, \alpha \in R$, which is just the root lattice of $R^\vee$.

Also we define the **weight lattice** $P \subset E$ to be the dual lattice to $Q^\vee$: $P = (Q^\vee)^*$, and the **coweight lattice** $P^\vee \subset E^*$ to be the dual lattice to Q: $P^\vee = Q^*$, so $P^\vee$ is the weight lattice of $R^\vee$. Thus

$$P = \{\lambda \in E : (\lambda, \alpha^\vee) \in \mathbb{Z}\, \forall \alpha \in R\},\ P^\vee = \{\lambda \in E^* : (\lambda, \alpha) \in \mathbb{Z}\, \forall \alpha \in R\}.$$

Since for $\alpha, \beta \in R$ we have $(\alpha^\vee, \beta) = n_{\alpha\beta} \in \mathbb{Z}$, we have $Q \subset P$, $Q^\vee \subset P^\vee$.

Given a system of simple roots $\Pi = \{\alpha_1, ..., \alpha_r\}$, we define **fundamental coweights** $\omega_i^\vee$ to be the dual basis to α_i and **fundamental weights** ω_i to be the dual basis to $\alpha_i^\vee$: $(\omega_i, \alpha_j^\vee) = (\omega_i^\vee, \alpha_j) = \delta_{ij}$. Thus P is generated by ω_i and $P^\vee$ by $\omega_i^\vee$.

Example 21.20. Let R be of type A_1. Then $(\alpha, \alpha^\vee) = 2$ for the unique positive root α, so $\omega = \frac{1}{2}\alpha$, thus $P/Q = \mathbb{Z}/2$. More generally, if R is of type A_{n-1} and we identify $Q \cong Q^\vee, P \cong P^\vee$, then P becomes the set of

$\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$ such that $\sum_i \lambda_i = 0$ and $\lambda_i - \lambda_j = \mathbb{Z}$. So we have a homomorphism $\phi : P \to \mathbb{R}/\mathbb{Z}$ given by $\phi(\lambda) = \lambda_i \bmod \mathbb{Z}$ (for any i). Since $\sum_i \lambda_i = 0$, we have $\phi : P \to \mathbb{Z}/n$, and $\mathrm{Ker}\phi = Q$ (integer vectors with sum zero). Also it is easy to see that ϕ is surjective (we may take $\lambda_i = \frac{k}{n}$ for $i \neq n$ and $\lambda_n = \frac{k}{n} - k$, then $\phi(\lambda) = \frac{k}{n}$). Thus $P/Q \cong \mathbb{Z}/n$.

22. **Properties of the Weyl group**

22.1. **Weyl chambers.** Suppose we have two polarizations of a root system R defined by $t, t' \in E$, and Π, Π' are the corresponding systems of simple roots. Are Π, Π' equivalent in a suitable sense? The answer turns out to be yes. To show this, we will need the notion of a Weyl chamber.

Note that the polarization defined by t depends only on the signs of (t, α), so does not change when t is continuously deformed without crossing the hyperplanes $(t, \alpha) = 0$. This motivates the following definition:

Definition 22.1. A **Weyl chamber** is a connected component of the complement of the root hyperplanes L_α given by the equations $(\alpha, x) = 0$ in E ($\alpha \in R$).

Thus a Weyl chamber is defined by a system of strict homogeneous linear inequalities $\pm(\alpha, x) = 0$, $\alpha \in R$. More precisely, the set of solutions of such a system is either empty of a Weyl chamber.

Thus the polarization defined by t depends only on the Weyl chamber containing t.

The following lemma is geometrically obvious.

Lemma 22.2. *(i) The closure $\overline{C}$ of a Weyl chamber C is a convex cone.*

(ii) The boundary of $\overline{C}$ is a union of codimension 1 faces F_i which are convex cones inside one of the root hyperplanes defined inside it by a system of non-strict homogeneous linear inequalities.

The root hyperplanes containing the faces F_i are called the **walls** of C.

We have seen above that every Weyl chamber defines a polarization of R. Conversely, every polarization defines the corresponding **positive Weyl chamber** C_+ defined by the conditions $(\alpha, x) > 0$ for $\alpha \in R_+$ (this set is nonempty since it contains t, hence is a Weyl chamber). Thus C_+ is the set of vectors of the form $\sum_{i=1}^r c_i\omega_i$ with $c_i > 0$. So C_+ has r faces $L_{\alpha_1} \cap C_+, ..., L_{\alpha_r} \cap C_+$.

Lemma 22.3. *These assignments are mutually inverse bijections between polarizations of R and Weyl chambers.*

Exercise 22.4. Prove Lemma 22.3.

Since the Weyl group W permutes the roots, it acts on the set of Weyl chambers.

Theorem 22.5. *W acts transitively on the set of Weyl chambers.*

Proof. Let us say that Weyl chambers C, C' are **adjacent** if they share a common face $F \subset L_\alpha$. In this case it is easy to see that $s_\alpha(C) = C'$. Now given any Weyl chambers C, C', pick generic $t \in C, t' \in C'$ and connect them with a straight segment. This will define a sequence of Weyl chambers visited by this segment: $C_0 = C, C_1, ..., C_m = C'$, and C_i, C_{i+1} are adjacent for each i. So C_i, C_{i+1} lie in the same W-orbit. Hence so do C, C'. □

Corollary 22.6. *Every Weyl chamber has r walls.*

Proof. This follows since it is true for the positive Weyl chamber and by Theorem 22.5 the Weyl group acts transitively on the Weyl chambers. □

Corollary 22.7. *Any two polarizations of R are related by the action of an element $w \in W$. Thus if Π, Π' are systems of simple roots corresponding to two polarizations then there is $w \in W$ such that $w(\Pi) = \Pi'$.*

22.2. **Simple reflections.** Given a polarization of R and the corresponding system of simple roots $\Pi = \{\alpha_1, ..., \alpha_r\}$, the **simple reflections** are the reflections s_{α_i}, denoted by s_i.

Lemma 22.8. *For every Weyl chamber C there exist $i_1, ..., i_m$ such that $C = s_{i_1}...s_{i_m}(C_+)$.*

Proof. Pick $t \in C, t_+ \in C_+$ generically and connect them with a straight segment as before. Let m be the number of chamber walls crossed by this segment. The proof is by induction in m (with obvious base). Let C' be the chamber entered by our segment from C and L_α the wall separating C, C', so that $C = s_\alpha(C')$. By the induction assumption $C' = u(C_+)$, where $u = s_{i_1}...s_{i_{m-1}}$. So $L_\alpha = u(L_{\alpha_j})$ for some j. Thus $s_\alpha = us_ju^{-1}$. Hence $C = s_\alpha(C') = s_\alpha u(C_+) = us_j(C_+)$, so we get the result with $i_m = j$. □

Corollary 22.9. *(i) The simple reflections s_i generate W;*
(ii) $W(\Pi) = R$.

Proof. (i) This follows since for any root α, the hyperplane L_α is a wall of some Weyl chamber, so s_α is a product of s_i.
(ii) Follows from (i). □

Thus R can be reconstructed from Π as $W(\Pi)$, where W is the subgroup of $O(E)$ generated by s_i.

Example 22.10. For root system A_{n-1} part (i) says that any element of S_n is a product of transpositions of neighbors.

22.3. **Length of an element of the Weyl group.** Let us say that a root hyperplane L_α **separates** two Weyl chambers C, C' if they lie on different sides of L_α.

Definition 22.11. The **length** $\ell(w)$ of $w \in W$ is the number of root hyperplanes separating C_+ and $w(C_+)$.

We have $t \in C_+, w(t) \in w(C_+)$, so $\ell(w)$ is the number of roots α such that $(t, \alpha) > 0$ but $(w(t), \alpha) = (t, w^{-1}\alpha) < 0$. Note that if α is a root satisfying this condition then $\beta = -w^{-1}\alpha$ satisfies the conditions $(t, \beta) > 0$, $(t, w\beta) < 0$. Thus $\ell(w) = \ell(w^{-1})$ and $\ell(w)$ is the number of positive roots which are mapped by w to negative roots. Note also that the notion of length depends on the polarization of R (as it refers to the positive chamber C_+ defined using the polarization).

Example 22.12. Let s_i be a simple reflection. Then $s_i(C_+)$ is adjacent to C_+, with the only separating hyperplane being L_{α_i}. Thus $\ell(s_i) = 1$. It follows that the only positive root mapped by s_i to a negative root is α_i (namely, $s_i(\alpha_i) = -\alpha_i$), and thus s_i permutes $R_+ \setminus \{\alpha_i\}$.

Proposition 22.13. *Let $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$. Then $(\rho, \alpha_i^\vee) = 1$ for all i. Thus $\rho = \sum_{i=1}^r \omega_i$.*

Proof. We have $\rho = \frac{1}{2}\alpha_i + \frac{1}{2}\sum_{\alpha \in R_+, \alpha \neq \alpha_i} \alpha$. Since s_i permutes $R_+ \setminus \{\alpha_i\}$, we get $s_i\rho = \rho - \alpha_i$. But for any λ, $s_i\lambda = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$. This implies the statement. $\square$

The weight ρ plays an important role in representation theory of semisimple Lie algebras. For instance, it occurs in the Weyl character formula for these representations which we will soon derive.

Theorem 22.14. *Let $w = s_{i_1}...s_{i_l}$ be a representation of $w \in W$ as a product of simple reflections that has minimal possible length. Then $l = \ell(w)$.*

Proof. As before, define a chain of Weyl chambers $C_k = s_{i_1}...s_{i_k}(C_+)$, so that $C_0 = C_+$ and $C_k = w(C_+)$. We have seen that C_k and C_{k-1} are adjacent. So there is a zigzag path from C_+ to $w(C_+)$ that intersects at most l root hyperplanes (namely, the segment from C_{k-1} to C_k intersects only one hyperplane). Thus $\ell(w) \leq l$. On the other hand, pick generic points in C_+ and $w(C_+)$ and connect them with a straight segment. This segment intersects every separating root hyperplane exactly once and does not intersect other root hyperplanes, so produces an expression of w as a product of $\ell(w)$ simple reflections. This implies the statement. $\square$

An expression $w = s_{i_1}...s_{i_l}$ is called **reduced** if $l = \ell(w)$.

Proposition 22.15. *The Weyl group W acts simply transitively on Weyl chambers.*

Proof. By Theorem 22.5 the action is transitive, so we just have to show that if $w(C_+) = C_+$ then $w = 1$. But in this case $\ell(w) = 0$, so w has to be a product of zero simple reflections, i.e., indeed $w = 1$. $\square$

Thus we see that $\overline{C}_+$ is a *fundamental domain* of the action of W on E.

Moreover, we have

Proposition 22.16. *$E/W = \overline{C}_+$, i.e., every W-orbit on E has a unique representative in $\overline{C}_+$.*

Proof. Suppose $\lambda, \mu \in \overline{C}_+$ and $\lambda = w\mu$, where $w \in W$ is shortest possible. Assume the contrary, that $w \neq 1$. Pick a reduced decomposition $w = s_{i_l}...s_{i_1}$. Let γ be the positive root which is mapped to a negative root by w but not by $s_{i_l}w$, i.e., $\gamma = s_{i_1}...s_{i_{l-1}}\alpha_{i_l}$. Then $0 \leq (\mu, \gamma) = (\lambda, w\gamma) \leq 0$. so $(\mu, \gamma) = 0$. Thus

$$\lambda = w\mu = s_{i_l}...s_{i_1}\mu = s_{i_{l-1}}...s_{i_1}s_\gamma\mu = s_{i_{l-1}}...s_{i_1}\mu$$

which is a contradiction since w was the shortest possible. $\square$

Corollary 22.17. *Let $C_- = -C_+$ be the* **negative Weyl chamber***. Then there exists a unique $w_0 \in W$ such that $w_0(C_+) = C_-$. We have $\ell(w_0) = |R_+|$ and for any $w \neq w_0$, $\ell(w) < \ell(w_0)$. Also $w_0^2 = 1$.*

Exercise 22.18. Prove Corollary 22.17.

The element w_0 is therefore called the **longest element** of W.

Example 22.19. For the root system A_{n-1} the element w_0 is the order reversing involution: $w_0(1, 2, ..., n) = (n, ..., 2, 1)$.

23. **Dynkin diagrams**

23.1. **Cartan matrices and Dynkin diagrams.** Our goal now is to classify reduced root systems, which is a key step in the classification of semisimple Lie algebras. We have shown that classifying root systems is equivalent to classifying sets Π of simple roots. So we need to classify such sets Π. Before doing so, note that we have a nice notion of **direct product** of root systems.

Namely, let $R_1 \subset E_1$ and $R_2 \subset E_2$ be two root systems. Let $E = E_1 \oplus E_2$ (orthogonal decomposition) and $R = R_1 \sqcup R_2$ (with $R_1 \perp R_2$). If $t_1 \in E_1, t_2 \in E_2$ define polarizations of R_1, R_2 with systems of simple roots Π_1, Π_2 then $t = t_1 + t_2$ defines a polarization of R with $\Pi = \Pi_1 \sqcup \Pi_2$ (with $\Pi_1 \perp \Pi_2$ and $\Pi_i = \Pi \cap R_i$).

Definition 23.1. A root system R is **irreducible** if it cannot be written (nontrivially) in this way.

Lemma 23.2. *If R is a root system with system of simple roots $\Pi = \Pi_1 \sqcup \Pi_2$ with $\Pi_1 \perp \Pi_2$ then $R = R_1 \sqcup R_2$ where R_i is the root system generated by Π_i.*

Proof. If $\alpha \in \Pi_1, \beta \in \Pi_2$ then $s_\alpha(\beta) = \beta$, $s_\beta(\alpha) = \alpha$ and s_α and s_β commute. So if W_i is the group generated by $s_\alpha, \alpha \in \Pi_i$ then $W = W_1 \times W_2$, with W_1 acting trivially on Π_2 and W_2 on Π_1. Thus

$$R = W(\Pi) = W(\Pi_1 \sqcup \Pi_2) = W_1(\Pi_1) \sqcup W_2(\Pi_2) = R_1 \sqcup R_2.$$

□

Proposition 23.3. *Any root system is uniquely a union of irreducible ones.*

Proof. The decomposition is given by the maximal decomposition of Π into mutually orthogonal systems of simple roots. □

Thus it suffices to classify irreducible root systems.

As noted above, a root system is determined by pairwise inner products of positive roots. However, it is more convenient to encode them by the **Cartan matrix** A defined by

$$a_{ij} = n_{\alpha_j \alpha_i} = (\alpha_i^\vee, \alpha_j).$$

The following properties of the Cartan matrix follow immediately from Lemma 21.15, Theorem 21.9 and Theorem 21.16:

Proposition 23.4. *(i) $a_{ii} = 2$;*

(ii) a_{ij} is a nonpositive integer;

(iii) for any $i \neq j$, $a_{ij} a_{ij} = 4\cos^2 \phi \in \{0, 1, 2, 3\}$, where ϕ is the angle between α_i and α_j;

(iv) Let $d_i = |\alpha_i|^2$. Then the matrix $d_i a_{ij}$ is symmetric and positive definite.

We will see later that conversely, any such matrix defines a root system.

Example 23.5. 1. Type A_{n-1}: $a_{ii} = 2, a_{i,i+1} = a_{i+1,i} = -1$, $a_{ij} = 0$ otherwise.

2. Type B_n: $a_{ii} = 2$, $a_{i,i+1} = a_{i+1,i} = -1$ except that $a_{n,n-1} = -2$.

3. Type C_n: transposed to B_n.

4. Type D_n: same as B_n but $a_{n-1,n-2} = a_{n,n-2} = a_{n-2,n} = a_{n-2,n-1} = -1$, $a_{n,n-1} = a_{n-1,n} = 0$.

5. Type G_2: $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

It is convenient to encode such matrices by **Dynkin diagrams:**
- Indices i are vertices;
- Vertices i and j are connected by $a_{ij}a_{ji}$ lines;
- If $a_{ij} \neq a_{ji}$, i.e., $|\alpha_i|^2 \neq |\alpha_j|^2$, then the arrow on the lines goes from long root to short root ("less than" sign).

It is clear that such diagram completely determines the Cartan matrix (if we fix the labeling of vertices), and vice versa. Also it is clear that the root system is irreducible if and only if its Dynkin diagram is connected.

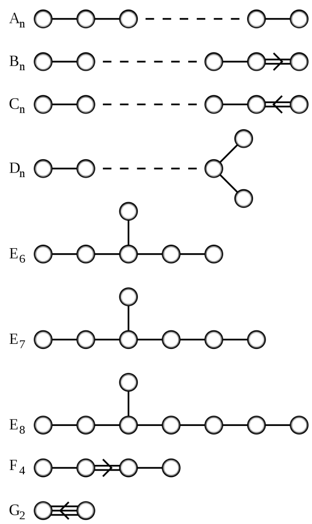
Proposition 23.6. *The Cartan matrix determines the root system uniquely.*

Proof. We may assume the Dynkin diagram is connected. The Cartan matrix determines, for any pair of simple roots, the angle between them (which is right or obtuse) and the ratio of their lengths if they are not orthogonal. By the classification of rank 2 root systems, this determines the inner product on simple roots, up to scaling, which implies the statement. □

23.2. **Classification of Dynkin diagrams.** The following theorem gives a complete classification of irreducible root systems.

Theorem 23.7. *(i) Connected Dynkin diagrams are classified by the list given i in the picture below, i.e., they are A_n, B_n, C_n, D_n, G_2 which we have already met, along with four more: F_4, E_6, E_7, E_8.*

(ii) Every matrix satisfying the conditions of Proposition 23.4 is a Cartan matrix of some root system.



The proof of Theorem 23.7 is rather long but direct. It consists of several steps. The first step is construction of the remaining root systems F_4, E_6, E_7, E_8.

23.3. **The root system F_4.**

Definition 23.8. The root system F_4 is the union of the root system $B_4 \subset \mathbb{R}^4$ with the vectors

$$\left(\pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}, \pm\tfrac{1}{2}\right) = \sum_{i=1}^{4}(\pm\tfrac{1}{2}\mathbf{e}_i),$$

for all choices of signs.

Thus besides the roots of B_4, which are $\pm\mathbf{e}_i \pm \mathbf{e}_j$ (24 of them, squared length 2) and $\pm\mathbf{e}_i$ (8 of them, squared length 1), we have the 16 new roots $\sum_{i=1}^{4}(\pm\frac{1}{2}\mathbf{e}_i)$ (squared length 1); this gives a total of 48.

Exercise 23.9. Check that this is an irreducible root system.

To give a polarization of the F_4 root system, pick $t = (t_1, t_2, t_3, t_4)$ with $t_1 \gg t_2 \gg t_3 \gg t_4$.

Exercise 23.10. Check that for this polarization, the simple positive roots are, $\alpha_1 = \frac{1}{2}(\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4)$, $\alpha_2 = \mathbf{e}_4$, $\alpha_3 = \mathbf{e}_3 - \mathbf{e}_4$, $\alpha_4 = \mathbf{e}_2 - \mathbf{e}_3$. Thus $\alpha_1^\vee = \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3 - \mathbf{e}_4$, $\alpha_2^\vee = 2\mathbf{e}_4$, $\alpha_3^\vee = \mathbf{e}_3 - \mathbf{e}_4$, $\alpha_4^\vee = \mathbf{e}_2 - \mathbf{e}_3$. So the Cartan matrix has the form

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

which gives the Dynkin diagram of F_4.

23.4. **The root system E_8.**

Definition 23.11. The root system E_8 is the union of the root system $D_8 \subset \mathbb{R}^8$ with the vectors $\sum_{i=1}^{8}(\pm\frac{1}{2}\mathbf{e}_i)$, for all choices of signs with even number of minuses.

Thus besides the roots of D_8, $\pm\mathbf{e}_i \pm \mathbf{e}_j$ (112 of them), we have 128 new roots $\sum_{i=1}^{8}(\pm\frac{1}{2}\mathbf{e}_i)$. So in total we have 240 roots. All roots have squared length 2.

Exercise 23.12. Show that it is an irreducible root system.

To give a polarization of the E_8 root system, pick t so that $t_i \gg t_{i+1}$.

Exercise 23.13. Check that for this polarization, the simple positive roots are, $\alpha_1 = \frac{1}{2}(\mathbf{e}_1+\mathbf{e}_8-\sum_{i=2}^{7}\mathbf{e}_i)$, $\alpha_2 = \mathbf{e}_7+\mathbf{e}_8$ and $\alpha_i = \mathbf{e}_{10-i}-\mathbf{e}_{11-i}$ for $3 \le i \le 8$. Thus the roots $\alpha_2, ..., \alpha_8$ generate the root system D_7, while $a_{13} = -1$ and $a_{1i} = 0$ for all $i \neq 1, 3$. In other words, the Cartan matrix has the form

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

This recovers the Dynkin diagram E_8.

23.5. **The root system E_7.**

Definition 23.14. The root system E_7 is the subsystem of E_8 generated by $\alpha_1, ..., \alpha_7$.

Note that these roots (unlike $\alpha_8 = \mathbf{e}_2 - \mathbf{e}_3$) satisfy the equation $x_1 + x_2 = 0$. Thus E_7 is the intersection of E_8 with this subspace. So it includes the roots $\pm\mathbf{e}_i \pm \mathbf{e}_j$ with $3 \le i, j \le 8$ distinct (60 roots), $\pm(\mathbf{e}_1 - \mathbf{e}_2)$ (2 roots) and $\sum_{i=1}^{8}(\pm\frac{1}{2}\mathbf{e}_i)$ with even number of minuses and the opposite signs for $\mathbf{e}_1$ and $\mathbf{e}_2$ (64 roots). Altogether we get 126 roots. The Cartan matrix is the upper left corner 7 by 7 submatrix of the Cartan matrix of E_8, so it is

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

23.6. **The root system E_6.**

Definition 23.15. The root system E_6 is the subsystem of E_8 and E_7 generated by $\alpha_1, ..., \alpha_6$.

Note that these roots (unlike $\alpha_8 = \mathbf{e}_2 - \mathbf{e}_3$ and $\alpha_7 = \mathbf{e}_3 - \mathbf{e}_4$) satisfy the equations $x_1 + x_2 = 0, x_2 - x_3 = 0$. Thus E_6 is the intersection of E_8 with this subspace. So it includes the roots $\pm\mathbf{e}_i \pm \mathbf{e}_j$ with $4 \le i, j \le 8$ distinct (40 roots), and $\sum_{i=1}^{8}(\pm\frac{1}{2}\mathbf{e}_i)$ with even number of minuses and the opposite signs for $\mathbf{e}_1$ and $\mathbf{e}_2$ and for $\mathbf{e}_1$ and $\mathbf{e}_3$ (32 roots). Altogether we get 72 roots. The Cartan matrix is the upper left corner 6 by 6 submatrix of the Cartan matrix of E_8, so it is

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

This recovers the Dynkin diagram E_6.

23.7. **The elements ρ and $\rho^\vee$.** Recall that the elements $\rho \in \mathfrak{h}^*$ and $\rho^\vee \in \mathfrak{h}$ for a simple Lie algebra $\mathfrak{g}$ are defined by the conditions $(\rho, \alpha_i^\vee) = (\rho^\vee, \alpha_i) = 1$ for all i. So for classical Lie algebras they can be computed from Example 21.18. Namely, we get

$$\rho_{A_{n-1}} = \rho^\vee_{A_{n-1}} = (\tfrac{n-1}{2}, \tfrac{n-3}{2}, ..., -\tfrac{n-1}{2}),$$
$$\rho_{B_n} = \rho^\vee_{C_n} = (\tfrac{2n-1}{2}, ..., \tfrac{3}{2}, \tfrac{1}{2}),$$
$$\rho_{C_n} = \rho^\vee_{B_n} = (n, n-1, ..., 1),$$
$$\rho_{D_n} = \rho^\vee_{D_n} = (n-1, n-2, ..., 0).$$

Exercise 23.16. Show that the elements ρ and $\rho^\vee$ for exceptional root systems (in the above realizations) are as follows:

$$\rho_{G_2} = 3\alpha + 5\beta,\ \rho^\vee_{G_2} = 5\alpha^\vee + 3\beta^\vee,$$

$$\rho_{F_4} = (\tfrac{11}{2}, \tfrac{5}{2}, \tfrac{3}{2}, \tfrac{1}{2}),\ \rho^\vee_{F_4} = (8,3,2,1),$$
$$\rho_{E_8} = \rho^\vee_{E_8} = (23,6,5,4,3,2,1,0),$$
$$\rho_{E_7} = \rho^\vee_{E_7} = (\tfrac{17}{2}, -\tfrac{17}{2}, 5,4,3,2,1,0),$$
$$\rho_{E_6} = \rho^\vee_{E_6} = (4,-4,-4,4,3,2,1,0).$$
(recall that we realized E_6, E_7, E_8 inside $\mathbb{R}^8$).

23.8. **Proof of Theorem 23.7.** Now that we have shown that there exist root systems attached to all Cartan matrices, it remains to classify Cartan matrices (or Dynkin diagrams), i.e. show that there are no others than those we have considered. For this purpose we consider Dynkin diagrams as graphs with certain kind of special edges (with one, two or three lines and a possible orientation). Note first that any subgraph of a Dynkin diagram must itself be a Dynkin diagram, since a principal submatrix of a positive definite symmetric matrix is itself positive definite. On the other hand, consider **untwisted and twisted affine Dynkin diagrams** depicted on the first picture at `https://en.wikipedia.org/wiki/Affine_Lie_algebra`. These are not Dynkin diagrams since the corresponding matrix A is degenerate, hence not positive definite.

Exercise 23.17. Prove this by showing that in each case there exists a nonzero vector v such that $Av = 0$. For example, in the simply laced case (only simple edges), this amounts to finding a labeling of the vertices by nonzero numbers such that the sum of labels of the neighbors to each vertex is twice the label of that vertex, and in the non-simply laced case it's a weighted version of that.

Thus they cannot occur inside a Dynkin diagram.

We conclude that a Dynkin diagram is a tree. Indeed, it cannot have a loop with simple edges, since this is the affine diagram $\widetilde{A}_{n-1}$, which has a null vector $(1,...,1)$. If there is a loop with non-simple edges, this is even worse - this vector will have a negative inner product with itself.

Further, it cannot have vertices with more than four simple edges coming out since it cannot have a subdiagram $\widetilde{D}_4$ (and for non-simple edges it is even worse, as before). Thus all the vertices of our tree are i-valent for $i \le 3$.

Also we cannot have a subdiagram $\widetilde{D}_n$, $n \ge 5$, which implies that there is at most one trivalent vertex.

Further, if there is a triple edge then the diagram is G_2. There is no way to attach any edge to the G_2 diagram because $D_4^{(3)}$ and $\widetilde{G}_2$ are forbidden.

Next, if there is a trivalent vertex then there cannot be a non-simple edge anywhere in the diagram (as we have forbidden affine diagrams $A^{(2)}_{2k-1}, \widetilde{B}_n$). So in this case the diagram is simply laced, so it must be on our list (D_n, E_6, E_7, E_8) since it cannot contain affine diagrams $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$.

It remains to consider chain-shaped diagrams. They can't contain two double edges (affine diagrams $A^{(2)}_{2k}, D^{(2)}_{k+1}, \widetilde{C}_n$). Thus if the double edge is at the end, we can only get B_n and C_n.

Finally, if the double edge is in the middle, we can't have affine subdiagram $\widetilde{F}_4$ or $E^{(2)}_6$, so our diagram must be F_4. Theorem 23.7 is proved.

Remark 23.18. Note that we have **exceptional isomorphisms** $D_2 \cong A_1 \times A_1$, $D_3 \cong A_3$, $B_2 \cong C_2$. Otherwise the listed root systems are distinct.

23.9. **Simply laced and non-simply laced diagrams.** As we already mentioned, a Dynkin diagram (or the corresponding root system) is called **simply laced** if all the edges are simple, i.e. $a_{ij} = 0, -1$ for $i \neq j$. This is equivalent to the Cartan matrix being symmetric, or to all roots having the same length. The connected simply-laced diagrams are $A_n, n \geq 1; D_n, n \geq 4; E_6, E_7, E_8$. The remaining diagrams B_n, C_n, F_4, G_2 are not simply laced, but they contain roots of only two squared lengths, whose ratio is 2 for double edge (B_n, C_n, F_4) and 3 for triple edge (G_2). The roots of the bigger length are called **long** and of the smaller length are called **short**.

It is easy to see that long and short roots form a root system of the same rank (but not necessarily irreducible). For instance, in G_2 both form a root system of type A_2, and in B_2 both are $A_1 \times A_1$. In B_3 long roots form D_3 and short ones form $A_1 \times A_1 \times A_1$. However, only long roots form a root subsystem, since a long positive root can be the sum of two short ones, but not vice versa.

24. **Construction of a semisimple Lie algebra from a Dynkin diagram**

24.1. **Serre relations.** Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. We would like to show that any reduced root system gives rise to a semisimple Lie algebra over $\mathbf{k}$, and moreover a unique one. To this end, it suffices to show that any reduced *irreducible* root system gives rise to a unique (finite dimensional) *simple* Lie algebra.

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbf{k}$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and root system $R \subset \mathfrak{h}^*$ (which is thus reduced and irreducible). Fix a polarization of R with the set of simple roots $\Pi = (\alpha_1, ..., \alpha_r)$, and let $A = (a_{ij})$ be the Cartan matrix of R. We have a decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{n}_\pm := \oplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$ are the Lie subalgebras spanned by positive, respectively negative root vectors. Pick elements $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ so that $e_i, f_i, h_i = [e_i, f_i]$ form an $\mathfrak{sl}_2$-triple.

Theorem 24.1. *(Serre relations) (i) The elements e_i, f_i, h_i, $i = 1, ..., r$ generate $\mathfrak{g}$.*

(ii) These elements satisfy the following relations:

$$[h_i, h_j] = 0,\ [h_i, e_j] = a_{ij} e_j,\ [h_i, f_j] = -a_{ij} f_j,\ [e_i, f_j] = \delta_{ij} h_i,$$

$$(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0,\ (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0,\ i \neq j.$$

The last two sets of relations are called **Serre relations**. Note that if $a_{ij} = 0$ then the Serre relations just say that $[e_i, e_j] = [f_i, f_j] = 0$.

Proof. (i) We know that h_i form a basis of $\mathfrak{h}$, so it suffices to show that e_i generate $\mathfrak{n}_+$ and f_i generate $\mathfrak{n}_-$. We only prove the first statement, the second being the same for the opposite polarization.

Let $\mathfrak{n}_+' \subset \mathfrak{n}_+$ be the Lie subalgebra generated by e_i. It is clear that $\mathfrak{n}_+' = \oplus_{\alpha \in R_+'} \mathfrak{g}_\alpha$ where $R_+' \subset R_+$. Assume the contrary, that $R_+' \neq R_+$. Pick $\alpha \in R_+ \setminus R_+'$ with the smallest height (it is not a simple root). Then $\mathfrak{g}_{\alpha - \alpha_i} \subset \mathfrak{n}_+'$, so $[e_i, \mathfrak{g}_{\alpha-\alpha_i}] = 0$. Let $x \in \mathfrak{g}_{-\alpha}$ be a nonzero element. We have

$$([x, e_i], y) = (x, [e_i, y]) = 0$$

for any $y \in \mathfrak{g}_{\alpha - \alpha_i}$. Thus $[x, e_i] = 0$ for all i, which implies, by the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4), that $(\alpha, \alpha_i^\vee) \leq 0$ for all i, hence $(\alpha, \alpha_i) \leq 0$ for all i. This would imply that $(\alpha, \alpha) \leq 0$, a contradiction. This proves (i).

(ii) All the relations except the Serre relations follow from the definition and properties of root systems. So only the Serre relations require proof. We prove only the relation involving f_i, the other one being the same for the opposite polarization. Consider the $(\mathfrak{sl}_2)_i$-submodule M_{ij} of $\mathfrak{g}$ generated by f_j. It is finite dimensional and we have $[h_i, f_j] = -a_{ij} f_j$, $[f_i, f_j] = 0$. Thus by the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4) we must have $M_{ij} \cong V_{-a_{ij}}$. Hence $(\mathrm{ad} f_i)^{-a_{ij}+1} f_j = 0$. $\square$

24.2. **The Serre presentation for semisimple Lie algebras.** Now for any reduced root system R let $\mathfrak{g}(R)$ be the Lie algebra generated by $e_i, f_i, h_i, i = 1, ..., r$, with **defining relations** being the relations of Theorem 24.1. Precisely, this means that $\mathfrak{g}(R)$ is the quotient of the free Lie algebra FL_{3r} with generators e_i, f_i, h_i modulo the Lie ideal generated by the differences of the left and right hand sides of these relations.

Theorem 24.2. *(Serre) (i) The Lie subalgebra $\mathfrak{n}_+$ of $\mathfrak{g}(R)$ generated by e_i has the Serre relations $(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0$ as the defining relations. Similarly, the Lie subalgebra $\mathfrak{n}_-$ of $\mathfrak{g}(R)$ generated by f_i has the Serre relations $(\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0$ as the defining relations. In particular, $e_i, f_i \neq 0$ in $\mathfrak{g}(R)$. Moreover, h_i are linearly independent.*

(ii) $\mathfrak{g}(R)$ is a sum of finite dimensional modules over every simple root subalgebra $(\mathfrak{sl}_2)_i = (e_i, f_i, h_i)$.

(iii) $\mathfrak{g}(R)$ is finite dimensional.

(iv) $\mathfrak{g}(R)$ is semisimple and has root system R.

Proof. It is easy to see that $\mathfrak{g}(R_1 \sqcup R_2) = \mathfrak{g}(R_1) \oplus \mathfrak{g}(R_2)$, so it suffices to prove the theorem for irreducible root systems.

(i) Consider the (in general, infinite dimensional) Lie algebra $\widetilde{\mathfrak{g}(R)}$ generated by e_i, f_i, h_i with the defining relations of Theorem 24.1 without the Serre relations. This Lie algebra is $\mathbb{Z}$-graded, with $\deg(e_i) = 1$, $\deg(f_i) = -1$, $\deg(h_i) = 0$. Thus we have a decomposition

$$\widetilde{\mathfrak{g}(R)} = \widetilde{\mathfrak{n}_+} \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}_-},$$

where $\widetilde{\mathfrak{n}_+}$, $\widetilde{\mathfrak{h}}$ and $\widetilde{\mathfrak{n}_-}$ are Lie subalgebras spanned by elements of positive, zero and negative degree, respectively. Moreover, it is easy to see that $\widetilde{\mathfrak{n}_+}$ is generated by e_i, $\widetilde{\mathfrak{n}_-}$ is generated by f_i, and $\widetilde{\mathfrak{h}}$ is spanned by h_i (indeed, any commutator can be simplified to have only e_i, only f_i, or only a single h_i).

Lemma 24.3. *(i) The Lie algebra $\widetilde{\mathfrak{n}_+}$ is free on the generators e_i and $\widetilde{\mathfrak{n}_-}$ is free on the generators f_i.*

(ii) h_i are linearly independent in $\widetilde{\mathfrak{h}}$ (i.e., $\widetilde{\mathfrak{h}} \cong \mathfrak{h}$).

Proof. (i) We prove only the first statement, the second being the same for the opposite polarization. Let $\mathfrak{h}'$ be a vector space with basis h_i',

$i = 1, ..., r$ and consider the Lie algebra $\mathfrak{a} := \mathfrak{h}' \ltimes FL_r$, where FL_r is freely generated by $f_1', ..., f_r'$ and

$$[h_i', f_j'] = -a_{ij} f_j',\ [h_i', h_j'] = 0.$$

Consider the universal enveloping algebra

$$U = U(\mathfrak{a}) = \mathbf{k}[h_1', ..., h_r'] \ltimes \mathbf{k}\langle f_1', ..., f_r' \rangle,$$

which as a vector space is naturally identified with the tensor product $\mathbf{k}\langle f_1, ..., f_r \rangle \otimes \mathbf{k}[h_1', ..., h_r']$, via $f \otimes h \mapsto fh$ (by Proposition 14.4). Now define an action of $\widetilde{\mathfrak{g}(R)}$ on the space U as follows. For $P \in \mathbf{k}[h_1', ..., h_r']$ and w a word in f_i' of weight $-\alpha$, we set

$$h_i(w \otimes P) = w \otimes (h_i' - \alpha(h_i))P,\ f_i(w \otimes P) = f_i' w \otimes P,$$

$$e_i(f_{j_1} ... f_{j_s} \otimes P) = \sum_{k: j_k = i} f_{j_1}' .... \widehat{f_{j_k}'} ... f_{j_s}' \otimes (h_{j_k}' - (\alpha_{j_{k+1}} + ... + \alpha_{j_s})(h_i'))P$$

(where the hat means that the corresponding factor is omitted). It is easy to check that this indeed defines an action, i.e., the relations of $\widetilde{\mathfrak{g}(R)}$ are satisfied (check it!). Thus we have a linear map $\widetilde{\mathfrak{g}(R)} \to U$ given by $x \mapsto x(1)$. The restriction of this map to the Lie subalgebra $\widetilde{\mathfrak{n}_-}$ is a map $\phi : \widetilde{\mathfrak{n}}_- \to FL_r$ which sends every iterated commutator of f_i to itself. This implies that ϕ is an isomorphism, i.e., $\widetilde{\mathfrak{n}_-}$ is free.

(ii) The elements $h_i(1) = h_i'$ are linearly independent, hence so are h_i. □

Now consider the element $S_{ij}^+ := (\mathrm{ad} e_i)^{1-a_{ij}} e_j$ in $\widetilde{\mathfrak{n}_+}$ and $S_{ij}^- := (\mathrm{ad} f_i)^{1-a_{ij}} f_j$ in $\widetilde{\mathfrak{n}_-}$. It is easy to check that $[f_k, S_{ij}^+] = 0$ (this follows easily from the representation theory of $\mathfrak{sl}_2$, Subsection 11.4,–check it!). Therefore, setting I_+ to be the ideal in the Lie algebra $\widetilde{\mathfrak{n}_+}$ generated by S_{ij}^+, and I_- to be the ideal in the Lie algebra $\widetilde{\mathfrak{n}_-}$ generated by S_{ij}^-, we see that the ideal of Serre relations in $\widetilde{\mathfrak{g}(R)}$ is $I_+ \oplus I_-$. Lemma 24.3 now implies (i).

(ii) The Serre relations imply that e_j generates the representation $V_{-a_{ij}}$ of $(\mathfrak{sl}_2)_i$ for $j \neq i$, and so does f_j. Also any element of $\mathfrak{h}$ generates V_0 or V_2 or the sum of the two, and e_i, f_i generate V_2. This implies (ii) since $\mathfrak{g}(R)$ is generated by e_i, f_i, h_i, and if x generates a representation X of $(\mathfrak{sl}_2)_i$ and y generates a representation Y then $[x, y]$ generates a quotient of $X \otimes Y$.

(iii) We have $\mathfrak{g}(R) = \oplus_{\alpha \in Q} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ are the subspaces of $\mathfrak{g}(R)$ of weight α, and $\mathfrak{g}_0 = \mathfrak{h}$. Let Q_+ be the $\mathbb{Z}_+$-span of α_i. Then $\mathfrak{g}_\alpha$ is zero unless $\alpha \in Q_+$ or $-\alpha \in Q_+$, and is finite dimensional for any α.

We will now show that if $\mathfrak{g}_\alpha \neq 0$ then $\alpha \in R$ or $\alpha = 0$, which implies (iii). It suffices to consider $\alpha \in Q_+$. We prove the statement by induction in the height $\mathrm{ht}(\alpha) = \sum_i k_i$ where $\alpha = \sum_i k_i\alpha_i$. The base case (height 1) is obvious, so we only need to justify the inductive step. We have $(\alpha, \omega_i^\vee) = k_i \geq 0$ for all i. If there is only one i with $k_i \geq 0$ then the statement is clear since $\mathfrak{g}_{m\alpha_i} = 0$ if $m \geq 2$. (as $\mathfrak{n}_+$ is generated by e_i). So assume that there are at least two such indices i. Since $(\alpha, \alpha) > 0$, there exists i such that $(\alpha, \alpha_i^\vee) > 0$. By the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4), $\mathfrak{g}_{s_i\alpha} \neq 0$. Clearly, $s_i\alpha = \alpha - (\alpha, \alpha_i^\vee)\alpha_i \notin -Q_+$ (since $k_j > 0$ for at least two indices j), so $s_i\alpha \in Q_+$ but has height smaller than α (as $(\alpha, \alpha_i^\vee) > 0$). So by the induction assumption $s_i\alpha \in R$, which implies $\alpha \in R$. This proves (iii).

(iv) We see that $\mathfrak{g}(R) = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ are 1-dimensional (this follows from (ii),(iii) since every root can be mapped to a simple root by a composition of simple reflections). Let I be a nonzero ideal in $\mathfrak{g}$. Then $I \supset \mathfrak{g}_\alpha$ for some $\alpha \neq 0$. Also, by the representation theory of $\mathfrak{sl}_2$, $I_\beta \neq 0$ implies $I_{w\beta} \neq 0$ for all $w \in W$. Thus $I_{\alpha_i} \neq 0$ for some i, i.e., $e_i \in I$. Hence $h_i, f_i \in I$. Now let J be the set of indices j for which $e_j, f_j, h_j \in I$ (or, equivalently, just $e_j \in I$); we have shown it is nonempty. Since $[h_j, e_k] = a_{jk}e_k$, we find that if $j \in J$ and $a_{jk} \neq 0$ (i.e., k is connected to j in the Dynkin diagram) then $k \in J$. Since the Dynkin diagram is connected, $J = [1, ..., r]$ and $I = \mathfrak{g}$. Thus $\mathfrak{g}$ is simple and clearly has root system R. This proves (iv) and completes the proof of Serre's theorem. $\square$

Corollary 24.4. *Isomorphism classes of simple Lie algebras over* $\mathbf{k}$ *are in bijection with Dynkin diagrams* A_n, $n \geq 1$, B_n, $n \geq 2$, C_n, $n \geq 3$, D_n, $n \geq 4$, E_6, E_7, E_8, F_4 *and* G_2.

25. **Representation theory of semisimple Lie algebras**

25.1. **Representations of semisimple Lie algebras.** We will now develop representation theory of complex semisimple Lie algebras. The representation theory of semisimple Lie algebras over an algebraically closed field of characteristic zero is completely parallel, so we will stick to the complex case. So all representations will be over $\mathbb{C}$. We will mostly be interested in finite dimensional representations; as we know, they can be exponentiated to holomorphic representations of the corresponding simply connected Lie group G, which defines a bijection between isomorphism classes of such representations of $\mathfrak{g}$ and G.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Recall that by Theorem 18.9, every finite dimensional representation of $\mathfrak{g}$ is completely reducible, so to classify finite dimensional representations it suffices to classify irreducible representations.

As in the simplest case of $\mathfrak{sl}_2$, a crucial tool is the decomposition of a representation in a direct sum of eigenspaces of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$.

Definition 25.1. Let $\lambda \in \mathfrak{h}^*$, and V a representation of $\mathfrak{g}$ (possibly infinite dimensional). Then a vector $v \in V$ is said to have **weight** λ if $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$; such vectors are called **weight vectors**. The subspace of such vectors is called the **weight subspace of** V **of weight** λ and denoted by $V[\lambda]$. If $V[\lambda] \neq 0$, we say that λ is a weight of V, and the set of weights of V is denoted by $P(V)$.

It is easy to see that $\mathfrak{g}_\alpha V[\lambda] \subset V[\lambda + \alpha]$.

Let $V' \subset V$ be the span of all weight vectors in V. Then it is clear that $V' = \oplus_{\lambda \in \mathfrak{h}} V[\lambda]$.

Definition 25.2. We say that V **has a weight decomposition** (with respect to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$) if $V' = V$, i.e., if $V = \oplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$.

Note that not every representation of $\mathfrak{g}$ has a weight decomposition (e.g., for $V = U(\mathfrak{g})$ with $\mathfrak{g}$ acting by left multiplication all weight subspaces are zero).

Proposition 25.3. *Any finite dimensional representation V of $\mathfrak{g}$ has a weight decomposition. Moreover, all weights of V are* **integral**, *i.e., $P(V)$ is a finite subset of the weight lattice $P \subset \mathfrak{h}^*$ of $\mathfrak{g}$.*

Proof. For each $i = 1, ..., r$, V is a finite dimensional representation of the root subalgebra $(\mathfrak{sl}_2)_i$, so its element h_i acts semisimply on V. Thus $\mathfrak{h}$ acts semisimply on V, hence V has a weight decomposition. Also eigenvalues of h_i are integers, so for any $\lambda \in P(V)$ we have $\lambda(h_i) = (\lambda, \alpha_i^\vee) \in \mathbb{Z}$, hence $\lambda \in P$. □

Definition 25.4. A vector v in $V[\lambda]$ is called a **highest weight vector of weight** λ if $e_i v = 0$ for all i, i.e., if $\mathfrak{n}_+ v = 0$. A representation V of $\mathfrak{g}$ is a **highest weight representation with highest weight** λ if it is generated by such a nonzero vector.

Proposition 25.5. *Any finite dimensional representation $V \neq 0$ contains a nonzero highest weight vector of some weight λ. Thus every irreducible finite dimensional representation of $\mathfrak{g}$ is a highest weight representation.*

Proof. Note that $P(V)$ is a finite set. Let $\rho^\vee = \sum_{i=1}^r \omega_i^\vee$. Pick $\lambda \in P(V)$ so that $(\lambda, \rho^\vee)$ is maximal. Then $\lambda + \alpha_i \notin P(V)$ for any i, since $(\lambda+\alpha_i, \rho^\vee) = (\lambda, \rho^\vee)+1$. Hence for any nonzero $v \in V[\lambda]$ (which exists as $\lambda \in P(V)$) we have $e_i v = 0$.

The second statement follows since an irreducible representation is generated by each its nonzero vector. $\square$

25.2. **Verma modules.** Even though we are mostly interested in finite dimensional representations of $\mathfrak{g}$, it is useful to consider some infinite dimensional representations, which are called **Verma modules**.

The Verma module M_λ is defined as "the largest highest weight representation with highest weight λ". Namely, it is generated by a single highest weight vector v_λ with **defining relations** $hv = \lambda(h)v$ for $h \in \mathfrak{h}$ and $e_i v = 0$. More formally speaking, we make the following definition.

Definition 25.6. Let $I_\lambda \subset U(\mathfrak{g})$ be the left ideal generated by the elements $h - \lambda(h), h \in \mathfrak{h}$ and e_i, $i = 1, ..., r$. Then the **Verma module** M_λ is the quotient $U(\mathfrak{g})/I_\lambda$.

In this realization, the highest weight vector v_λ is just the class of the unit 1 of $U(\mathfrak{g})$.

Proposition 25.7. *The map $\phi : U(\mathfrak{n}_-) \to M_\lambda$ given by $\phi(x) = xv_\lambda$ is an isomorphism of left $U(\mathfrak{n}_-)$-modules.*

Proof. By the PBW theorem, the multiplication map

$$\xi : U(\mathfrak{n}_-) \otimes U(\mathfrak{h} \oplus \mathfrak{n}_+) \to U(\mathfrak{g})$$

is a linear isomorphism. It is easy to see that $\xi^{-1}(I_\lambda) = U(\mathfrak{n}_-) \otimes K_\lambda$, where

$$K_\lambda := \sum_i U(\mathfrak{h} \oplus \mathfrak{n}_+)(h_i - \lambda(h_i)) + \sum_i U(\mathfrak{h} \oplus \mathfrak{n}_+)e_i$$

is the kernel of the homomorphism $\lambda_+ : U(\mathfrak{h} \oplus \mathfrak{n}_+) \to \mathbb{C}$ given by $\lambda_+(h) = \lambda(h)$, $h \in \mathfrak{h}$, $\lambda_+(e_i) = 0$. Thus, we have a natural isomorphism of left $U(\mathfrak{n}_-)$-modules

$$U(\mathfrak{n}_-) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h} \oplus \mathfrak{n}_+)/K_\lambda \to M_\lambda,$$

as claimed. $\square$

Remark 25.8. The definition of M_λ means that it is the **induced module** $U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h} \oplus \mathfrak{n}_+$ on which it acts via λ_+.

Recall that Q_+ denotes the set of elements $\sum_{i=1}^r k_i\alpha_i$ where $k_i \in \mathbb{Z}_{\geq 0}$. We obtain

Corollary 25.9. *M_λ has a weight decomposition with $P(M_\lambda) = \lambda - Q_+$, $\dim M_\lambda[\lambda] = 1$, and weight subspaces of M_λ are finite dimensional.*

Proposition 25.10. *(i) (Universal property of Verma modules) If V is a representation of $\mathfrak{g}$ and $v \in V$ is a vector such that $hv = \lambda(h)v$ for $h \in h$ and $e_iv = 0$ for $1 \leq i \leq r$ then there is a unique homomorphism $\eta : M_\lambda \to V$ such that $\eta(v_\lambda) = v$. In particular, if V is generated by such $v \neq 0$ (i.e., V is a highest weight representation with highest weight vector v) then V is a quotient of M_λ.*

(ii) Every highest weight representation has a weight decomposition into finite dimensional weight subspaces.

Proof. (i) Uniqueness follows from the fact that v_λ generates M_λ. To construct η, note that we have a natural homomorphism of $\mathfrak{g}$-modules $\widetilde{\eta} : U(\mathfrak{g}) \to V$ given by $\widetilde{\eta}(x) = xv$. Moreover, $\widetilde{\eta}|_{I_\lambda} = 0$ thanks to the relations satisfied by v, so $\widetilde{\eta}$ descends to a map $\eta : U(\mathfrak{g})/I_\lambda = M_\lambda \to V$. Moreover, if V is generated by v then this map is surjective, as desired.

(ii) This follows from (i) since a quotient of any representation with a weight decomposition must itself have a weight decomposition. $\square$

Corollary 25.11. *Every highest weight representation V has a unique highest weight generator, up to scaling.*

Proof. Suppose v, w are two highest weight generators of V of weights λ, μ. If $\lambda = \mu$ then they are proportional since $\dim V[\lambda] \leq \dim M_\lambda[\lambda] = 1$, as V is a quotient of M_λ. On the other hand, if $\lambda \neq \mu$, then we can assume without loss of generality that $\lambda - \mu \notin Q_+$ (otherwise switch λ, μ). Then $\mu \notin \lambda - Q_+$, hence $\mu \notin P(V)$, a contradiction. $\square$

Proposition 25.12. *For every $\lambda \in \mathfrak{h}^*$, the Verma module M_λ has a unique irreducible quotient L_λ. Moreover, L_λ is a quotient of every highest weight $\mathfrak{g}$-module V with highest weight λ.*

Proof. Let $Y \subset M_\lambda$ be a proper submodule. Then Y has a weight decomposition, and cannot contain a nonzero multiple of v_λ (as otherwise

$Y = M_\lambda$), so $P(Y) \subset (\lambda - Q_+) \setminus \{\lambda\}$. Now let J_λ be the sum of all proper submodules $Y \subset M_\lambda$. Then $P(J_\lambda) \subset (\lambda - Q_+) \setminus \{\lambda\}$, so J_λ is also a proper submodule of M_λ (the maximal one). Thus, $L_\lambda := M_\lambda / J_\lambda$ is an irreducible highest weight module with highest weight λ. Moreover, if V is any nonzero quotient of M_λ then the kernel K of the map $M_\lambda \to V$ is a proper submodule, hence contained in J_λ. Thus the surjective map $M_\lambda \to L_\lambda$ descends to a surjective map $V \to L_\lambda$. The kernel of this map is a proper submodule of V, hence zero if V is irreducible. Thus in the latter case $V \cong L_\lambda$. □

Corollary 25.13. *Irreducible highest weight $\mathfrak{g}$-modules are classified by their highest weight $\lambda \in \mathfrak{h}^*$, via the bijection $\lambda \mapsto L_\lambda$.*

25.3. **Finite dimensional modules.** Since every finite dimensional irreducible $\mathfrak{g}$-module is highest weight, it is of the form L_λ for λ belonging to some subset $P_F \subset P$, the set of weights λ such that L_λ is finite dimensional. So to obtain a final classification of finite dimensional irreducible representations of $\mathfrak{g}$, we should determine the subset P_F.

Let $P_+ \subset P$ be the intersection of P with the closure of the dominant Weyl chamber C_+; i.e., P_+ is the set of nonnegative integer linear combinations of the fundamental weights ω_i. In other words, P_+ is the set of $\lambda \in P$ such that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}_+$ for $1 \le i \le r$. Weights belonging to P_+ are called **dominant integral**.

Proposition 25.14. *We have $P_F \subset P_+$.*

Proof. The vector v_λ is highest weight for $(\mathfrak{sl}_2)_i$ with highest weight $\lambda(h_i) = (\lambda, \alpha_i^\vee)$. This must be a nonnegative integer for the corresponding $\mathfrak{sl}_2$-module to be finite dimensional. □

Lemma 25.15. *If $\lambda \in P_+$ then in L_λ, we have $f_i^{\lambda(h_i)+1} v_\lambda = 0$.*

Proof. By the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4), we have $e_i f_i^{\lambda(h_i)+1} v_\lambda = 0$. Also $e_j f_i^{\lambda(h_i)+1} v_\lambda = 0$ for $j \neq i$ since $[e_j, f_i] = 0$. Thus, $w := f_i^{\lambda(h_i)+1} v_\lambda$ is a highest weight vector in L_λ. So w cannot be a generator (as the highest weight generator is unique up to scaling). Thus w generates a proper submodule in L_λ, which must be zero since L_λ is irreducible. □

Lemma 25.16. *Let V be a $\mathfrak{g}$-module with weight decomposition into finite dimensional weight subspaces. If V is a sum of finite dimensional $(\mathfrak{sl}_2)_i$-modules for each $i = 1, ..., r$, then for each $\lambda \in P$ and $w \in W$, $\dim V[\lambda] = \dim V[w\lambda]$. In particular, $P(V)$ is W-invariant.*

Proof. Since the Weyl group W is generated by the simple reflections s_i, it suffices to prove the statement for $w = s_i$, and in fact to prove that $\dim V[\lambda] \le \dim V[s_i\lambda]$ (as $s_i^2 = 1$).

If $(\lambda, \alpha_i^\vee) = m \ge 0$ then consider the operator $f_i^m : V[\lambda] \to V[s_i\lambda]$. We claim that this operator is injective, which implies the desired inequality. Indeed, let $v \in V[\lambda]$ be a nonzero vector and E be the representation of $(\mathfrak{sl}_2)_i$ generated by v. Then E is finite dimensional, and $v \in E[m]$, so by the representation theory of $\mathfrak{sl}_2$ (Subsection 11.4), $f_i^m v \neq 0$, as claimed.

Similarly, if $(\lambda, \alpha_i^\vee) = -m \le 0$ then the operator $e_i^m : V[\lambda] \to V[s_i\lambda]$ is injective. This proves the lemma. □

Now we are ready to state the main classification theorem.

Theorem 25.17. *For any $\lambda \in P_+$, L_λ is finite dimensional; i.e., $P_F = P_+$. Thus finite dimensional irreducible representations of $\mathfrak{g}$ are classified, up to an isomorphism, by their highest weight $\lambda \in P_+$, via the bijection $\lambda \mapsto L_\lambda$. Moreover, for any $\mu \in P$ and $w \in W$,* $\dim L_\lambda[\mu] = \dim L_\lambda[w\mu]$.

Proof. Since $f_i^{\lambda(h_i)+1} v_\lambda = 0$, we see that v_λ generates the irreducible finite dimensional $(\mathfrak{sl}_2)_i$-module of highest weight $\lambda(h_i)$. Also, every nonzero element of $\mathfrak{g}$ generates a finite dimensional $(\mathfrak{sl}_2)_i$-module. Hence every vector in L_λ generates a finite dimensional $(\mathfrak{sl}_2)_i$-module. Thus by Lemma 25.16, $P(L_\lambda)$ is W-invariant.

Now let $\mu \in P(L_\lambda) \cap P_+$. Then $\mu = \lambda - \beta$, $\beta \in Q_+$, so

$$(\mu, \rho^\vee) = (\lambda, \rho^\vee) - (\beta, \rho^\vee) \le (\lambda, \rho^\vee).$$

So if $\mu = \sum_i m_i \omega_i$, $m_i \in \mathbb{Z}_+$ then $\sum_i m_i(\omega_i, \rho^\vee) \le (\lambda, \rho^\vee)$. Since $(\omega_i, \rho^\vee) \ge \frac{1}{2}$, this implies that $P(L_\lambda) \cap P_+$ is finite. But we know that $WP_+ = P$, hence $W(P(L_\lambda) \cap P_+) = P(L_\lambda)$, as $P(L_\lambda)$ is W-invariant. It follows that $P(L_\lambda)$ is finite, hence L_λ is finite dimensional. □

Example 25.18. For $\mathfrak{g} = \mathfrak{sl}_2$ the dominant integral weights are positive integers $n \in \mathbb{Z}_{\ge 0}$, and it is easy to see that $L_n = V_n$.

26. **The Weyl character formula**

26.1. **Characters.** Let V be a finite dimensional representation of a semisimple Lie algebra $\mathfrak{g}$. Recall that the action of $\mathfrak{g}$ on V can be exponentiated to the action of the corresponding simply connected complex Lie group G. Recall also that the **character** of a finite dimensional representation V of any group G is the function

$$\chi_V(g) = \mathrm{Tr}|_V(g).$$

Let us compute this character in our case. To this end, let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, $h \in \mathfrak{h}$, and let us compute $\chi_V(e^h)$. Note that this completely determines χ_V since it determines $\chi_V(e^x)$ for any semisimple element $x \in \mathfrak{g}$, and semisimple elements form a dense open set in $\mathfrak{g}$ (complement of zeros of some polynomial). So elements of the form e^x as above form a dense open set at least in some neighborhood of 1 in G, and an analytic function on G is determined by its values on any nonempty open set.

We know that V has a weight decomposition: $V = \oplus_{\mu\in P} V[\mu]$. Thus we have

$$\chi_V(e^h) = \sum_{\mu\in P} \dim V[\mu] e^{\mu(h)}.$$

Consider the group algebra $\mathbb{Z}[P]$. It sits naturally inside the algebra of analytic functions on $\mathfrak{h}$ via $\lambda \mapsto e^\lambda$, where $e^\lambda(h) := e^{\lambda(h)}$, and we see that $\chi_V \in \mathbb{Z}[P]$, namely

$$\chi_V = \sum_{\mu\in P} \dim V[\mu] e^{\mu}.$$

We will call the element χ_V the **character** of V.

26.2. **Category $\mathcal{O}$.** Note that the above definition of character is a purely formal algebraic definition, i.e., χ_V is simply the generating function of dimensions of weight subspaces of V. So it makes sense for any (possibly infinite dimensional) representation V with a weight decomposition into finite dimensional weight subspaces, except we may obtain an infinite sum. More precisely, we make the following definition.

Definition 26.1. The category $\mathcal{O}_{\mathrm{int}}$ is the category of representations V of $\mathfrak{g}$ with weight decomposition into finite dimensional weight spaces $V = \oplus_{\mu\in P} V[\mu]$, such that $P(V)$ is contained in the union of sets $\lambda^i - Q_+$ for a finite collection of weights $\lambda^1, ..., \lambda^N \in P$ (depending on V).[11]

[11] Usually one also adds the condition that V is a finitely generated $U(\mathfrak{g})$-module, but we don't need this condition here, so we won't impose it.

Here the subscript "int" indicates that we consider only integral weights (i.e., ones in P). However, for brevity we will drop this subscript in this section and just denote this category by $\mathcal{O}$.

For example, any highest weight module belongs to $\mathcal{O}$.

Let $\mathcal{R}$ be the ring of series $a := \sum_{\mu\in P} a_\mu e^\mu$ ($a_\mu \in \mathbb{Z}$) such that the set $P(a)$ of μ with $a_\mu \neq 0$ is contained in the union of sets $\lambda^i - Q_+$ for a finite collection of weights $\lambda^1, ..., \lambda^N \in P$. Then for every $V \in \mathcal{O}$ we can define the character $\chi_V \in \mathcal{R}$. Moreover, it is easy to see that if

$$0 \to X \to Y \to Z \to 0$$

is a short exact sequence in $\mathcal{O}$ then $\chi_Y = \chi_X + \chi_Z$, and that for any $V, U \in \mathcal{O}$ we have $V \otimes U \in \mathcal{O}$ and $\chi_{V\otimes U} = \chi_V \chi_U$.

Example 26.2. Let $V = M_\lambda$ be the Verma module. Recall that as a vector space $M_\lambda = U(\mathfrak{n}_-)v_\lambda$, and that $U(\mathfrak{n}_-) = \otimes_{\alpha\in R_+}\mathbb{C}[e_{-\alpha}]$ (using the PBW theorem). Thus

$$\sum_\mu U(\mathfrak{n}_-)[\mu]e^\mu = \frac{1}{\prod_{\alpha\in R_+}(1-e^{-\alpha})}$$

and hence

$$\chi_{M_\lambda} = \frac{e^\lambda}{\prod_{\alpha\in R_+}(1-e^{-\alpha})}.$$

It is convenient to rewrite this formula as follows:

$$\chi_{M_\lambda} = \frac{e^{\lambda+\rho}}{\Delta},\ \Delta := \prod_{\alpha\in R_+}(e^{\alpha/2} - e^{-\alpha/2}).$$

The (trigonometric) polynomial Δ is called the **Weyl denominator**.

Note that we have a homomorphism $\varepsilon : W \to \mathbb{Z}/2$ given by the formula $w \mapsto \det(w|_{\mathfrak{h}})$, i.e. $w \mapsto (-1)^{\ell(w)}$; it is defined on simple reflections by $s_i \mapsto -1$. This homomorphism is called the **sign character**. For example, for type A_{n-1} this is the sign of a permutation in S_n. We will say that an element of $f \in \mathbb{C}[P]$ is **anti-invariant** under W if $w(f) = (-1)^{\ell(w)}f$ for all $w \in W$.

Proposition 26.3. *The Weyl denominator Δ is anti-invariant under W.*

Proof. Since s_i permutes positive roots not equal to α_i and send α_i to $-\alpha_i$, it follows that $s_i\Delta = -\Delta$. $\square$

26.3. **The Weyl character formula.**

Theorem 26.4. *(Weyl character formula) For any $\lambda \in P_+$ the character $\chi_\lambda := \chi_{L_\lambda}$ of the irreducible finite dimensional representation L_λ is given by*

$$\chi_\lambda = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\Delta}.$$

The proof of this theorem is in the next subsection.

Corollary 26.5. *(Weyl denominator formula) One has*

$$\Delta = \sum_{w\in W}(-1)^{\ell(w)}e^{w\rho}.$$

Proof. This follows from the Weyl character formula by setting $\lambda = 0$ (as $L_0 = \mathbb{C}$ is the trivial representation). $\square$

For example, for $\mathfrak{g} = \mathfrak{sl}_n$ Corollary 26.5 reduces to the usual product formula for the Vandermonde determinant.

26.4. **Proof of the Weyl character formula.** Consider the product $\Delta\chi_\lambda \in \mathbb{Z}[P]$. We know that χ_λ is W-invariant, so this product is W-anti-invariant. Thus,

$$\Delta\chi_\lambda = \sum_{\mu\in P} c_\mu e^\mu,$$

where $c_{w\mu} = (-1)^{\ell(w)}c_\mu$. Moreover, $c_\mu = 0$ unless $\mu \in \lambda + \rho - Q_+$, and $c_{\lambda+\rho} = 1$. Thus to prove the Weyl character formula, we need to show that $c_\mu = 0$ if $\mu \in P_+ \cap (\lambda + \rho - Q_+)$ and $\mu \neq \lambda + \rho$.

To this end, we will construct the above decomposition $\Delta\chi_\lambda$ using representation theory, so that this vanishing property is apparent from the construction.

First recall from Subsection 18.3 that we have the Casimir element C of $U(\mathfrak{g})$ given by the formula $C = \sum_i a_i a^i$ for a basis $a_i \in \mathfrak{g}$ with dual basis a^i of $\mathfrak{g}$ under the Killing form. This element is central, so acts by a scalar on every highest weight (in particular, finite dimensional irreducible) representation. We can write C in the form

$$C = \sum_j x_j^2 + \sum_{\alpha\in R_+}(e_{-\alpha}e_\alpha + e_\alpha e_{-\alpha}),$$

for an orthonormal basis x_j of $\mathfrak{h}$. Since $[e_\alpha, e_{-\alpha}] = h_\alpha$, we find that

$$C = \sum_j x_j^2 + 2\sum_{\alpha\in R_+} e_{-\alpha}e_\alpha + \sum_{\alpha\in R_+} h_\alpha.$$

Thus we get

Lemma 26.6. *If V is a highest weight representation with highest weight λ then $C|_V = (\lambda, \lambda + 2\rho) = |\lambda + \rho|^2 - |\rho|^2$.*

Now we will define a sequence of modules $K(b)$ from category $\mathcal{O}$ parametrized by some binary strings b. This is done inductively. We set $K(\emptyset) = L_\lambda$. Now suppose $K(b)$ is already defined. If $K(b) = 0$, we do not define $K(b')$ for any longer string b' starting with b. Otherwise, pick a nonzero vector $v_b \in K(b)$, of some weight $\nu(b) \in \lambda - Q_+$ such that the height of $\lambda - \nu(b)$ is minimal possible. Then v_b is a highest weight vector, and we can consider the corresponding the homomorphism

$$\xi_b : M_{\nu_b} \to K(b).$$

Let $K(b1), K(b0)$ be the kernel and cokernel of ξ_b. We have

$$\chi_{K(b1)} - \chi_{M_{\nu(b)}} + \chi_{K(b)} - \chi_{K(b0)} = 0.$$

Thus we have

$$\chi_{K(b)} = \chi_{M_{\nu(b)}} - \chi_{K(b1)} + \chi_{K(b0)}.$$

It is clear that for every b and μ, there is b' starting with b such that $K_{b'}[\mu] = 0$. So iterating this formula starting with $b = \emptyset$, we will get

$$\chi_\lambda = \sum_b (-1)^{\Sigma(b)} \chi_{M_{\nu(b)}}$$

where $\Sigma(b)$ is the sum of digits of b. So

$$\Delta \chi_\lambda = \sum_b (-1)^{\Sigma(b)} e^{\nu(b)+\rho}.$$

Also note that by induction in the length of b we can conclude that the eigenvalue of C on $M_{\nu(b)}$ is $|\lambda + \rho|^2 - |\rho|^2$ regardless of b, which implies that

$$|\nu(b) + \rho|^2 = |\lambda + \rho|^2$$

for all b.

So it remains to show that if $\mu = \lambda + \rho - \beta \in P_+$ with $\beta \in Q_+$ and $\beta \neq 0$ then $|\mu|^2 < |\lambda + \rho|^2$. Indeed,

$$|\lambda + \rho|^2 - |\mu|^2 = |\lambda + \rho|^2 - |\lambda - \beta + \rho|^2 =$$
$$2(\lambda + \rho, \beta) - |\beta|^2 > (\lambda + \rho, \beta) - |\beta|^2 = (\lambda + \rho - \beta, \beta) \geq 0.$$

This completes the proof of the Weyl character formula.

Exercise 26.7. Let Q be the root lattice of a simple Lie algebra $\mathfrak{g}$, Q_+ its positive part. Define the **Kostant partition function** to be the function $p : Q \to \mathbb{Z}_{\geq 0}$ which attaches to $\beta \in Q_+$ the number of ways to write β as a sum of positive roots of $\mathfrak{g}$ (where the order does not matter), and $p(\beta) = 0$ if $\beta \notin Q_+$.

(i) Show that

$$\sum_{\beta \in Q_+} p(\beta)e^{-\beta} = \frac{1}{\prod_{\alpha \in R_+}(1 - e^{-\alpha})}.$$

(ii) Prove the **Kostant multiplicity formula**

$$\dim L_\lambda[\gamma] = \sum_{w \in W} (-1)^{\ell(w)} p(w(\lambda + \rho) - \rho - \gamma).$$

(iii) Compute $p(k_1\alpha_1 + k_2\alpha_2)$ for $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{sp}_4$.

(iv) Use (iii) to compute explicitly the weight multiplicities of the irreducible representations L_λ for $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{sp}_4$. (You should get a sum of 6, respectively 8 terms, not particularly appealing, but easily computable in each special case).

26.5. **The Weyl dimension formula.** Recall that the Weyl character formula can be written as a trace formula: for $h \in \mathfrak{h}$

$$\chi_\lambda(e^h) = \mathrm{Tr}|_{L_\lambda}(e^h) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{(w(\lambda+\rho),h)}}{\prod_{\alpha \in R_+}(e^{\frac{1}{2}(\alpha,h)} - e^{-\frac{1}{2}(\alpha,h)})}.$$

The dimension of L_λ should be obtained from this formula when $h = 0$. However, we don't not immediately get the answer since this formula gives the character as a ratio of two trigonometric polynomials which both vanish at $h = 0$, giving an indeterminacy. We know the limit exists since the character is a trigonometric polynomial, but we need to compute it. This can be done as follows.

Let us restrict attention to $h = 2th_\rho$ where $t \in \mathbb{R}$ and $h_\rho \in \mathfrak{h}$ corresponds to $\rho \in \mathfrak{h}^*$ using the identification induced by the invariant form. We have

$$\chi_\lambda(e^{2th_\rho}) = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{2t(w(\lambda+\rho),\rho)}}{\prod_{\alpha \in R_+}(e^{t(\alpha,\rho)} - e^{-t(\alpha,\rho)})}.$$

The key idea is that for this specialization the numerator can also be factored using the denominator formula, which will allow us to resolve the indeterminacy. Namely, we have

$$\chi_{L_\lambda}(e^{2th_\rho}) = \frac{\prod_{\alpha \in R_+}(e^{t(\alpha,\lambda+\rho)} - e^{-t(\alpha,\lambda+\rho)})}{\prod_{\alpha \in R_+}(e^{t(\alpha,\rho)} - e^{-t(\alpha,\rho)})}. \tag{26.1}$$

Now sending $t \to 0$, we obtain

Proposition 26.8. *We have*

$$\dim L_\lambda = \frac{\prod_{\alpha \in R_+}(\alpha, \lambda + \rho)}{\prod_{\alpha \in R_+}(\alpha, \rho)}.$$

Note that this number is an integer, but this is not obvious without its intertpretation as the dimension of a representation.

Formula (26.1) has a meaning even before taking the limit. Namely, the eigenvalues of the element $2h_\rho$ define a $\mathbb{Z}$-grading on the representation L_λ called the **principal grading**, and we obtain a product formula for the Poincaré polynomial of this grading.

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