Problems from "An introduction to Lie groups and Lie algebras" by A. Kirillov Jr.

Homework 1

2.1. Let G be a Lie group and H — a closed Lie subgroup.

(1) Let $\overline{H}$ be the closure of H in G. Show that $\overline{H}$ is a subgroup in G.
(2) Show that each coset $Hx, x \in \overline{H}$, is open and dense in $\overline{H}$.
(3) Show that $\overline{H} = H$, that is, every Lie subgroup is closed.

2.2.

(1) Show that every discrete normal subgroup of a connected Lie group is central (hint: consider the map $G \to N\colon g \mapsto ghg^{-1}$ where h is a fixed element in N).
(2) By applying part (a) to kernel of the map $\widetilde{G} \to G$, show that for any connected Lie group G, the fundamental group $\pi_1(G)$ is commutative.

2.3. Let $f\colon G_1 \to G_2$ be a morphism of connected Lie groups such that $f_*\colon T_1G_1 \to T_1G_2$ is an isomorphism (such a morphism is sometimes called *local isomorphism*). Show that f is a covering map, and $\operatorname{Ker} f$ is a discrete central subgroup.

2.4. Let $\mathcal{F}_n(\mathbb{C})$ be the set of all flags in $\mathbb{C}^n$. Show that

$$\mathcal{F}_n(\mathbb{C}) = \mathrm{GL}(n, \mathbb{C})/B(n, \mathbb{C}) = \mathrm{U}(n)/T(n)$$

where $B(n, \mathbb{C})$ is the group of invertible complex upper triangular matrices, and $T(n)$ is the group of diagonal unitary matrices (which is easily shown to be the n-dimensional torus $(\mathbb{R}/\mathbb{Z})^n$). Deduce from this that $\mathcal{F}_n(\mathbb{C})$ is a compact complex manifold and find its dimension over $\mathbb{C}$.

2.5. Let $G_{n,k}$ be the set of all dimension k subspaces in $\mathbb{R}^n$ (usually called the Grassmanian). Show that $G_{n,k}$ is a homogeneous space for the group $\mathrm{O}(n, \mathbb{R})$ and thus can be identified with coset space $\mathrm{O}(n, \mathbb{R})/H$ for appropriate H. Use it to prove that $G_{n,k}$ is a manifold and find its dimension.

2.6. Show that if $G = \mathrm{GL}(n, \mathbb{R}) \subset \mathrm{End}(\mathbb{R}^n)$ so that each tangent space is canonically identified with $\mathrm{End}(\mathbb{R}^n)$, then $(L_g)_*v = gv$ where the product in the right-hand side is the usual product of matrices, and similarly for the right action. Also, the adjoint action is given by $\operatorname{Ad} g(v) = gvg^{-1}$.

Homework 2

2.7. Define a bilinear form on $\mathfrak{su}(2)$ by $(a,b)=\frac{1}{2}\operatorname{tr}(a\bar{b}^t)$. Show that this form is symmetric, positive definite, and invariant under the adjoint action of $\mathrm{SU}(2)$.

2.8. Define a basis in $\mathfrak{su}(2)$ by

$$i\sigma_1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}\qquad i\sigma_2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\qquad i\sigma_3=\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix}$$

Show that the map

$$\varphi\colon \mathrm{SU}(2)\to \mathrm{GL}(3,\mathbb{R}),\ g\mapsto \text{matrix of } \operatorname{Ad} g \text{ in the basis } i\sigma_1, i\sigma_2, i\sigma_3$$

gives a morphism of Lie groups $\mathrm{SU}(2)\to \mathrm{SO}(3,\mathbb{R})$.

2.9. Let $\varphi\colon \mathrm{SU}(2)\to \mathrm{SO}(3,\mathbb{R})$ be the morphism defined in the previous problem. Compute explicitly the map of tangent spaces $\varphi_*\colon \mathfrak{su}(2)\to\mathfrak{so}(3,\mathbb{R})$ and show that φ_* is an isomorphism. Deduce from this that $\operatorname{Ker}\varphi$ is a discrete normal subgroup in $\mathrm{SU}(2)$, and that $\operatorname{Im}\varphi$ is an open subgroup in $\mathrm{SO}(3,\mathbb{R})$.

2.10. Prove that the map φ used in two previous exercises establishes an isomorphism $\mathrm{SU}(2)/\mathbb{Z}_2\to \mathrm{SO}(3,\mathbb{R})$ and thus, since $\mathrm{SU}(2)\simeq S^3$, $\mathrm{SO}(3,\mathbb{R})\simeq \mathbb{RP}^3$.

2.11. Show that for $n\geq 1$, we have $\pi_0(\mathrm{SU}(n+1))=\pi_0(\mathrm{SU}(n))$, $\pi_0(\mathrm{U}(n+1))=\pi_0(\mathrm{U}(n))$ and deduce from it that groups $\mathrm{U}(n)$, $\mathrm{SU}(n)$ are connected for all n. Similarly, show that for $n\geq 2$, we have $\pi_1(\mathrm{SU}(n+1))=\pi_1(\mathrm{SU}(n))$, $\pi_1(\mathrm{U}(n+1))=\pi_1(\mathrm{U}(n))$ and deduce from it that for $n\geq 2$, $\mathrm{SU}(n)$ is simply-connected and $\pi_1(\mathrm{U}(n))=\mathbb{Z}$.

2.12. Show that for $n\geq 2$, we have $\pi_0(\mathrm{SO}(n+1,\mathbb{R}))=\pi_0(\mathrm{SO}(n,\mathbb{R}))$ and deduce from it that groups $\mathrm{SO}(n)$ are connected for all $n\geq 2$. Similarly, show that for $n\geq 3$, $\pi_1(\mathrm{SO}(n+1,\mathbb{R}))=\pi_1(\mathrm{SO}(n,\mathbb{R}))$ and deduce from it that for $n\geq 3$, $\pi_1(\mathrm{SO}(n,\mathbb{R}))=\mathbb{Z}_2$.

2.13. Using Gram-Schmidt orthogonalization process, show that $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n,\mathbb{R})$ is diffeomorphic to the space of upper-triangular matrices with positive entries on the diagonal. Deduce from this that $\mathrm{GL}(n,\mathbb{R})$ is homotopic (as a topological space) to $\mathrm{O}(n,\mathbb{R})$.

2.14. Let L_n be the set of all Lagrangian subspaces in $\mathbb{R}^{2n}$ with the standard symplectic form ω. (A subspace V is Lagrangian if $\dim V=n$ and $\omega(x,y)=0$ for any $x,y\in V$.)

Show that the group $\mathrm{Sp}(n,\mathbb{R})$ acts transitively on L_n and use it to define on L_n a structure of a smooth manifold and find its dimension.

2.15. Let $\mathbb{H}=\{a+bi+cj+dk \mid a,b,c,d\in\mathbb{R}\}$ be the algebra of quaternions, defined by $ij=k=-ji$, $jk=i=-kj$, $ki=j=-ik$, $i^2=j^2=k^2=-1$, and let $\mathbb{H}^n=\{(h_1,\dots,h_n)\mid h_i\in\mathbb{H}\}$. In particular, the subalgebra generated by $1,i$ coincides with the field $\mathbb{C}$ of complex numbers.

Note that $\mathbb{H}^n$ has a structure of both left and right module over $\mathbb{H}$ defined by

$$h(h_1,\dots,h_n)=(hh_1,\dots,hh_n),\qquad (h_1,\dots,h_n)h=(h_1h,\dots,h_nh)$$

(1) Let $\operatorname{End}_{\mathbb{H}}(\mathbb{H}^n)$ be the algebra of endomorphisms of $\mathbb{H}^n$ considered as right $\mathbb{H}$-module:

$$\operatorname{End}_{\mathbb{H}}(\mathbb{H}^n)=\{A\colon\mathbb{H}^n\to\mathbb{H}^n \mid A(\mathbf{h}+\mathbf{h}')=A(\mathbf{h})+A(\mathbf{h}'),\ A(\mathbf{h}h)=A(\mathbf{h})h\}$$

Show that $\operatorname{End}_{\mathbb{H}}(\mathbb{H}^n)$ is naturally identified with the algebra of $n\times n$ matrices with quaternion entries.

(2) Define an $\mathbb{H}$–valued form $(\ ,\ )$ on $\mathbb{H}^n$ by

$$(\mathbf{h},\mathbf{h}')=\sum_i \overline{h_i}h_i'$$

where $\overline{a+bi+cj+dk}=a-bi-cj-dk$. (Note that $\overline{uv}=\overline{v}\overline{u}$.)

Let $\mathrm{U}(n,\mathbb{H})$ be the group of "unitary quaternionic transformations":

$$\mathrm{U}(n,\mathbb{H})=\{A\in\operatorname{End}_{\mathbb{H}}(\mathbb{H}^n)\mid (A\mathbf{h},A\mathbf{h}')=(\mathbf{h},\mathbf{h}')\}.$$

Show that this is indeed a group and that a matrix A is in $\mathrm{U}(n,\mathbb{H})$ iff $A^*A=1$, where $(A^*)_{ij}=\overline{A_{ji}}$.

(3) Define a map $\mathbb{C}^{2n} \simeq \mathbb{H}^n$ by

$$(z_1, \dots, z_{2n}) \mapsto (z_1 + jz_{n+1}, \dots, z_n + jz_{2n})$$

Show that it is an isomorphism of complex vector spaces (if we consider $\mathbb{H}^n$ as a complex vector space by $z(h_1, \dots h_n) = (h_1 z, \dots, h_n z)$) and that this isomorphism identifies

$$\mathrm{End}_{\mathbb{H}}(\mathbb{H}^n) = \{A \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{2n}) \mid \overline{A} = J^{-1}AJ\}$$

where $J := \begin{pmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{pmatrix}$. (Hint: use $jz = \overline{z}j$ for any $z \in \mathbb{C}$ to show that $\mathbf{h} \mapsto \mathbf{h}j$ is identified with $\mathbf{z} \mapsto J\overline{\mathbf{z}}$.)

(4) Show that under identification $\mathbb{C}^{2n} \simeq \mathbb{H}^n$ defined above, the quaternionic form $(\, , \,)$ is identified with

$$(\mathbf{z}, \mathbf{z}') - j\langle \mathbf{z}, \mathbf{z}' \rangle$$

where $(\mathbf{z}, \mathbf{z}') = \sum \overline{z_i} z'_i$ is the standard Hermitian form in $\mathbb{C}^{2n}$ and

$$\langle \mathbf{z}, \mathbf{z}' \rangle = \sum_{i=1}^{n} (z_{i+n} z'_i - z_i z'_{i+n})$$

is the standard bilinear skew-symmetric form in $\mathbb{C}^{2n}$. Deduce from this that the group $\mathrm{U}(n, \mathbb{H})$ is identified with $\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{SU}(2n)$.

2.16.

(1) Show that $\mathrm{Sp}(1) \simeq \mathrm{SU}(2) \simeq S^3$.

(2) Using the previous exercise, show that we have a natural transitive action of $\mathrm{Sp}(n)$ on the sphere S^{4n-1} and a stabilizer of a point is isomorphic to $\mathrm{Sp}(n-1)$.

(3) Deduce that $\pi_1(\mathrm{Sp}(n+1)) = \pi_1(\mathrm{Sp}(n))$, $\pi_0(\mathrm{Sp}(n+1)) = \pi_0(\mathrm{Sp}(n))$.

Homework 4

3.1. Consider the group $\mathrm{SL}(2,\mathbb{R})$. Show that the element $X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not in the image of the exponential map. (Hint: if $X = \exp(x)$, what are the eigenvalues of x?).

3.5.

(1) Prove that $\mathbb{R}^3$ with the commutator given by the cross-product is a Lie algebra. Show that this Lie algebra is isomorphic to $\mathfrak{so}(3,\mathbb{R})$.

(2) Let $\varphi\colon \mathfrak{so}(3,\mathbb{R}) \to \mathbb{R}^3$ be the isomorphism of part (1). Prove that under this isomorphism, the standard action of $\mathfrak{so}(3)$ on $\mathbb{R}^3$ is identified with the action of $\mathbb{R}^3$ on itself given by the cross-product:

$$a \cdot \vec{v} = \varphi(a) \times \vec{v}, \qquad a \in \mathfrak{so}(3), \vec{v} \in \mathbb{R}^3$$

where $a \cdot \vec{v}$ is the usual multiplication of a matrix by a vector.

This problem explains common use of cross-products in mechanics: angular velocities and angular momenta are actually elements of Lie algebra $\mathfrak{so}(3,\mathbb{R})$ (to be precise, angular momenta are elements of the dual vector space, $(\mathfrak{so}(3,\mathbb{R}))^*$, but we can ignore this difference). To avoid explaining this, most textbooks write angular velocities as vectors in $\mathbb{R}^3$ and use cross-product instead of commutator. Of course, this would completely fail in dimensions other than 3, where $\mathfrak{so}(n,\mathbb{R})$ is not isomorphic to $\mathbb{R}^n$ even as a vector space.

3.6. Let P_n be the space of polynomials with real coefficients of degree $\leq n$ in variable x. The Lie group $G = \mathbb{R}$ acts on P_n by translations of the argument: $\rho(t)(x) = x + t, t \in G$. Show that the corresponding action of the Lie algebra $\mathfrak{g} = \mathbb{R}$ is given by $\rho(a) = a\partial_x, a \in \mathfrak{g}$, and deduce from this the Taylor formula for polynomials:

$$f(x+t) = \sum_{n\geq 0} \frac{(t\partial_x)^n}{n!} f.$$

3.7. Let G be the Lie group of all maps $A\colon \mathbb{R} \to \mathbb{R}$ having the form $A(x) = ax + b, a \neq 0$. Describe explicitly the corresponding Lie algebra. [There are two ways to do this problem. The easy way is to embed $G \subset \mathrm{GL}(2,\mathbb{R})$, which makes the problem trivial. More straightforward way is to explicitly construct some basis in the tangent space, construct the corresponding one-parameter subgroups, and compute the commutator using the formula

$$\exp(x)\exp(y)\exp(-x)\exp(-y) = \exp([x,y] + ...).$$

The second way is recommended to those who want to understand how the correspondence between Lie groups and Lie algebras works.]

3.8. Let $\mathrm{SL}(2,\mathbb{C})$ act on $\mathbb{CP}^1$ in the usual way:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (x : y) = (ax + by : cx + dy).$$

This defines an action of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ by vector fields on $\mathbb{CP}^1$. Write explicitly vector fields corresponding to h, e, f in terms of coordinate $t = x/y$ on the open cell $\mathbb{C} \subset \mathbb{CP}^1$.

3.9. Let G be a Lie group with Lie algebra $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$, and $\mathrm{Der}(\mathfrak{g})$ be the Lie algebra of derivations of $\mathfrak{g}$.

(1) Show that $g \mapsto \mathrm{Ad}\, g$ gives a morphism of Lie groups $G \to \mathrm{Aut}(\mathfrak{g})$; similarly, $x \mapsto \mathrm{ad}\, x$ is a morphism of Lie algebras $\mathfrak{g} \to \mathrm{Der}\,\mathfrak{g}$. (The automorphisms of the form $\mathrm{Ad}\, g$ are called *inner* automorphisms; the derivations of the form $\mathrm{ad}\, x, x \in \mathfrak{g}$ are called inner derivations.)

(2) Show that for $f \in \mathrm{Der}\,\mathfrak{g}$, $x \in \mathfrak{g}$, one has $[f, \mathrm{ad}\, x] = \mathrm{ad}\, f(x)$ as operators in $\mathfrak{g}$, and deduce from this that $\mathrm{ad}(\mathfrak{g})$ is an ideal in $\mathrm{Der}\,\mathfrak{g}$.

3.11. Let J_x, J_y, J_z be the standard basis in $\mathfrak{so}(3,\mathbb{R}) \cong \mathbb{R}^3$ (the Lie bracket is the cross product). The standard action of $\mathrm{SO}(3,\mathbb{R})$ on $\mathbb{R}^3$ defines an action of $\mathfrak{so}(3,\mathbb{R})$ by vector fields on $\mathbb{R}^3$. Abusing the language, we will use the same notation J_x, J_y, J_z for the corresponding vector fields on $\mathbb{R}^3$. Let

$\Delta_{sph} = J_x^2 + J_y^2 + J_z^2$; this is a second order differential operator on $\mathbb{R}^3$, which is usually called the *spherical Laplace operator*, or the *Laplace operator on the sphere*.

(1) Write Δ_{sph} in terms of $x, y, z, \partial_x, \partial_y, \partial_z$.

(2) Show that Δ_{sph} is well defined as a differential operator on a sphere $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, i.e., if f is a function on $\mathbb{R}^3$ then $(\Delta_{sph} f)|_{S^2}$ only depends on $f|_{S^2}$.

(3) Show that the usual Laplace operator $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ can be written in the form $\Delta = \frac{1}{r^2}\Delta_{sph} + \Delta_{radial}$, where Δ_{radial} is a differential operator written in terms of $r = \sqrt{x^2 + y^2 + z^2}$ and $r\partial_r = x\partial_x + y\partial_y + z\partial_z$.

(4) Show that Δ_{sph} is rotation invariant: for any function f and $g \in \mathrm{SO}(3, \mathbb{R})$, $\Delta_{sph}(gf) = g(\Delta_{sph} f)$.

Homework 5

3.13.

(1) Let $\mathfrak{g}$ be a three-dimensional real Lie algebra with basis x, y, z and commutation relations $[x, y] = z$, $[z, x] = [z, y] = 0$ (this algebra is called *Heisenberg algebra*). Without using Campbell-Hausdorff formula, show that in the corresponding Lie group, one has $\exp(tx)\exp(sy) = \exp(tsz)\exp(sy)\exp(tx)$ and construct explicitly the connected, simply connected Lie group corresponding to $\mathfrak{g}$.

(2) Generalize the previous part to the Lie algebra $\mathfrak{g} = V \oplus \mathbb{R}z$, where V is a real vector space with non-degenerate skew-symmetric form ω and the commutation relations are given by $[v_1, v_2] = \omega(v_1, v_2)z$, $[z, v] = 0$.

3.15. Let G be a complex connected simply-connected Lie group, with Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$, and let $\mathfrak{k} \subset \mathfrak{g}$ be a real form of $\mathfrak{g}$.

(1) Define the $\mathbb{R}$-linear map $\theta\colon \mathfrak{g} \to \mathfrak{g}$ by $\theta(x + iy) = x - iy$, $x, y \in \mathfrak{k}$. Show that θ is an automorphism of $\mathfrak{g}$ (considered as a real Lie algebra), and that it can be uniquely lifted to an automorphism $\theta\colon G \to G$ of the group G (considered as a real Lie group).

(2) Let $K = G^\theta$. Show that K is a real Lie group with Lie algebra $\mathfrak{k}$.

3.16. Let $\mathrm{Sp}(n)$ be the unitary quaternionic group. Show that $\mathfrak{sp}(n)_\mathbb{C} = \mathfrak{sp}(n, \mathbb{C})$. Thus $\mathrm{Sp}(n)$ is a compact real form of $\mathrm{Sp}(n, \mathbb{C})$.

3.17. Let $\mathfrak{so}(p, q) = \mathrm{Lie}(\mathrm{SO}(p, q))$. Show that its complexification is $\mathfrak{so}(p, q)_\mathbb{C} = \mathfrak{so}(p + q, \mathbb{C})$.

3.18. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}).$$

(1) Show that $S = \exp\Big(\frac{\pi}{2}(f - e)\Big)$, where $e, f \in \mathfrak{sl}(2, \mathbb{C})$ are standard basis elements.

(2) Compute $\mathrm{Ad}\, S$ in the basis e, f, h.

3.19. Let G be a complex connected Lie group.

(1) Show that $g \mapsto \mathrm{Ad}\, g$ is an analytic map $G \to \mathfrak{gl}(\mathfrak{g})$.

(2) Assume that G is compact. Show that then $\mathrm{Ad}\, g = 1$ for any $g \in G$.

(3) Show that any connected compact complex group must be commutative.

(4) Show that if G is a connected complex compact group, then the exponential map gives an isomorphism of Lie groups

$$\mathfrak{g}/L \simeq G$$

for some lattice $L \subset \mathfrak{g}$ (i.e. a free abelian group of rank equal to $2\dim\mathfrak{g}$).

Homework 6

4.2. Let $V = \mathbb{C}^2$ be the standard 2-dimensional representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ with basis e_1, e_2, and let $S^k V$ be the symmetric power of V.

(1) Write explicitly the action of $e, f, h \in \mathfrak{sl}(2, \mathbb{C})$ in the basis $e_1^i e_2^{k-i}$ if $ee_1 = 0$, $ee_2 = e_1$, $fe_1 = e_2$, $fe_2 = 0$, $he_1 = e_1, he_2 = -e_2$.
(2) Show that $S^2 V$ is isomorphic to the adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$.
(3) Recall that each representation of $\mathfrak{sl}(2, \mathbb{C})$ can be considered as a representation of $\mathfrak{so}(3, \mathbb{R})$. Which of representations $S^k V$ can be lifted to a representation of $\mathrm{SO}(3, \mathbb{R})$?

4.4. Let V be a representation of $\mathfrak{sl}(2, \mathbb{C})$, and let $C \in \mathrm{End}(V)$ be defined by

$$C = \rho(e)\rho(f) + \rho(f)\rho(e) + \frac{1}{2}\rho(h)^2.$$

(1) Show that C commutes with the action of $\mathfrak{sl}(2, \mathbb{C})$: for any $x \in \mathfrak{sl}(2, \mathbb{C})$, we have $[\rho(x), C] = 0$. [Hint: use that for any a, b, $c \in \mathrm{End}(V)$, one has $[a, bc] = [a, b]c + b[a, c]$.]
(2) Show that if $V = V_k$ is an irreducible representation with highest weight k, then C is a scalar operator: $C = c_k \,\mathrm{id}$. Compute the constant c_k.
(3) Recall that we have an isomorphism $\mathfrak{so}(3, \mathbb{C}) \simeq \mathfrak{sl}(2, \mathbb{C})$. Show that this isomorphism identifies operator C above with a multiple of $\rho(J_x)^2 + \rho(J_y)^2 + \rho(J_z)^2$.

The element C introduced here is a special case of more general notion of Casimir element for a simple Lie algebra.

4.7. Let $\mathfrak{g}$ be a Lie algebra, and $(\, , \,)$ — a symmetric ad-invariant bilinear form on $\mathfrak{g}$. Show that the element $\omega \in (\mathfrak{g}^*)^{\otimes 3}$ given by

$$\omega(x, y, z) = ([x, y], z)$$

is skew-symmetric and ad-invariant.

4.9. Let C be the standard cube in $\mathbb{R}^3$: $C = \{|x_i| \leq 1\}$, and let S be the set of faces of C (thus, S consists of 6 elements). Consider the 6-dimensional complex vector V space of functions on S, and define $A\colon V \to V$ by

$$(Af)(\sigma) = \frac{1}{4} \sum_{\sigma'} f(\sigma')$$

where the sum is taken over all faces σ' which are neighbors of σ (i.e., have a common edge with σ). The goal of this problem is to diagonalize A.

(1) Let $G = \{g \in \mathrm{O}(3, \mathbb{R}) \mid g(C) = C\}$ be the group of symmetries of C. Show that A commutes with the natural action of G on V.
(2) Let $z = -I \in G$. Show that as a representation of G, V can be decomposed in the direct sum

$$V = V_+ \oplus V_-, \qquad V_\pm = \{f \in V \mid zf = \pm f\}.$$

(3) Show that as a representation of G, V_+ can be decomposed in the direct sum

$$V_+ = V_+^0 \oplus V_+^1, \quad V_+^0 = \{f \in V_+ \mid \sum_\sigma f(\sigma) = 0\}, \quad V_+^1 = \mathbb{C} \cdot 1$$

where 1 denotes the constant function on S whose value at every $\sigma \in S$ is 1.
(4) Find the eigenvalues of A on V_-, V_+^0, V_+^1.

[Note: in fact, each of V_-, V_+^0, V_+^1 is an irreducible representation of G, but you do not need this fact.]

Homework 7

4.1. Let $\varphi\colon \mathrm{SU}(2) \to \mathrm{SO}(3,\mathbb{R})$ be the covering map.

(1) Show that $\operatorname{Ker}\varphi = \{1, -1\} = \{1, e^{\pi i h}\}$.

(2) Using this, show that representations of $\mathrm{SO}(3,\mathbb{R})$ are the same as representations of $\mathfrak{sl}(2,\mathbb{C})$ satisfying $e^{\pi i \rho(h)} = \mathrm{id}$.

4.11. Show that if V is a finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$, then $V \simeq \bigoplus n_k V_k$, and $n_k = \dim V[k] - \dim V[k+2]$. Show also that $\sum n_{2k} = \dim V[0]$, $\sum n_{2k+1} = \dim V[1]$.

4.12. Show that the symmetric power representation $S^k\mathbb{C}^2$ is isomorphic to the irreducible representation V_k with highest weight k.

5.1.

(1) Let V be a representation of $\mathfrak{g}$ and $W \subset V$ be a subrepresentation. Then $B_V = B_W + B_{V/W}$, where $B_V(x,y) = \operatorname{tr}(\rho_V(x)\rho_V(y))$.

(2) Let $I \subset \mathfrak{g}$ be an ideal. Then the restriction of the Killing form of $\mathfrak{g}$ to I coincides with the Killing form of I.

5.2. Show that for $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, the Killing form is given by $K(x,y) = 2n\operatorname{tr}(xy)$.

5.3. Let $\mathfrak{g} \subset \mathfrak{gl}(n,\mathbb{C})$ be the subspace consisting of block-triangular matrices:

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \right\}$$

where A is a $k \times k$ matrix, B is a $k \times (n-k)$ matrix, and D is a $(n-k)\times(n-k)$ matrix.

(1) Show that $\mathfrak{g}$ is a Lie subalgebra (this is a special case of so-called *parabolic subalgebras*).

(2) Show that radical of $\mathfrak{g}$ consists of matrices of the form $\begin{pmatrix} \lambda \cdot I & B \\ 0 & \mu \cdot I \end{pmatrix}$, and describe $\mathfrak{g}/\operatorname{rad}(\mathfrak{g})$.

5.4. Show that the bilinear form $\operatorname{tr}(xy)$ on $\mathfrak{sp}(n,\mathbb{K})$ is non-degenerate.

5.5. Let $\mathfrak{g}$ be a real Lie algebra with a positive definite Killing form. Show that then $\mathfrak{g} = 0$. [Hint: $\mathfrak{g} \subset \mathfrak{so}(\mathfrak{g})$.]

5.6. Let $\mathfrak{g}$ be a simple Lie algebra.

(1) Show that the invariant bilinear form is unique up to a factor.

(2) Show that $\mathfrak{g} \simeq \mathfrak{g}^*$ as representations of $\mathfrak{g}$.

5.7. Let V be a finite-dimensional complex vector space and let $A\colon V \to V$ be an upper-triangular operator. Let $F^k \subset \operatorname{End}(V)$, $-n \le k \le n$ be the subspace spanned by matrix units E_{ij} with $i - j \le k$. Show that then $\operatorname{ad} A.F^k \subset F^{k-1}$ and thus, $\operatorname{ad} A\colon \operatorname{End}(V) \to \operatorname{End}(V)$ is nilpotent.

Homework 8

6.1. Show that the Casimir operator for $\mathfrak{g} = \mathfrak{so}(3, \mathbb{R})$ is given by $C = \frac{1}{2}(J_x^2 + J_y^2 + J_z^2)$; thus, it follows that $J_x^2 + J_y^2 + J_z^2 \in U\mathfrak{so}(3, \mathbb{R})$ is central.

6.2. Show that for $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$, the definition of a semisimple element (an element x such as $\mathrm{ad} x$ is a semisimple operator) coincides to the usual definition of a semisimple operator.

6.3. Show that if $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra in a complex semisimple Lie algebra, then $\mathfrak{h}$ is a nilpotent subalgebra which coincides with its normalizer $n(\mathfrak{h}) = \{x \in g \mid \mathrm{ad}\, x.\mathfrak{h} \subset \mathfrak{h}\}$. (This is the usual definition of a Cartan subalgebra which can be used for any Lie algebra, not necessarily a semisimple one.)

6.4. Let $\mathfrak{g}$ be a complex Lie algebra which has a root decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where R is a finite subset in $\mathfrak{h}^* - \{0\}$, $\mathfrak{h}$ is commutative and for $h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha$, we have $[h, x] = \langle h, \alpha \rangle x$. Show that then $\mathfrak{g}$ is semisimple, and $\mathfrak{h}$ is a Cartan subalgebra.

6.5. Let $\mathfrak{h} \subset \mathfrak{so}(4, \mathbb{C})$ be the subalgebra consisting of matrices of the form

$$\begin{bmatrix} & a & & \\ -a & & & \\ & & & b \\ & & -b & \end{bmatrix}$$

(entries not shown are zeros). Show that then $\mathfrak{h}$ is a Cartan subalgebra and find the corresponding root decomposition.

6.6.

(1) Define a bilinear form B on $W = \Lambda^2 \mathbb{C}^4$ by $\omega_1 \wedge \omega_2 = B(\omega_1, \omega_2) e_1 \wedge e_2 \wedge e_3 \wedge e_4$. Show that B is a symmetric non-degenerate form and construct an orthonormal basis for B.
(2) Let $\mathfrak{g} = \mathfrak{so}(W, B) = \{x \in \mathfrak{gl}(W) \mid B(x\omega_1, \omega_2) + B(\omega_1, x\omega_2) = 0\}$. Show that $\mathfrak{g} \simeq \mathfrak{so}(6, \mathbb{C})$.
(3) Show that the form B is invariant under the natural action of $\mathfrak{sl}(4, \mathbb{C})$ on $\Lambda^2 \mathbb{C}^4$.
(4) Using results of the previous parts, construct a homomorphism $\mathfrak{sl}(4, \mathbb{C}) \to \mathfrak{so}(6, \mathbb{C})$ and prove that it is an isomorphism.

Homework 10

7.1. Let $R \subset \mathbb{R}^n$ be given by

$$R = \{\pm e_i, \pm 2e_i \mid 1 \le i \le n\} \cup \{\pm e_i \pm e_j \mid 1 \le i, j \le n, i \ne j\}$$

where e_i is the standard basis in $\mathbb{R}^n$. Show that R is a non-reduced root system. (This root system is usually denoted BC_n.)

7.2.

(1) Let $R \subset E$ be a root system. Show that the set

$$R^\vee = \{\alpha^\vee \mid \alpha \in R\} \subset E^*$$

where $\alpha^\vee \in E^*$ is the coroot corresponding to α is also a root system. It is usually called the *dual root system* of R.

(2) Let $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset R$ be the set of simple roots. Show that the set $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_r^\vee\} \subset R^\vee$ is the set of simple roots of $R^\vee$. [Note: this is not completely trivial, as $\alpha \mapsto \alpha^\vee$ is not a linear map.]

7.4. Show that $|P/Q| = |\det A|$, where A is the Cartan matrix: $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$.

7.5. Compute explicitly the group P/Q for root systems A_n, D_n.

7.8. Let $\overline{C}_+$ be the closure of the positive Weyl chamber, and $\lambda \in \overline{C}_+$, $w \in W$ be such that $w(\lambda) \in \overline{C}_+$.

(1) Show that $\lambda \in \overline{C}_+ \cap w^{-1}(\overline{C}_+)$.

(2) Let $L_\alpha \subset E$ be a root hyperplane which separates C_+ and $w^{-1}C_+$. Show that then $\lambda \in L_\alpha$.

(3) Show that $w(\lambda) = \lambda$.

Deduce from this that every W-orbit in E contains a unique element from $\overline{C}_+$.

7.9. Let $w_0 \in W$ be the longest element in the Weyl group W. Show that then for any $w \in W$, we have $l(ww_0) = l(w_0 w) = l(w_0) - l(w)$.

7.10. Let $W = S_n$ be the Weyl group of root system A_{n-1}. Show that the longest element $w_0 \in W$ is the permutation $w_0 = (n\ n-1\ \dots 1)$.

7.11.

(1) Let R be a reduced root system of rank 2, with simple roots α_1, α_2. Show that the longest element in the corresponding Weyl group is

$$w_0 = s_1 s_2 s_1 \dots = s_2 s_1 s_2 \dots \qquad (m \text{ factors in each of the products})$$

where m depends on the angle φ between α_1, α_2: $\varphi = \pi - \frac{\pi}{m}$ (so $m = 2$ for $A_1 \times A_1$, $m = 3$ for A_2, $m = 4$ for B_2, $m = 6$ for G_2). If you can not think of any other proof, give a case-by-case proof.

(2) Show that the following relations hold in W (these are called *Coxeter relations*):

$$s_i^2 = 1,\ (s_i s_j)^{m_{ij}} = 1,$$

where m_{ij} is determined by the angle between α_i, α_j in the same way as in the previous part.

(It can be shown that the Coxeter relations is a defining set of relations for the Weyl group: W could be defined as the group generated by elements s_i subject to Coxeter relations. A proof of this fact can be found in the book of Humphreys "Reflection groups and Coxeter groups".)

Homework 11

7.7. Let $w = s_{i_1} \dots s_{i_l}$ be a reduced expression. Show that then

$$\{\alpha \in R_+ \mid w(\alpha) \in R_-\} = \{\beta_1, \dots, \beta_l\}$$

where $\beta_k = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$

7.12. Let $\varphi\colon R_1 \xrightarrow{\sim} R_2$ be an isomorphism between irreducible root systems. Show that then φ is a composition of an isometry and a scalar operator: $(\varphi(v), \varphi(w)) = c(v, w)$ for any $v, w \in E_1$.

7.13.

(1) Let $\mathfrak{n}_\pm$ be the positive and negative nilpotent subalgebras in a semisimple complex Lie algebra. Show that $\mathfrak{n}_\pm$ are indeed nilpotent.

(2) Let $\mathfrak{b} = \mathfrak{n}_+ \oplus \mathfrak{h}$. Show that $\mathfrak{b}$ is solvable.

7.14.

(1) Show that if two vertices in a Dynkin diagram are connected by a single edge, then the corresponding simple roots are in the same W-orbit.

(2) Show that for a reduced irreducible root system, the Weyl group acts transitively on the set of all roots of the same length.

7.15. Let $R \subset E$ be an irreducible root system. Show that then E is an irreducible representation of the Weyl group W.

7.16. Let G be a connected complex Lie group such that $\mathfrak{g} = \mathrm{Lie}(G)$ is semisimple. Fix a root decomposition of $\mathfrak{g}$.

(1) Choose $\alpha \in R$ and let $i_\alpha\colon \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ be the embedding corresponding to the root α. This embedding can be lifted to a morphism $i_\alpha\colon \mathrm{SL}(2,\mathbb{C}) \to G$.

Let

$$S_\alpha = i_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp\Big(\frac{\pi}{2}(f_\alpha - e_\alpha)\Big) \in G$$

Show that $\mathrm{Ad}\, S_\alpha(h_\alpha) = -h_\alpha$ and that $\mathrm{Ad}\, S_\alpha(h) = h$ if $h \in \mathfrak{h}$, $\langle h, \alpha \rangle = 0$. Deduce from this that the action of S_α on $\mathfrak{g}^*$ preserves $\mathfrak{h}^*$ and that restriction of $\mathrm{Ad}\, S_\alpha$ to $\mathfrak{h}^*$ coincides with the reflection s_α.

(2) Show that the Weyl group W acts on $\mathfrak{h}^*$ by inner automorphisms: for any $w \in W$, there exists an element $\tilde{w} \in G$ such that $\mathrm{Ad}\, \tilde{w}|_{\mathfrak{h}^*} = w$. [Note, however, that in general, $\widetilde{w_1 w_2} \neq \tilde{w}_1 \tilde{w}_2$.]