Problems from “An introduction to Lie groups and Lie algebras” by A. Kirillov Jr.

Homework 1

2.1. Let $G$ be a Lie group and $H$ — a closed Lie subgroup.

1. Let $\overline{H}$ be the closure of $H$ in $G$. Show that $\overline{H}$ is a subgroup in $G$.
2. Show that each coset $Hx, x \in \overline{H}$, is open and dense in $\overline{H}$.
3. Show that $\overline{H} = H$, that is, every Lie subgroup is closed.

2.2.

1. Show that every discrete normal subgroup of a connected Lie group is central (hint: consider the map $G \to N : g \mapsto ghg^{-1}$ where $h$ is a fixed element in $N$).
2. By applying part (a) to kernel of the map $\tilde{G} \to G$, show that for any connected Lie group $G$, the fundamental group $\pi_1(G)$ is commutative.

2.3. Let $f : G_1 \to G_2$ be a morphism of connected Lie groups such that $f_* : T_1G_1 \to T_1G_2$ is an isomorphism (such a morphism is sometimes called local isomorphism). Show that $f$ is a covering map, and $\text{Ker } f$ is a discrete central subgroup.

2.4. Let $F_n(\mathbb{C})$ be the set of all flags in $\mathbb{C}^n$. Show that

$$F_n(\mathbb{C}) = \frac{\text{GL}(n, \mathbb{C})}{\text{B}(n, \mathbb{C})} = \frac{\text{U}(n)}{\text{T}(n)}$$

where $\text{B}(n, \mathbb{C})$ is the group of invertible complex upper triangular matrices, and $\text{T}(n)$ is the group of diagonal unitary matrices (which is easily shown to be the $n$-dimensional torus $(\mathbb{R}/\mathbb{Z})^n$). Deduce from this that $F_n(\mathbb{C})$ is a compact complex manifold and find its dimension over $\mathbb{C}$.

2.5. Let $G_{n,k}$ be the set of all dimension $k$ subspaces in $\mathbb{R}^n$ (usually called the Grassmanian). Show that $G_{n,k}$ is a homogeneous space for the group $O(n, \mathbb{R})$ and thus can be identified with coset space $O(n, \mathbb{R})/H$ for appropriate $H$. Use it to prove that $G_{n,k}$ is a manifold and find its dimension.

2.6. Show that if $G = \text{GL}(n, \mathbb{R}) \subset \text{End}(\mathbb{R}^n)$ so that each tangent space is canonically identified with $\text{End}(\mathbb{R}^n)$, then $(L_g)_* v = gv$ where the product in the right-hand side is the usual product of matrices, and similarly for the right action. Also, the adjoint action is given by $\text{Ad } g(v) = gvg^{-1}$.
Homework 2

2.7. Define a bilinear form on $\mathfrak{su}(2)$ by $(a, b) = \frac{1}{2} \text{tr}(ab^\dagger)$. Show that this form is symmetric, positive definite, and invariant under the adjoint action of $SU(2)$.

2.8. Define a basis in $\mathfrak{su}(2)$ by

$$i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

Show that the map

$$\varphi : SU(2) \to GL(3, \mathbb{R}), \quad g \mapsto \text{matrix of Ad} \, g \text{ in the basis } i\sigma_1, i\sigma_2, i\sigma_3$$

gives a morphism of Lie groups $SU(2) \to SO(3, \mathbb{R})$.

2.9. Let $\varphi : SU(2) \to SO(3, \mathbb{R})$ be the morphism defined in the previous problem. Compute explicitly the map of tangent spaces $\varphi_* : \mathfrak{su}(2) \to \mathfrak{so}(3, \mathbb{R})$ and show that $\varphi_*$ is an isomorphism. Deduce from this that $\ker \varphi$ is a discrete normal subgroup in $SU(2)$, and that $\text{Im} \varphi$ is an open subgroup in $SO(3, \mathbb{R})$.

2.10. Prove that the map $\varphi$ used in two previous exercises establishes an isomorphism $SU(2)/\mathbb{Z}_2 \to SO(3, \mathbb{R})$ and thus, since $SU(2) \cong S^3$, $SO(3, \mathbb{R}) \cong \mathbb{RP}^3$.

2.11. Show that for $n \geq 1$, we have $\pi_0(SU(n+1)) = \pi_0(SU(n))$, $\pi_0(U(n+1)) = \pi_0(U(n))$ and deduce from it that groups $U(n)$, $SU(n)$ are connected for all $n$. Similarly, show that for $n \geq 2$, we have $\pi_1(SU(n+1)) = \pi_1(SU(n))$, $\pi_1(U(n+1)) = \pi_1(U(n))$ and deduce from it that for $n \geq 2$, $SU(n)$ is simply-connected and $\pi_1(U(n)) = \mathbb{Z}$.

2.12. Show that for $n \geq 2$, we have $\pi_0(SO(n+1, \mathbb{R})) = \pi_0(SO(n, \mathbb{R}))$ and deduce from it that groups $SO(n)$ are connected for all $n \geq 2$. Similarly, show that for $n \geq 3$, $\pi_1(SO(n+1, \mathbb{R})) = \pi_1(SO(n, \mathbb{R}))$ and deduce from it that for $n \geq 3$, $\pi_1(SO(n, \mathbb{R})) = \mathbb{Z}_2$.

2.13. Using Gram-Schmidt orthogonalization process, show that $GL(n, \mathbb{R})/O(n, \mathbb{R})$ is diffeomorphic to the space of upper-triangular matrices with positive entries on the diagonal. Deduce from this that $GL(n, \mathbb{R})$ is homotopic (as a topological space) to $O(n, \mathbb{R})$.

2.14. Let $L_n$ be the set of all Lagrangian subspaces in $\mathbb{R}^{2n}$ with the standard symplectic form $\omega$. (A subspace $V$ is Lagrangian if $\dim V = n$ and $\omega(x, y) = 0$ for any $x, y \in V$.)

Show that the group $Sp(n, \mathbb{R})$ acts transitively on $L_n$ and use it to define on $L_n$ a structure of a smooth manifold and find its dimension.

2.15. Let $\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$ be the algebra of quaternions, defined by $ij = k = -ji$, $jk = i = -kj$, $ki = j = -ik$, $i^2 = j^2 = k^2 = -1$, and let $\mathbb{H}^n = \{h_1, \ldots, h_n \mid h_i \in \mathbb{H}\}$. In particular, the subalgebra generated by $1, i$ coincides with the field $\mathbb{C}$ of complex numbers.

Note that $\mathbb{H}^n$ has a structure of both left and right module over $\mathbb{H}$ defined by

$$h(h_1, \ldots, h_n) = (hh_1, \ldots, hh_n), \quad (h_1, \ldots, h_n)h = (h_1h, \ldots, h_nh)$$

1. Let $\text{End}_\mathbb{H}(\mathbb{H}^n)$ be the algebra of endomorphisms of $\mathbb{H}^n$ considered as right $\mathbb{H}$-module:

$$\text{End}_\mathbb{H}(\mathbb{H}^n) = \{ A : \mathbb{H}^n \to \mathbb{H}^n \mid A(h + h') = A(h) + A(h'), \quad A(hh) = A(h)h \}$$

Show that $\text{End}_\mathbb{H}(\mathbb{H}^n)$ is naturally identified with the algebra of $n \times n$ matrices with quaternionic entries.

2. Define an $\mathbb{H}$–valued form $(\cdot, \cdot)$ on $\mathbb{H}^n$ by

$$(h, h') = \sum_i \overline{h_i}h'_i$$

where $a + bi + cj + dk = a - bi - cj - dk$. (Note that $\overline{vw} = \overline{v}\overline{w}$.)

Let $U(n, \mathbb{H})$ be the group of “unitary quaternionic transformations”:

$$U(n, \mathbb{H}) = \{ A \in \text{End}_\mathbb{H}(\mathbb{H}^n) \mid (Ah, Ah') = (h, h') \}.$$
(3) Define a map $\mathbb{C}^{2n} \cong \mathbb{H}^n$ by
\[
(z_1, \ldots, z_{2n}) \mapsto (z_1 + jz_{n+1}, \ldots, z_n + jz_{2n})
\]
Show that it is an isomorphism of complex vector spaces (if we consider $\mathbb{H}^n$ as a complex vector space by $z(h_1, \ldots, h_n) = (h_1z, \ldots, h_nz)$) and that this isomorphism identifies $\text{End}_{\mathbb{H}}(\mathbb{H}^n) = \{ A \in \text{End}_{\mathbb{C}}(\mathbb{C}^{2n}) \mid \overline{A} = J^{-1}AJ \}$
where $J := \begin{pmatrix} 0 & \text{Id}_n \\ -\text{Id}_n & 0 \end{pmatrix}$. (Hint: use $jz = \overline{z}j$ for any $z \in \mathbb{C}$ to show that $h \mapsto hj$ is identified with $z \mapsto Jz$.)

(4) Show that under identification $\mathbb{C}^{2n} \cong \mathbb{H}^n$ defined above, the quaternionic form $(\cdot, \cdot)$ is identified with
\[
(z, z') - j\langle z, z' \rangle
\]
where $(z, z') = \sum \overline{z}_i' \overline{z}_i$ is the standard Hermitian form in $\mathbb{C}^{2n}$ and
\[
\langle z, z' \rangle = \sum_{i=1}^{n}(z_{i+n}z_i' - z_iz_{i+n}')
\]
is the standard bilinear skew-symmetric form in $\mathbb{C}^{2n}$. Deduce from this that the group $U(n, \mathbb{H})$ is identified with $\text{Sp}(n) = \text{Sp}(n, \mathbb{C}) \cap \text{SU}(2n)$.

2.16.
(1) Show that $\text{Sp}(1) \cong \text{SU}(2) \cong S^3$.
(2) Using the previous exercise, show that we have a natural transitive action of $\text{Sp}(n)$ on the sphere $S^{4n-1}$ and a stabilizer of a point is isomorphic to $\text{Sp}(n-1)$.
(3) Deduce that $\pi_1(\text{Sp}(n+1)) = \pi_1(\text{Sp}(n))$, $\pi_0(\text{Sp}(n+1)) = \pi_0(\text{Sp}(n))$. 
Homework 4

3.1. Consider the group $\text{SL}(2, \mathbb{R})$. Show that the element $X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$ is not in the image of the exponential map. (Hint: if $X = \exp(x)$, what are the eigenvalues of $x$?)

3.5.

(1) Prove that $\mathbb{R}^3$ with the commutator given by the cross-product is a Lie algebra. Show that this Lie algebra is isomorphic to $\mathfrak{so}(3, \mathbb{R})$.

(2) Let $\varphi: \mathfrak{so}(3, \mathbb{R}) \to \mathbb{R}^3$ be the isomorphism of part (1). Prove that under this isomorphism, the standard action of $\mathfrak{so}(3)$ on $\mathbb{R}^3$ is identified with the action of $\mathbb{R}^3$ on itself given by the cross-product:

$$a \cdot \vec{v} = \varphi(a) \times \vec{v}, \quad a \in \mathfrak{so}(3), \vec{v} \in \mathbb{R}^3$$

where $a \cdot \vec{v}$ is the usual multiplication of a matrix by a vector.

This problem explains common use of cross-products in mechanics: angular velocities and angular momenta are actually elements of Lie algebra $\mathfrak{so}(3, \mathbb{R})$ (to be precise, angular momenta are elements of the dual vector space, $(\mathfrak{so}(3, \mathbb{R}))^*$, but we can ignore this difference). To avoid explaining this, most textbooks write angular velocities as vectors in $\mathbb{R}^3$ and use cross-product instead of commutator. Of course, this would completely fail in dimensions other than 3, where $\mathfrak{so}(n, \mathbb{R})$ is not isomorphic to $\mathbb{R}^n$ even as a vector space.

3.6. Let $P_n$ be the space of polynomials with real coefficients of degree $\leq n$ in variable $x$. The Lie group $G = \mathbb{R}$ acts on $P_n$ by translations of the argument: $\rho(t)(x) = x + t, t \in G$. Show that the corresponding action of the Lie algebra $\mathfrak{g} = \mathbb{R}$ is given by $\rho(a) = a \partial_x, a \in \mathfrak{g}$, and deduce from this the Taylor formula for polynomials:

$$f(x + t) = \sum_{n \geq 0} \frac{(t \partial_x)^n}{n!} f.$$

3.7. Let $G$ be the Lie group of all maps $A: \mathbb{R} \to \mathbb{R}$ having the form $A(x) = ax + b, a \neq 0$. Describe explicitly the corresponding Lie algebra. (There are two ways to do this problem. The easy way is to embed $G \subset \text{GL}(2, \mathbb{R})$, which makes the problem trivial. More straightforward way is to explicitly construct some basis in the tangent space, construct the corresponding one-parameter subgroups, and compute the commutator using the formula

$$\exp(x) \exp(y) \exp(-x) \exp(-y) = \exp([x, y] + \ldots).$$

The second way is recommended to those who want to understand how the correspondence between Lie groups and Lie algebras works.)

3.8. Let $\text{SL}(2, \mathbb{C})$ act on $\mathbb{CP}^1$ in the usual way:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (x : y) = (ax + by : cx + dy).$$

This defines an action of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ by vector fields on $\mathbb{CP}^1$. Write explicitly vector fields corresponding to $h, e, f$ in terms of coordinate $t = x/y$ on the open cell $C \subset \mathbb{CP}^1$.

3.9. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$, $\text{Aut}(\mathfrak{g})$ the group of automorphisms of $\mathfrak{g}$, and $\text{Der}(\mathfrak{g})$ be the Lie algebra of derivations of $\mathfrak{g}$.

(1) Show that $g \mapsto \text{Ad} g$ gives a morphism of Lie groups $G \to \text{Aut}(\mathfrak{g})$; similarly, $x \mapsto \text{ad} x$ is a morphism of Lie algebras $\mathfrak{g} \to \text{Der} \mathfrak{g}$. (The automorphisms of the form $\text{Ad} g$ are called inner automorphisms; the derivations of the form $\text{ad} x, x \in \mathfrak{g}$ are called inner derivations.)

(2) Show that for $f \in \text{Der} \mathfrak{g}$, $x \in \mathfrak{g}$, one has $[f, \text{ad} x] = \text{ad} f(x)$ as operators in $\mathfrak{g}$, and deduce from this that $\text{ad}(g)$ is an ideal in $\text{Der} \mathfrak{g}$.

3.11. Let $J_x, J_y, J_z$ be the standard basis in $\mathfrak{so}(3, \mathbb{R}) \cong \mathbb{R}^3$ (the Lie bracket is the cross product). The standard action of $\text{SO}(3, \mathbb{R})$ on $\mathbb{R}^3$ defines an action of $\mathfrak{so}(3, \mathbb{R})$ by vector fields on $\mathbb{R}^3$. Abusing the language, we will use the same notation $J_x, J_y, J_z$ for the corresponding vector fields on $\mathbb{R}^3$. Let
\[ \Delta_{\text{sph}} = J_x^2 + J_y^2 + J_z^2; \] this is a second order differential operator on \( \mathbb{R}^3 \), which is usually called the \textit{spherical Laplace operator}, or the \textit{Laplace operator on the sphere}.

(1) Write \( \Delta_{\text{sph}} \) in terms of \( x, y, z, \partial_x, \partial_y, \partial_z \).

(2) Show that \( \Delta_{\text{sph}} \) is well defined as a differential operator on a sphere \( S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\} \), i.e., if \( f \) is a function on \( \mathbb{R}^3 \) then \( (\Delta_{\text{sph}} f)_{|S^2} \) only depends on \( f|_{S^2} \).

(3) Show that the usual Laplace operator \( \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2 \) can be written in the form
\[
\Delta = \frac{1}{r^2} \Delta_{\text{sph}} + \Delta_{\text{radial}},
\]
where \( \Delta_{\text{radial}} \) is a differential operator written in terms of \( r = \sqrt{x^2 + y^2 + z^2} \) and \( r \partial_r = x \partial_x + y \partial_y + z \partial_z \).

(4) Show that \( \Delta_{\text{sph}} \) is rotation invariant: for any function \( f \) and \( g \in \text{SO}(3, \mathbb{R}) \), \( \Delta_{\text{sph}}(gf) = g(\Delta_{\text{sph}} f) \).
Homework 5

3.13. Let $\mathfrak{g}$ be a three-dimensional real Lie algebra with basis $x, y, z$ and commutation relations $[x, y] = z$, $[z, x] = [z, y] = 0$ (this algebra is called Heisenberg algebra). Without using Campbell-Hausdorff formula, show that in the corresponding Lie group, one has
$$\exp(tx) \exp(sy) = \exp(tsz) \exp(sy) \exp(tx)$$
and construct explicitly the connected, simply connected Lie group corresponding to $\mathfrak{g}$.

2. Generalize the previous part to the Lie algebra $\mathfrak{g} = V \oplus \mathbb{R}z$, where $V$ is a real vector space with non-degenerate skew-symmetric form $\omega$ and the commutation relations are given by $[v_1, v_2] = \omega(v_1, v_2)z$, $[z, v] = 0$.

3.15. Let $G$ be a complex connected simply-connected Lie group, with Lie algebra $\mathfrak{g} = \text{Lie}(G)$, and let $\mathfrak{k} \subset \mathfrak{g}$ be a real form of $\mathfrak{g}$.

1. Define the $\mathbb{R}$-linear map $\theta: \mathfrak{g} \to \mathfrak{g}$ by $\theta(x + iy) = x - iy$, $x, y \in \mathfrak{k}$. Show that $\theta$ is an automorphism of $\mathfrak{g}$ (considered as a real Lie algebra), and that it can be uniquely lifted to an automorphism $\theta: G \to G$ of the group $G$ (considered as a real Lie group).

2. Let $K = G^\theta$. Show that $K$ is a real Lie group with Lie algebra $\mathfrak{k}$.

3.16. Let $\text{Sp}(n)$ be the unitary quaternionic group. Show that $\mathfrak{sp}(n, \mathbb{C}) = \mathfrak{sp}(n, \mathbb{C})$. Thus $\text{Sp}(n)$ is a compact real form of $\text{Sp}(n, \mathbb{C})$.

3.17. Let $\mathfrak{so}(p, q) = \text{Lie}(\text{SO}(p, q))$. Show that its complexification is $\mathfrak{so}(p, q)_{\mathbb{C}} = \mathfrak{so}(p + q, \mathbb{C})$.

3.18. Let
$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \text{SL}(2, \mathbb{C}).$$

1. Show that $S = \exp\left( \frac{\pi}{2} (f - e) \right)$, where $e, f \in \mathfrak{sl}(2, \mathbb{C})$ are standard basis elements.

2. Compute $\text{Ad} S$ in the basis $e, f, h$.

3.19. Let $G$ be a complex connected Lie group.

1. Show that $\mathfrak{g} \to \text{Ad} \mathfrak{g}$ is an analytic map $G \to \mathfrak{gl}(\mathfrak{g})$.

2. Assume that $G$ is compact. Show that then $\text{Ad} g = 1$ for any $g \in G$.

3. Show that any connected compact complex group must be commutative.

4. Show that if $G$ is a connected complex compact group, then the exponential map gives an isomorphism of Lie groups
$$\mathfrak{g}/L \simeq G$$
for some lattice $L \subset \mathfrak{g}$ (i.e. a free abelian group of rank equal to $2 \dim \mathfrak{g}$).
**Homework 6**

4.2. Let $V = \mathbb{C}^2$ be the standard 2-dimensional representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ with basis $e_1, e_2$, and let $S^k V$ be the symmetric power of $V$.

1) Write explicitly the action of $e, f, h \in \mathfrak{sl}(2, \mathbb{C})$ in the basis $e_1^i e_2^j$ if $ee_1 = 0, ee_2 = e_1, f e_1 = e_2$, $f e_2 = 0$, $he_1 = e_1$, $he_2 = -e_2$.

2) Show that $S^2 V$ is isomorphic to the adjoint representation of $\mathfrak{sl}(2, \mathbb{C})$.

3) Recall that each representation of $\mathfrak{sl}(2, \mathbb{C})$ can be considered as a representation of $\mathfrak{so}(3, \mathbb{R})$.

4.4. Let $V$ be a representation of $\mathfrak{sl}(2, \mathbb{C})$, and let $C \in \text{End}(V)$ be defined by

$$C = \rho(e) \rho(f) + \rho(f) \rho(e) + \frac{1}{2} \rho(h)^2.$$ 

1) Show that $C$ commutes with the action of $\mathfrak{sl}(2, \mathbb{C})$: for any $x \in \mathfrak{sl}(2, \mathbb{C})$, we have $[\rho(x), C] = 0$.

[Hint: use that for any $a, b, c \in \text{End}(V)$, one has $[a, b c] = [a, b] c + b [a, c]$.]

2) Show that if $V = V_k$ is an irreducible representation with highest weight $k$, then $C$ is a scalar operator: $C = c_k \text{id}$. Compute the constant $c_k$.

3) Recall that we have an isomorphism $\mathfrak{so}(3, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C})$. Show that this isomorphism identifies operator $C$ above with a multiple of $\rho(J_x)^2 + \rho(J_y)^2 + \rho(J_z)^2$.

The element $C$ introduced here is a special case of more general notion of Casimir element for a simple Lie algebra.

4.7. Let $\mathfrak{g}$ be a Lie algebra, and $\langle \ , \ \rangle$ — a symmetric ad-invariant bilinear form on $\mathfrak{g}$. Show that the element $\omega \in (\mathfrak{g}^*)^\otimes 3$ given by

$$\omega(x, y, z) = \langle [x, y], z \rangle$$

is skew-symmetric and ad-invariant.

4.9. Let $C$ be the standard cube in $\mathbb{R}^3$: $C = \{ |x_i| \leq 1 \}$, and let $S$ be the set of faces of $C$ (thus, $S$ consists of 6 elements). Consider the 6-dimensional complex vector $V$ space of functions on $S$, and define $A: V \to V$ by

$$(Af)(\sigma) = \frac{1}{4} \sum_{\sigma'} f(\sigma')$$

where the sum is taken over all faces $\sigma'$ which are neighbors of $\sigma$ (i.e., have a common edge with $\sigma$).

The goal of this problem is to diagonalize $A$.

1) Let $G = \{ g \in O(3, \mathbb{R}) \mid g(C) = C \}$ be the group of symmetries of $C$. Show that $G$ commutes with the natural action of $G$ on $V$.

2) Let $z = -I \in G$. Show that as a representation of $G$, $V$ can be decomposed in the direct sum

$$V = V_+ \oplus V_-,$$

$$V_\pm = \{ f \in V \mid zf = \pm f \}.$$ 

3) Show that as a representation of $G$, $V_+$ can be decomposed in the direct sum

$$V_+ = V^0_+ \oplus V^1_+,$$ 

$$V^0_+ = \{ f \in V_+ \mid \sum_{\sigma} f(\sigma) = 0 \}, \quad V^1_+ = \mathbb{C} \cdot 1$$

where 1 denotes the constant function on $S$ whose value at every $\sigma \in S$ is 1.

4) Find the eigenvalues of $A$ on $V_-, V^0_+, V^1_+$.

[Note: in fact, each of $V_-, V^0_+, V^1_+$ is an irreducible representation of $G$, but you do not need this fact.]
Homework 7

4.1. Let \( \varphi : \text{SU}(2) \to \text{SO}(3, \mathbb{R}) \) be the covering map.

(1) Show that \( \text{Ker} \varphi = \{1, -1\} = \{1, e^{\pi i h}\} \).

(2) Using this, show that representations of \( \text{SO}(3, \mathbb{R}) \) are the same as representations of \( \mathfrak{sl}(2, \mathbb{C}) \) satisfying \( e^{\pi i \rho(h)} = \text{id} \).

4.11. Show that if \( V \) is a finite-dimensional representation of \( \mathfrak{sl}(2, \mathbb{C}) \), then \( V \cong \bigoplus n_k V_k \), and \( n_k = \dim V[k] - \dim V[k + 2] \). Show also that \( \sum n_{2k} = \dim V[0] \), \( \sum n_{2k+1} = \dim V[1] \).

4.12. Show that the symmetric power representation \( S^k \mathbb{C}^2 \) is isomorphic to the irreducible representation \( V_k \) with highest weight \( k \).

5.1. Let \( V \) be a representation of \( \mathfrak{g} \) and \( W \subset V \) be a subrepresentation. Then \( B_V = B_W + B_{V/W} \), where \( B_V(x, y) = \text{tr}(\rho_V(x) \rho_V(y)) \).

5.2. Let \( I \subset \mathfrak{g} \) be an ideal. Then the restriction of the Killing form of \( \mathfrak{g} \) to \( I \) coincides with the Killing form of \( I \).

5.3. Let \( \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C}) \) be the subspace consisting of block-triangular matrices:

\[
\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \right\}
\]

where \( A \) is a \( k \times k \) matrix, \( B \) is a \( k \times (n-k) \) matrix, and \( D \) is a \( (n-k) \times (n-k) \) matrix.

(1) Show that \( \mathfrak{g} \) is a Lie subalgebra (this is a special case of so-called \textit{parabolic subalgebras}).

(2) Show that radical of \( \mathfrak{g} \) consists of matrices of the form \( \begin{pmatrix} \lambda \cdot I & B \\ 0 & \mu \cdot I \end{pmatrix} \), and describe \( \mathfrak{g}/\text{rad}(\mathfrak{g}) \).

5.4. Show that the bilinear form \( \text{tr}(xy) \) on \( \mathfrak{sp}(n, \mathbb{K}) \) is non-degenerate.

5.5. Let \( \mathfrak{g} \) be a real Lie algebra with a positive definite Killing form. Show that then \( \mathfrak{g} = 0 \).

[Hint: \( \mathfrak{g} \subset \mathfrak{so}(\mathfrak{g}) \).]

5.6. Let \( \mathfrak{g} \) be a simple Lie algebra.

(1) Show that the invariant bilinear form is unique up to a factor.

(2) Show that \( \mathfrak{g} \cong \mathfrak{g}^* \) as representations of \( \mathfrak{g} \).

5.7. Let \( V \) be a finite-dimensional complex vector space and let \( A : V \to V \) be an upper-triangular operator. Let \( F^k \subset \text{End}(V) \), \( -n \leq k \leq n \) be the subspace spanned by matrix units \( E_{ij} \) with \( i-j \leq k \).

Show that then \( \text{ad} A.F^k \subset F^{k-1} \) and thus, \( \text{ad} A : \text{End}(V) \to \text{End}(V) \) is nilpotent.
Homework 8

6.1. Show that the Casimir operator for \( g = so(3, \mathbb{R}) \) is given by
\[ C = \frac{1}{2}(J_z^2 + J_y^2 + J_x^2); \]
thus, it follows that \( J_z^2 + J_y^2 + J_x^2 \in Uso(3, \mathbb{R}) \) is central.

6.2. Show that for \( g = \mathfrak{gl}(n, \mathbb{C}) \), the definition of a semisimple element (an element \( x \) such as \( \text{ad} x \) is a semisimple operator) coincides to the usual definition of a semisimple operator.

6.3. Show that if \( \mathfrak{h} \subset g \) is a Cartan subalgebra in a complex semisimple Lie algebra, then \( \mathfrak{h} \) is a nilpotent subalgebra which coincides with its normalizer \( n(\mathfrak{h}) = \{ x \in g \mid \text{ad} x.\mathfrak{h} \subset \mathfrak{h} \} \). (This is the usual definition of a Cartan subalgebra which can be used for any Lie algebra, not necessarily a semisimple one.)

6.4. Let \( g \) be a complex Lie algebra which has a root decomposition:
\[ g = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha \]
where \( R \) is a finite subset in \( \mathfrak{h}^* - \{0\} \), \( \mathfrak{h} \) is commutative and for \( h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha \), we have \( [h, x] = (h, \alpha)(x) \). Show that then \( g \) is semisimple, and \( \mathfrak{h} \) is a Cartan subalgebra.

6.5. Let \( \mathfrak{h} \subset so(4, \mathbb{C}) \) be the subalgebra consisting of matrices of the form
\[ \begin{bmatrix} a & \\ -a & b \\ & & -b \end{bmatrix} \]
(entries not shown are zeros). Show that then \( \mathfrak{h} \) is a Cartan subalgebra and find the corresponding root decomposition.

6.6.
1. Define a bilinear form \( B \) on \( W = \Lambda^2 \mathbb{C}^4 \) by \( \omega_1 \wedge \omega_2 = B(\omega_1, \omega_2) e_1 \wedge e_2 \wedge e_3 \wedge e_4 \). Show that \( B \) is a symmetric non-degenerate form and construct an orthonormal basis for \( B \).
2. Let \( g = so(W, B) = \{ x \in \mathfrak{gl}(W) \mid B(x\omega_1, \omega_2) + B(\omega_1, x\omega_2) = 0 \} \). Show that \( g \simeq so(6, \mathbb{C}) \).
3. Show that the form \( B \) is invariant under the natural action of \( sl(4, \mathbb{C}) \) on \( \Lambda^2 \mathbb{C}^4 \).
4. Using results of the previous parts, construct a homomorphism \( sl(4, \mathbb{C}) \rightarrow so(6, \mathbb{C}) \) and prove that it is an isomorphism.
Homework 10

7.1. Let $R \subset \mathbb{R}^n$ be given by

$$R = \{ \pm e_i, \pm 2e_i \mid 1 \leq i \leq n \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i, j \leq n, i \neq j \}$$

where $e_i$ is the standard basis in $\mathbb{R}^n$. Show that $R$ is a non-reduced root system. (This root system is usually denoted $BC_n$.)

7.2. (1) Let $R \subset E$ be a root system. Show that the set $R^\vee = \{ \alpha^\vee \mid \alpha \in R \} \subset E^*$ where $\alpha^\vee \in E^*$ is the coroot corresponding to $\alpha$ is also a root system. It is usually called the dual root system of $R$.

(2) Let $\Pi = \{ \alpha_1, \ldots, \alpha_r \} \subset R$ be the set of simple roots. Show that the set $\Pi^\vee = \{ \alpha_1^\vee, \ldots, \alpha_r^\vee \} \subset R^\vee$ is the set of simple roots of $R^\vee$. [Note: this is not completely trivial, as $\alpha \mapsto \alpha^\vee$ is not a linear map.]

7.4. Show that $|P/Q| = |\det A|$, where $A$ is the Cartan matrix: $a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$.

7.5. Compute explicitly the group $P/Q$ for root systems $A_n, D_n$.

7.8. Let $C_+$ be the closure of the positive Weyl chamber, and $\lambda \in C_+, w \in W$ be such that $w(\lambda) \in C_+$. Show that

(1) $\lambda \in C_+ \cap w^{-1}(C_+)$. 
(2) Let $L_\alpha \subset E$ be a root hyperplane which separates $C_+$ and $w^{-1}C_+$. Show that then $\lambda \in L_\alpha$. 
(3) Show that $w(\lambda) = \lambda$.

Deduce from this that every $W$-orbit in $E$ contains a unique element from $C_+$.

7.9. Let $w_0 \in W$ be the longest element in the Weyl group $W$. Show that then for any $w \in W$, we have $l(ww_0) = l(w_0w) = l(w_0) - l(w)$.

7.10. Let $W = S_n$ be the Weyl group of root system $A_{n-1}$. Show that the longest element $w_0 \in W$ is the permutation $w_0 = (n \ n-1 \ \ldots 1)$.

7.11. (1) Let $R$ be a reduced root system of rank 2, with simple roots $\alpha_1, \alpha_2$. Show that the longest element in the corresponding Weyl group is

$$w_0 = s_1s_2s_1 \cdots s_2s_1s_2 \cdots \quad (m \text{ factors in each of the products})$$

where $m$ depends on the angle $\varphi$ between $\alpha_1, \alpha_2$: $\varphi = \pi - \frac{\pi}{m}$ (so $m = 2$ for $A_1 \times A_1$, $m = 3$ for $A_2$, $m = 4$ for $B_2$, $m = 6$ for $G_2$). If you cannot think of any other proof, give a case-by-case proof.

(2) Show that the following relations hold in $W$ (these are called Coxeter relations):

$$s_i^2 = 1, \quad (s_is_j)^{m_{ij}} = 1,$$

where $m_{ij}$ is determined by the angle between $\alpha_i, \alpha_j$ in the same way as in the previous part.

(It can be shown that the Coxeter relations is a defining set of relations for the Weyl group: $W$ could be defined as the group generated by elements $s_i$ subject to Coxeter relations. A proof of this fact can be found in the book of Humphreys “Reflection groups and Coxeter groups”.)
Homework 11

7.7. Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced expression. Show that then
\[ \{ \alpha \in R_+ \mid w(\alpha) \in R_- \} = \{ \beta_1, \ldots, \beta_l \} \]
where $\beta_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$

7.12. Let $\varphi: R_1 \sim R_2$ be an isomorphism between irreducible root systems. Show that then $\varphi$ is a composition of an isometry and a scalar operator: $(\varphi(v), \varphi(w)) = c(v, w)$ for any $v, w \in E_1$.

7.13.
(1) Let $n_\pm$ be the positive and negative nilpotent subalgebras in a semisimple complex Lie algebra. Show that $n_\pm$ are indeed nilpotent.
(2) Let $b = n_+ \oplus \mathfrak{h}$. Show that $b$ is solvable.

7.14.
(1) Show that if two vertices in a Dynkin diagram are connected by a single edge, then the corresponding simple roots are in the same $W$-orbit.
(2) Show that for a reduced irreducible root system, the Weyl group acts transitively on the set of all roots of the same length.

7.15. Let $R \subset E$ be an irreducible root system. Show that then $E$ is an irreducible representation of the Weyl group $W$.

7.16. Let $G$ be a connected complex Lie group such that $\mathfrak{g} = \text{Lie}(G)$ is semisimple. Fix a root decomposition of $\mathfrak{g}$.

(1) Choose $\alpha \in R$ and let $i_\alpha: \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ be the embedding corresponding to the root $\alpha$. This embedding can be lifted to a morphism $i_\alpha: \text{SL}(2, \mathbb{C}) \to G$.

Let
\[ S_\alpha = i_\alpha \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \exp \left( \frac{\pi i}{2} (f_\alpha - e_\alpha) \right) \in G \]

Show that $\text{Ad} S_\alpha(h) = -h$ and that $\text{Ad} S_\alpha(h) = h$ if $h \in \mathfrak{h}$, $\langle h, \alpha \rangle = 0$. Deduce from this that the action of $S_\alpha$ on $\mathfrak{g}^*$ preserves $\mathfrak{h}^*$ and that restriction of $\text{Ad} S_\alpha$ to $\mathfrak{h}^*$ coincides with the reflection $s_\alpha$.

(2) Show that the Weyl group $W$ acts on $\mathfrak{h}^*$ by inner automorphisms: for any $w \in W$, there exists an element $\tilde{w} \in G$ such that $\text{Ad} \tilde{w}|_{\mathfrak{h}^*} = w$. [Note, however, that in general, $\tilde{w}_1 \tilde{w}_2 \neq \tilde{w}_1 \tilde{w}_2$.]