28. **Representations of GL_n, II**

28.1. **Schur functors.**

Definition 28.1. For a partition λ of N we define the **Schur functor** S^λ on the category of complex vector spaces (or complex representations of any group or Lie algebra) by $S^\lambda V = \mathrm{Hom}_{S_N}(\pi_\lambda, V^{\otimes n})$.

Thus we have
$$V^{\otimes N} = \oplus_\lambda S^\lambda V \otimes \pi_\lambda,$$
and if λ has $\leq n$ parts and $V = \mathbb{C}^n$ then $S^\lambda V = L_\lambda$ as a representation of $GL(V) = GL_n(\mathbb{C})$.

Example 28.2. 1. We have $S^{(n)}V = S^n V$, $S^{(1^n)}V = \wedge^n V$.

2. We have
$$V \otimes V = S^{(2)}V \otimes \mathbb{C}_+ \oplus S^{(1,1)}V \otimes \mathbb{C}_- = S^2 V \oplus \wedge^2 V$$
where S_2 acts in the first summand trivially and in the second one by sign.

Consider now the decomposition of $V \otimes V \otimes V$. We have
$$V \otimes V \otimes V = S^{(3)}V \otimes \mathbb{C}_+ \oplus S^{(2,1)}V \otimes \mathbb{C}^2 \oplus S^{(1,1,1)}V \otimes \mathbb{C}_-$$
$$= S^3 V \oplus S^{(2,1)}V \otimes \mathbb{C}^2 \oplus \wedge^3 V.$$
Thus
$$S^2 V \otimes V = S^3 V \oplus S^{(2,1)}V, \ \wedge^2 V \otimes V = \wedge^3 V \oplus S^{(2,1)}V.$$
We conclude that $S^{(2,1)}V$ can be described as the space of tensors symmetric in the first two components whose full symmetrization is zero, or tensors antisymmetric on the first two components whose full antisymmetrization is zero.

Exercise 28.3. 1. Let $V = \mathbb{C}^n$, $n \geq 4$. Decompose $V^{\otimes 4}$ as a direct sum of irreducible representations of $GL_n(\mathbb{C}) \times S_4$. Characterize the occurring Schur functors as spaces of tensors with certain symmetry properties, similarly to the above description of $S^{(2,1)}V$. Compute the decompositions of $V \otimes S^3 V$, $V \otimes \wedge^3 V$, $S^2 V \otimes S^2 V, S^2 V \otimes \wedge^2 V$ and $\wedge^2 V \otimes \wedge^2 V$ into Schur functors.

2. Decompose $V \otimes V^*$, $V \otimes V \otimes V^*$ into a direct sum of irreducible representations. Describe the algebra $\mathrm{End}_{GL_n(\mathbb{C})}(V \otimes V^* \otimes V^*)$.

Let us compute the dimension of $S^\lambda V$ when $\dim V = N$ and λ has k parts. We have $\rho = (N-1, N-2, ..., 1, 0)$ (for SL_N), so the Weyl dimension formula tells us that
$$\dim S^\lambda V = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i} =$$

$$\prod_{1\le i<j\le k} \frac{\lambda_i - \lambda_j + j - i}{j-i} \prod_{1\le i\le k<j\le N} \frac{\lambda_i + j - i}{j-i} =$$

$$\prod_{1\le i<j\le k} \frac{\lambda_i - \lambda_j + j - i}{j-i} \prod_{i=1}^{k} \frac{(N+1-i)...(N+\lambda_i - i)}{(k+1-i)...(k+\lambda_i - i)}.$$

We obtain

Proposition 28.4. $\dim S^\lambda V = P_\lambda(N)$ *where P_λ is a polynomial of degree $|\lambda|$ with rational coefficients and integer roots. Moreover, the roots of P_λ are all the integers in the interval $[1-\lambda_1, k-1]$ (occurring with multiplicities).*

Moreover, we see that $P_\lambda(N)$ is an integer-valued polynomial, i.e., it takes integer values at integer points (this is equivalent to being an integer linear combination of $\binom{N}{j}$)).

Example 28.5.

$$P_{(n)}(N) = \dim S^n V = \binom{N+n-1}{n},\ P_{(1^n)}(N) = \dim \wedge^n V = \binom{N}{n}.$$

Also

$$P_{(a,b)}(N) = (a-b+1)\frac{N...(N+a-1)\cdot(N-1)...(N+b-2)}{(a+1)!b!} =$$

$$\frac{a-b+1}{a+1}\binom{N+a-1}{a}\binom{N+b-2}{b}$$

E.g., $P_{(2,1)}(N) = \dim S^{(2,1)}V = \frac{N(N+1)(N-1)}{3}$. Also,

$$P_{(a,a)}(N) = \frac{1}{a+1}\binom{N+a-1}{a}\binom{N+a-2}{a} =$$

$$\frac{1}{N+a-1}\binom{N+a-1}{N-1}\binom{N+a-2}{N-2} = \mathrm{Nar}(N+a-1, N-1),$$

the **Narayana numbers**.

Exercise 28.6. Let g_q be the diagonal matrix with diagonal elements $1, q, q^2, ..., q^{N-1}$. Compute the trace of g_q in $S^\lambda V$ in the product form. Write the answer explicitly (as a polynomial in q) with positive coefficients in the case $|\lambda| \le 3$.

Exercise 28.7. Draw the weights of the representation $S^{(2,2)}\mathbb{C}^3$ of $SL(3)$ on the hexagonal lattice, and indicate their multiplicities.

28.2. **The fundamental theorem of invariant theory.** Suppose we have a finite dimensional vector space V and a collection of tensors $T_i \in V^{\otimes m_i} \otimes V^{*\otimes n_i}$, $i = 1, ..., k$. An important problem is to describe "coordinate free" invariants of such a collection of tensors, i.e., polynomials functions $F(T_1, ..., T_k)$ which are invariant under the action of $GL(V)$. How can we classify such functions? This sounds formidably hard in such generality, but turns out to be very easy using Schur-Weyl duality.

It suffices to study such functions that have homogeneity degree d_i with respect to each T_i. To do so, we will depict each T_i by a vertex with m_i incoming and n_i outgoing arrows. We should think of incoming arrows as V-components and outgoing ones as V^*-components. Let us draw d_i such vertices for each i. To construct an invariant, let us connect the arrows preserving orientation so that all the arrows are used (this will only be possible if the number of incoming arrows equals the number of outgoing ones; otherwise every invariant of the multidegree $(d_1, ..., d_k)$ will be zero). To the obtained graph Γ we can assign the **convolution** of tensors, which gives an invariant function F_Γ of the correct multidegree.

Theorem 28.8. *The functions F_Γ for various Γ span the space of invariant functions.*

Proof. An invariant function may be viewed as an element of the space $\bigotimes_{i=1}^{k}(V^{*\otimes m_i} \otimes V^{\otimes n_i})^{\otimes d_i}$, which we may write as the space of linear maps $V^{\otimes M} \to V^{\otimes N}$, where $M = \sum d_i m_i$ is the number of incoming arrows and $N = \sum d_i n_i$ the number of outgoing arrows. If $M \neq N$, there are no nonzero invariant maps. Otherwise, by the Schur-Weyl duality, the space of such maps is spanned by maps defined by permutations. But any such permutation defines a graph Γ, so the corresponding invariant is just the convolution F_Γ, which implies the statement. $\square$

Remark 28.9. Note that this proof also implies that if $\dim V$ is large compared to m_i, n_i, d_i then the functions F_Γ for non-isomorphic graphs Γ are linearly independent, so they form a basis in the algebra of A of invariant functions. (Here the vertices of Γ are colored by k colors corresponding to the types of tensors, and at every vertex of color i the incoming edges are labeled by $[1, n_i]$ and outgoing edges by $[1, m_i]$. Isomorphisms are required to preserve these colorings and labelings).

Example 28.10. Assume that $m_i = n_i = 1$, i.e., $T_1, ..., T_k$ are just matrices with GL_n acting by conjugation. Then all graphs that we can get are unions of cycles, so Theorem 28.8 implies that the algebra $A_{k,n}$ of such invariants (where $n = \dim V$) is generated by traces of cyclic words

$$F_{j_1,...,j_r} = \mathrm{Tr}(T_{j_1}...T_{j_r})$$

(here "cyclic" means that words differing by a cyclic permutation are considered to be the same). Moreover, by Remark 28.9, these elements are "asymptotically algebraically independent", i.e. there is no nonzero polynomial of them that vanishes for all sizes of matrices n.

This implies that there are no universal polynomial identities for matrices of all sizes. Indeed, if $P(T_1, ..., T_k) = 0$ for square matrices $T_1, ..., T_k$ of any size n (where P is a fixed nonzero noncommutative polynomial) then adding another matrix T_{k+1}, we get $\mathrm{Tr}(P(T_1, ..., T_k)T_{k+1}) = 0$, which contradicts linear independence of $F_{j_1,...,j_r}$.

In particular, this implies that the universal Lie polynomials $\mu_n(x, y)$ of degree n occurring in the Baker-Campbell-Hausdorff formula, i.e., such that

$$\sum_{m\geq 1} \frac{\mu_m(x, y)}{n!} = \log(\exp(x)\exp(y))$$

for $x \in \mathrm{Lie}(G)$ for any Lie group G, are unique (in fact, they are already unique for the family of groups $GL_n(\mathbb{C})$ for all n).

This is false, however, if the size of matrices is fixed; in this case there are plenty of polynomial identities for each matrix size. For example, for matrices of size 1 we have $[X, Y] = 0$ and for matrices of size 2 we have $[Z, [X, Y]^2] = 0$. For general n there is the Amitsur-Levitzki identity given in Exercise 28.11.

Exercise 28.11. Let $X_1, ..., X_{2n}$ be complex n by n matrices. Let $\Lambda = \wedge(\xi_1, ..., \xi_{2n})$ be the exterior algebra generated by ξ_i with relations $\xi_i\xi_j = -\xi_j\xi_i, \xi_i^2 = 0$. Let X be the matrix over Λ given by

$$X := X_1\xi_1 + ... + X_{2n}\xi_{2n}.$$

(i) Let $Y = X^2$. Show that $Y \in \mathrm{Mat}_n(\Lambda_+)$ where Λ_+ is the commutative subalgebra of Λ spanned by the elements of even degrees. Compute Y^n.

(ii) Show that $\mathrm{Tr}(Y^k) = 0 \in \Lambda_+$ for $k = 1, ..., n$.

(iii) Deduce that $Y^n = 0$. This should yield the **Amitsur-Levitzki identity**

$$\sum_{\sigma\in S_{2n}} \mathrm{sign}(\sigma)X_{\sigma(1)}...X_{\sigma(2n)} = 0.$$

(iv) Deduce the same identity over any commutative ring R.

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