27. **Representations of GL_n, I**

We begin with a more detailed study of finite dimensional representations of semisimple Lie algebras and the corresponding complex Lie groups.

27.1. **Tensor products of fundamental representations.** The following result shows that if we understand fundamental representations of a semisimple Lie algebra $\mathfrak{g}$ (i.e., irreducible representations with fundamental highest weights ω_i), we can gain some insight into general finite dimensional representations.

Proposition 27.1. *Let $\lambda = \sum_{i=1}^r m_i\omega_i$ be a dominant integral weight for $\mathfrak{g}$. Consider the tensor product $T_\lambda := \otimes_i L_{\omega_i}^{\otimes m_i}$, and let $v := \otimes_i v_{\omega_i}^{\otimes m_i}$ be the tensor product of the highest weight vectors. Let V be the subrepresentation of T_λ generated by v. Then $V \cong L_\lambda$.*

Proof. We have $V = L_\lambda \oplus \bigoplus_{\mu\in(\lambda-Q_+)\cap P_+} N_{\lambda\mu}L_\mu$ where $N_{\lambda\mu}$ are positive integers. Let $C \in U(\mathfrak{g})$ be the Casimir element for $\mathfrak{g}$. Recall that $C|_{L_\mu} = (\mu, \mu + 2\rho)$. Thus $C|_V = (\lambda, \lambda + 2\rho)$. But we have seen in the proof of the Weyl character formula that for any $\mu \in (\lambda - Q_+) \cap P_+$ such that $\mu \neq \lambda$, we have $(\mu, \mu + 2\rho) < (\lambda, \lambda + 2\rho)$. Therefore we see that $N_{\lambda\mu} = 0$ for $\mu \neq \lambda$. $\square$

27.2. **Representations of $SL_n(\mathbb{C})$.** Let us now discuss more explicitly the representation theory of $SL_n(\mathbb{C})$. We will consider its finite dimensional complex analytic representations as a complex Lie group. We have shown that this is equivalent to considering finite dimensional representations of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. We have also seen that these are completely reducible and the irreducible representations are L_λ, where $\lambda = \sum_{i=1}^{n-1} m_i\omega_i$, ω_i are the fundamental weights, and $m_i \in \mathbb{Z}_{\geq 0}$.

First let us compute ω_i. Recall that the standard Cartan subalgebra $\mathfrak{h}$ is the space $\mathbb{C}_0^n$ of vectors in $\mathbb{C}^n$ with zero sum of coordinates (diagonal matrices with trace zero). So elements of $\mathfrak{h}^*$ can be viewed as vectors $(x_1, ..., x_n) \in \mathbb{C}^n$ modulo simultaneous shift of all coordinates by the same number (i.e., $\mathfrak{h}^* = \mathbb{C}^n/\mathbb{C}_{\mathrm{diag}}$).

Recall that the simple roots are $\alpha_i^\vee = \mathbf{e}_i - \mathbf{e}_{i+1}$. Thus ω_i are determined by the conditions

$$(\omega_i, \mathbf{e}_j - \mathbf{e}_{j+1}) = \delta_{ij}.$$

This means that $\omega_i = (1, ..., 1, 0, ..., 0)$ where there are i copies of 1. Thus a dominant integral weight λ has the form

$$\lambda = (m_1 + ... + m_{n-1}, m_2 + ... + m_{n-1}, ..., m_{n-1}, 0).$$

So dominant integral weights are parametrized by non-increasing sequences $\lambda_1 \geq ... \geq \lambda_{n-1}$ of nonnegative integers. This agrees with the representation theory of $SL_2(\mathbb{C})$ that we worked out before: in this case the sequence has just one term.

Let us now describe explicitly the fundamental representations L_{ω_i}. Consider first the representation $V = \mathbb{C}^n$ with the usual action of matrices. It is called the **vector representation** or the **tautological representation** (as every matrix goes to itself). It is irreducible and has a standard basis $v_1, ..., v_n$. To find its highest weight, we have to find a vector $v \neq 0$ such that $e_i v = 0$. As $e_i = E_{i,i+1}$, we have $v = v_1$. It is easy to see that $hv = \omega_1(h)v$, so we see that v has weight ω_1, hence $L_{\omega_1} = V$.

To construct L_{ω_m} for $m > 1$, consider the exterior power $\wedge^m V$. It is easy to show that it is irreducible. A basis of $\wedge^m V$ consists of wedges $v_{i_1} \wedge ... \wedge v_{i_m}$ where $i_1 < ... < i_m$. The highest weight vector is clearly $v_1 \wedge ... \wedge v_m$, and it has weight ω_m. Thus $L_{\omega_m} = \wedge^m V$.

Note that $\wedge^n V = \mathbb{C}$ (the trivial representation) since every matrix in $SL_n(\mathbb{C})$ acts by its determinant, which is 1, and $\wedge^m V = 0$ for $m > n$. Also $V^* \cong \wedge^{n-1} V$ since the wedge pairing $V \otimes \wedge^{n-1} V \to \wedge^n V = \mathbb{C}$ is invariant and nondegenerate. Similarly, $\wedge^m V^* \cong \wedge^{n-m} V$.

We now see from Proposition 27.1 that the irreducible representation L_λ for $\lambda = \sum_i m_i \omega_i$ is generated inside $\otimes_{i=1}^{n-1} (\wedge^i V)^{\otimes m_i}$ by the tensor product of the highest weight vectors.

Example 27.2. $L_{N\omega_1} = S^N V$, generated by the vector $v_1^{\otimes N} \in V^{\otimes N}$.

27.3. **Representations of $GL_n(\mathbb{C})$.** Let us now explain how to extend these results to $GL_n(\mathbb{C})$. This is easy to do since $GL_n(\mathbb{C})$ is not very different from the direct product $\mathbb{C}^\times \times SL_n(\mathbb{C})$. Namely, $GL_n(\mathbb{C}) = (\mathbb{C}^\times \times SL_n(\mathbb{C}))/\mu_n$ where μ_n is the group of roots of unity of order n embedded as $z \mapsto (z^{-1}, z\mathbf{1}_n)$. Indeed, the corresponding covering homomorphism $\mathbb{C}^\times \times SL_n(\mathbb{C}) \to GL_n(\mathbb{C})$ is given by $(z, A) \mapsto zA$. So it suffices to classify irreducible holomorphic representations of the complex Lie group $\mathbb{C}^\times \times SL_n(\mathbb{C})$; the irreducible holomorphic representations of $GL_n(\mathbb{C})$ are a subset of them.

For $n = 1$ this is just the problem of describing the holomorphic representations of $\mathbb{C}^\times$. This is easy. The Lie algebra is spanned by a single element h such that $e^{2\pi i h} = 1$. This element must act in a representation by an operator H such that $e^{2\pi i H} = 1$. It follows that H is diagonalizable with integer eigenvalues. Thus representations of $\mathbb{C}^\times$ are completely reducible, with irreducibles χ_N one-dimensional and labeled by integers $N \in \mathbb{Z}$, $\chi_N(z) = z^N$.

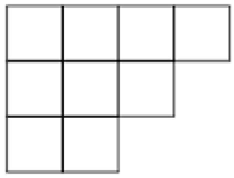
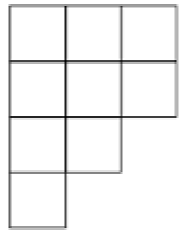
We have a similar answer for $\mathbb{C}^\times \times SL_n$: representations are completely reducible with irreducibles being $L_{\lambda,N} = \chi_N \otimes L_\lambda$. Moreover, the ones factoring through GL_n just have $N = nr + \sum_{i=1}^{n-1} \lambda_i$ for some integer r.

Recall that GL_n has reductive Lie algebra $\mathfrak{gl}_n$ with Cartan subalgebra $\mathfrak{h} = \mathbb{C}^n$. The highest weight of $L_{\lambda, nm_n + \sum_{i=1}^{n-1} \lambda_i}$ is easily computed and equals $(m_1 + ... + m_{n-1} + m_n, ..., m_{n-1} + m_n, m_n)$. Thus highest weights of finite dimensional representations are non-increasing sequences $(\lambda_1, ..., \lambda_n)$ of integers which don't have to be positive. The fundamental representations are still $L_{\omega_m} = \wedge^m V$, and the only difference with SL_n is that now the top exterior power $\wedge^n V$ is not trivial but rather is a 1-dimensional **determinant character** with highest weight $\omega_n = (1, ..., 1)$. The highest weight of a finite dimensional representation then has the form $\lambda = \sum_{i=1}^n m_i \omega_i$, where $m_i \geq 0$ for $i \neq n$, while m_n is an arbitrary integer. Consequently, L_λ is found inside $\otimes_{i=1}^n (\wedge^i V)^{\otimes m_i}$ as the representation generated by the product of highest weight vectors. Note that it makes sense to take $m_n < 0$, as for a one-dimensional representation and $k < 0$ it is natural to define $\chi^{\otimes k} := (\chi^*)^{\otimes -k}$.

The representations with $m_n \geq 0$ are especially important; it is easy to see that these are exactly the ones that occur inside $V^{\otimes N}$ for some N (check it!). These representations are called **polynomial** since their matrix coefficients are polynomial functions of the matrix entries x_{ij} of $X \in GL_n(\mathbb{C})$, and consequently they extend by continuity to representations of the semigroup $\mathrm{Mat}_n(\mathbb{C}) \supset GL_n(\mathbb{C})$. Note that any irreducible representation is a polynomial one tensored with a nonpositive power of the determinant character $\wedge^n V$.

27.4. **Schur-Weyl duality.** Note that highest weights of polynomial representations are non-increasing sequences of nonnegative integers $(\lambda_1, ..., \lambda_n)$, i.e. **partitions** with $\leq n$ parts. Namely, they are partitions of $|\lambda| = \sum_i \lambda_i$, which is just the eigenvalue of $\mathbf{1}_n \in \mathfrak{gl}_n$ on L_λ and can also be defined as the number N such that L_λ occurs in $V^{\otimes N}$.

Traditionally partitions are encoded by **Young diagrams.** Namely, the Young diagram of a partition $\lambda = (\lambda_1, ..., \lambda_n)$ consists of n rows of boxes, the i-th row consisting of λ_i boxes, so that row i is placed directly under row $i-1$ and all rows start on the same vertical line. For example, here are the Young diagrams of the partitions $(4, 3, 2)$ (left) and $(3, 3, 2, 1)$ (right):

Thus we have

$$V^{\otimes N} = \oplus_{\lambda:|\lambda|=N} L_\lambda \otimes \pi_\lambda,$$

where $\pi_\lambda := \mathrm{Hom}_{GL_n(\mathbb{C})}(L_\lambda, V^{\otimes N})$ are multiplicity spaces. Here the summation is over partitions of N, and $L_\lambda = 0$ if λ has more than n parts. To understand the spaces π_λ, note that the symmetric group S_N acts on $V^{\otimes N}$ and commutes with $GL_n(\mathbb{C})$, so it gets to act on each π_λ.

Let A be the image of $U(\mathfrak{gl}_n)$ in $\mathrm{End}_{\mathbb{C}}(V^{\otimes N})$, and B be the image there of $\mathbb{C}S_N$. The algebras A, B commute.

Theorem 27.3. *(Schur-Weyl duality) (i) The centralizer of A is B and vice versa.*

(ii) If λ has at most n parts then the representation π_λ of B (hence S_N) is irreducible, and such representations are pairwise non-isomorphic.

(iii) If $\dim V \geq N$ then π_λ exhaust all irreducible representations of S_N.

Proof. We start with

Lemma 27.4. *If U is a $\mathbb{C}$-vector space then $S^N U$ is spanned by elements $x \otimes ... \otimes x$, $x \in U$.*

Proof. It suffices to consider the case when U is finite dimensional. Then the span of these vectors is a nonzero subrepresentation in the irreducible $GL(U)$-representation $S^N U$, which implies the statement. □

Lemma 27.5. *For any associative algebra R over $\mathbb{C}$, the algebra $S^N R$ is generated by elements*

$$\Delta_N(x) := x \otimes 1 \otimes ... \otimes 1 + 1 \otimes x \otimes ... \otimes 1 + ... + 1 \otimes ... \otimes 1 \otimes x$$

for $x \in R$.

Proof. Let P_N be the Newton polynomial expressing $z_1...z_N$ via $p_k := \sum_{i=1}^N z_i^k$, $k = 1, ..., N$ (it exists and is unique by the fundamental theorem on symmetric functions). Then we have

$$x \otimes ... \otimes x = P_N(\Delta_N(x), ..., \Delta_N(x^N)).$$

Hence the lemma follows from Lemma 27.4. □

Let us now show that A is the centralizer Z_B of B. Note that $Z_B = S^N(\mathrm{End}V)$. Thus the statement follows from Lemma 27.5.

We will now use the following easy but important lemma (which actually holds over any field).

Lemma 27.6. *(Double centralizer lemma) Let V be a finite dimensional vector space and $A, B \subset \mathrm{End}V$ be subalgebras such that B is isomorphic to a direct sum of matrix algebras and A is the centralizer of B. Then A is also isomorphic to a direct sum of matrix algebras, and moreover*

$$V = \oplus_{i=1}^n W_i \otimes U_i,$$

where W_i run through all irreducible A-modules and U_i through irreducible B-modules. In particular, B is the centralizer of A and we have a natural bijection between irreducible A-modules and irreducible B-modules which matches W_i and U_i.

Proof. We have $V = \oplus_{i=1}^n W_i \otimes U_i$ where U_i run through irreducible representations of B and $W_i = \mathrm{Hom}_B(U_i, V) \neq 0$ are multiplicity spaces. Thus $A = \oplus_{i=1}^n \mathrm{End}W_i$ and $B = \oplus_{i=1}^n \mathrm{End}U_i$, which implies the statement. □

Since the algebra B is a direct sum of matrix algebras (by complete reducibility of representations of finite groups), Lemma 27.6 yields (i).[12]

To prove (ii), it suffices to note that if λ has $\leq n$ parts then L_λ occurs in $V^{\otimes N}$, so $\pi_\lambda \neq 0$. The rest follows from (i) and Lemma 27.6.

(iii) If $\dim V \geq N$ then pick N linearly independent vectors $v_1, ..., v_N \in V$. It is easy to see that the map $\mathbb{C}S_N \to V^{\otimes N}$ defined by $s \mapsto s(v_1 \otimes ... \otimes v_N)$ is injective. Thus $B = \mathbb{C}S_N$. This implies the statement. □

Remark 27.7. The algebra A is called the **Schur algebra** and B the **centralizer algebra**.

Thus we see that representations of S_N are labeled by partitions λ of N, and those that occur in $V^{\otimes N}$ correspond to the partitions that have $\leq \dim V$ parts. Moreover, we claim that this labeling of representations by partitions does not depend on $\dim V$. To show this, suppose λ has $\leq n$ parts and $V = \mathbb{C}^n$. We have the Schur-Weyl decomposition of $GL_{n+1}(\mathbb{C}) \times S_N$-modules

$$(V \oplus \mathbb{C})^{\otimes N} = \oplus_\mu L_\mu^{(n+1)} \otimes \pi_\mu^{(n+1)},$$

[12]This also gives another proof of the fact that A is a direct sum of matrix algebras, i.e. complete reducibility of $V^{\otimes N}$.

Let us restrict this sum to $GL_n(\mathbb{C}) \times S_N$, and consider what happens to the summand $L_\lambda^{(n+1)} \otimes \pi_\lambda^{(n+1)}$. The highest weight vector v in $L_\lambda^{(n+1)}$ tensored with any element w of $\pi_\lambda^{(n+1)}$ sits in $V^{\otimes N} \subset (V \oplus \mathbb{C})^{\otimes N}$, since the $n+1$-th component of its weight is zero. Hence $v \otimes w$ generates a copy of $L_\lambda^{(n)} \otimes \pi_\lambda^{(n)}$ as a $GL_n(\mathbb{C}) \times S_N$-module. This implies that $\pi_\lambda^{(n+1)} \cong \pi_\lambda^{(n)}$.

Exercise 27.8. Let $R = \mathbb{C}[x_1, ..., x_N, y_1, ..., y_N]^{S_N}$ (the algebra of invariant polynomials). Show that R is generated by the elements $Q_{rs} := \sum_{i=1}^N x_i^r y_i^s$ where $1 \le r + s \le N$.

Exercise 27.9. Let $\lambda = (\lambda_1, ..., \lambda_n)$ be a partition. Let us fill the Young diagram of λ with numbers, placing $c(i,j) := i - j$ in the j-th box in the i-th row. Thus the number written in each box depends only of its position (i,j); it is called the **content** of this box. The **content of** λ is the sum $c(\lambda)$ of contents of all its boxes:

$$c(\lambda) = \sum_{(i,j)\in\lambda} c(i,j).$$

(i) Show that

$$c(\lambda) = \sum_{i=1}^n \frac{\lambda_i(\lambda_i - 2i + 1)}{2}.$$

(ii) Let $\mathbf{c} = \sum_{1 \le i < j \le N} (ij) \in \mathbb{C}S_N$ be the **Jucys-Murphy element** (the sum of all transpositions). Show that $\mathbf{c}$ is a central element of $\mathbb{C}S_N$ which acts on the irreducible representation π_λ of S_N by the scalar $c(\lambda)$. (**Hint:** Consider the action of $\mathbf{c}$ on $V^{\otimes N}$ and use Schur-Weyl duality to relate it to the diagonal action of the quadratic Casimir of $\mathfrak{gl}_n$).

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