29. **Representations of GL_n, III**

29.1. **Schur polynomials and characters of representations of the symmetric group.** Using Schur-Weyl duality and the character formula for representations of GL_n, we can obtain information about characters of the symmetric group. Namely, it follows from the Weyl character formula that the characters of representations of GL_n are given by the formula

$$s_\lambda(x_1, ..., x_n) = \frac{\sum_{\sigma \in S_n} \mathrm{sign}(\sigma) x_{\sigma(1)}^{\lambda_1+N-1} ... x_{\sigma(n)}^{\lambda_n}}{\prod_{i<j}(x_i - x_j)} = \frac{\det(x_i^{\lambda_j+N-j})}{\prod_{i<j}(x_i - x_j)}.$$

These symmetric polynomials are called **Schur polynomials**. For example, the character of $S^m V$ is

$$s_{(m)}(x_1, ..., x_n) = \sum_{1 \le j_1 \le ... \le j_m \le n} x_{j_1} ... x_{j_m} = h_m(x_1, ..., x_n),$$

the m-th **complete symmetric function**, and the character of $\wedge^m V$ is

$$s_{(1^m)}(x_1, ..., x_n) = \sum_{1 \le j_1 < ... < j_m \le n} x_{j_1} ... x_{j_m} = e_m(x_1, ..., x_m),$$

the m-th **elementary symmetric function**.

Let us now compute the trace in $V^{\otimes N}$ of $x \otimes \sigma$, where $x = \mathrm{diag}(x_1, ..., x_n)$ is a diagonal matrix and $\sigma \in S_N$ a permutation. Let σ have m_i cycles of length i. Then we have

$$\mathrm{Tr}|_{V^{\otimes N}}(x \otimes \sigma) = \prod_i (x_1^i + ... + x_n^i)^{m_i}.$$

On the other hand, using Schur-Weyl duality, we get

$$\mathrm{Tr}|_{V^{\otimes N}}(x \otimes \sigma) = \sum_\lambda \chi_\lambda(\sigma) s_\lambda(x),$$

where $\chi_\lambda(\sigma) = \mathrm{Tr}|_{\pi_\lambda}(\sigma)$ is the character of the representation π_λ of S_N. Thus we have

$$\sum_\lambda \chi_\lambda(\sigma) s_\lambda(x) = \prod_i (x_1^i + ... + x_n^i)^{m_i}.$$

Multiplying this by the discriminant, we get

$$\sum_\lambda \chi_\lambda(\sigma) \det(x_i^{\lambda_j+N-j}) = \prod_{i<j}(x_i - x_j) \cdot \prod_i (x_1^i + ... + x_n^i)^{m_i}.$$

Thus we get

Theorem 29.1. *(Frobenius character formula) The character value $\chi_\lambda(\sigma)$ is the coefficient of $x_1^{\lambda_1+N-1}...x_N^{\lambda_N}$ in the polynomial*

$$\prod_{i<j}(x_i - x_j) \cdot \prod_i (x_1^i + ... + x_n^i)^{m_i}.$$

Exercise 29.2. Let $V = \mathbb{C}^2$ be the 2-dimensional tautological representation of $GL_2(\mathbb{C})$. Decompose $V^{\otimes N}$ into a direct sum of irreducible representations of $GL_2(\mathbb{C}) \times S_N$ and compute the characters and dimensions of all the irreducible representations of GL_2 and S_N that occur.

29.2. **Howe duality.** Howe duality is another instance when we have a double centralizer property. Consider two finite dimensional complex vector spaces V, W, and consider the symmetric power $S^n(V \otimes W)$ as a representation of $GL(V) \times GL(W)$.

Theorem 29.3. *(Howe duality) We have a decomposition*

$$S^n(V \otimes W) = \oplus_{\lambda:|\lambda|=n} S^\lambda V \otimes S^\lambda W.$$

Note that if λ has more parts than $\dim V$ or $\dim W$ then the corresponding summand is zero.

Proof. We have

$$S^n(V \otimes W) = ((V \otimes W)^{\otimes n})^{S_n} = (V^{\otimes n} \otimes W^{\otimes n})^{S_n}$$

So using the Schur-Weyl duality, we get

$$S^n(V \otimes W) = ((\oplus_{\lambda:|\lambda|=n} S^\lambda V \otimes \pi_\lambda) \otimes (\oplus_{\mu:|\mu|=n} S^\mu W \otimes \pi_\mu))^{S_n} =$$

$$\oplus_{\lambda,\mu:|\lambda|=|\mu|=n} S^\lambda V \otimes S^\mu W \otimes (\pi_\lambda \otimes \pi_\mu)^{S_n}.$$

But the character of π_λ is integer-valued, so $\pi_\lambda = \pi_\lambda^*$. Thus by Schur's lemma $(\pi_\lambda \otimes \pi_\mu)^{S_n} = \mathbb{C}^{\delta_{\lambda\mu}}$, and we get

$$S^n(V \otimes W) = \oplus_{\lambda:|\lambda|=n} S^\lambda V \otimes S^\lambda W,$$

as claimed. ∎

Note that we never used that V, W were finite dimensional, so the statement is valid for any complex vector spaces V, W.

Corollary 29.4. *(Cauchy identity) If $x = (x_1, ..., x_r)$ and $y = (y_1, ..., y_s)$ then one has*

$$\sum_\lambda s_\lambda(x) s_\lambda(y) z^{|\lambda|} = \prod_{i=1}^r \prod_{j=1}^s \frac{1}{1 - z x_i y_j}.$$

Proof.

Lemma 29.5. *(Molien formula). Let $A : V \to V$ be a linear operator on a finite dimensional vector space V. Denote by $S^n A$ the induced linear operator $A^{\otimes n}$ on $S^n V$. Then*

$$\sum_{n=0}^{\infty} \mathrm{Tr}(S^n A) z^n = \frac{1}{\det(1 - zA)}.$$

Proof. Let A have eigenvalues $x_1, ..., x_r$. Then the eigenvalues of $S^n A$ are all possible monomials in x_i of degree r. Thus $\mathrm{Tr}(S^n A)$ is the sum of these monomials, which is the complete symmetric function $h_n(x_1, ..., x_r)$. So

$$\sum_{n=0}^{\infty} \mathrm{Tr}(S^n A) z^n = \sum_{n \geq 0} h_n(x_1, ..., x_r) z^n = \prod_{i=1}^{r} \frac{1}{1 - z x_i} = \frac{1}{\det(1 - zA)}.$$

$\square$

Now let $X \in GL(V)$ with eigenvalues $x_1, ..., x_r$ and $Y \in GL(W)$ with eigenvalues $y_1, ..., y_s$. Then by Howe duality

$$\mathrm{Tr}(S^n(X \otimes Y)) = \sum_{\lambda : |\lambda| = n} s_\lambda(x) s_\lambda(y).$$

On the other hand, by Molien's formula

$$\sum_{n \geq 0} \mathrm{Tr}(S^n(X \otimes Y)) z^n = \frac{1}{\det(1 - z(X \otimes Y))} = \prod_{i,j} \frac{1}{1 - z x_i y_j}.$$

Comparing the two formulas, we obtain the statement. $\square$

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