30. **Fundamental and minuscule weights**

30.1. **Minuscule weights.** Let $\mathfrak{g}$ be a simple complex Lie algebra. Minuscule weights for $\mathfrak{g}$ are highest weights for which irreducible representations are especially simple.

Definition 30.1. A dominant integral weight ω for $\mathfrak{g}$ is called **minuscule** if $(\omega, \beta) \le 1$ for all positive coroots β.

Equivalently, $|(\omega, \beta)| \le 1$ for any coroot β.

Obviously, $\omega = 0$ is minuscule, but there may exist other minuscule weights. For example, for $\mathfrak{g} = \mathfrak{sl}_n$, all fundamental weights are minuscule, since $(\omega_i, \mathbf{e}_j - \mathbf{e}_k) = 0$ if $j, k \le i$ or $j, k > i$ and $(\omega_i, \mathbf{e}_j - \mathbf{e}_k) = 1$ if $j \le i < k$.

It is easy to see that any minuscule weight $\omega \neq 0$ is fundamental. Indeed, we can have $(\omega, \alpha_i^\vee) = 1$ only for one i, and for all other simple coroots this inner product must be zero. Otherwise we will have $(\omega, \theta^\vee) \ge 2$, where $\theta^\vee$ is the maximal coroot (the maximal root of the dual root system $R^\vee$).[15]

On the other hand, not all fundamental weights are minuscule. In fact, we will see that the simple Lie algebras of types G_2, F_4 and E_8 do not have any nonzero minuscule weights. To formulate a criterion for a fundamental weight to be minuscule, recall that $\theta^\vee = \sum_i m_i \alpha_i^\vee$, where $m_i = (\omega_i, \theta^\vee)$ are strictly positive integers.

Lemma 30.2. *A fundamental weight ω_i is minuscule if and only if $m_i = 1$.*

Proof. The definition of minuscule means that $m_i \le 1$. On the other hand, if $m_i = 1$ then given a positive coroot $\beta = \sum_j n_j \alpha_j^\vee$, we have $n_j \le m_j$, in particular $n_i \le 1$, so ω_i is minuscule. □

Lemma 30.3. *Let $\omega \in Q$ and $|(\omega, \beta)| \le 1$ for all coroots β. Then $\omega = 0$.*

Proof. Assume the contrary. Choose a counterexample $\omega = \sum_i m_i \alpha_i$ so that $\sum_i |m_i|$ is minimal possible. We have

$$(\omega, \omega) = \sum_i m_i (\omega, \alpha_i) > 0.$$

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[15] The maximal coroot $\theta^\vee$ should not be confused with the coroot $\widetilde{\theta}^\vee$ corresponding to the maximal root θ (highest weight of the adjoint representation) under a W-invariant identification $\mathfrak{h}^* \cong \mathfrak{h}$. In the non-simply-laced case they are not even proportional: e.g., for the root system B_2, $\theta^\vee = (1, 1)$ while $\widetilde{\theta}^\vee = (2, 0)$. This may be confusing since according to the general coroot notation, $\widetilde{\theta}^\vee$ should be denoted by $\theta^\vee$.

So there exists j such that m_j and $(\omega, \alpha_j^\vee)$ are nonzero and have the same sign. Replacing ω with $-\omega$ if needed, we may assume that both are positive, then $(\omega, \alpha_j^\vee) = 1$. Then $s_j\omega = \omega - \alpha_j = \sum_j m_i'\alpha_i$ where $m_j' = m_j - 1$ and $m_i' = m_i$ for all $i \neq j$ is another counterexample. But we have $\sum_i |m_i'| = \sum_i |m_i| - 1$, a contradiction. □

Why are minuscule weights interesting? It is because of the following result.

Proposition 30.4. *The following conditions on a dominant integral weight ω are equivalent:*
(1) ω is minuscule;
(2) all weights of the representation L_ω belong to the orbit $W\omega$;
(3) if λ is a dominant integral weight such that $\omega - \lambda \in Q_+$ then $\lambda = \omega$.

Proof. Let us prove that (1) implies (3). If $\omega = 0$, there is nothing to prove, since then $-\lambda \in Q_+$, so $(\lambda, \rho) \leq 0$, hence $\lambda = 0$. So suppose that $\omega = \omega_i$ is minuscule. We have $\omega_i - \lambda = \sum_k m_k\alpha_k$ with $m_k \geq 0$. If $m_k = 0$ for some $k \neq i$ then the problem reduces to smaller rank by deleting the vertex k from the Dynkin diagram. So we may assume $m_k > 0$ for all $k \neq i$. Let β be a positive coroot. Then

$$(\omega_i - \lambda, \beta) = (\omega_i, \beta) - (\lambda, \beta) \leq (\omega_i, \beta) \leq 1$$

and if $\alpha_i^\vee$ does not occur in β then it is ≤ 0. So in particular we have $(\omega_i - \lambda, \alpha_j^\vee) \leq 0$ if $j \neq i$. If also $(\omega_i - \lambda, \alpha_i^\vee) \leq 0$ then $(\omega_i - \lambda, \omega_i - \lambda) \leq 0$, so $\omega_i = \lambda$, as claimed. Thus we may assume that $(\omega_i - \lambda, \alpha_i^\vee) = 1$, i.e., $m_i > 0$, so $m_j > 0$ for all j. Thus, $(\omega_i - \lambda, \theta^\vee) \geq 1$ (as $\theta^\vee$ is a dominant coweight). Hence $(\lambda, \theta^\vee) \leq 0$, i.e., $\lambda = 0$, as $\theta^\vee$ contains all $\alpha_j^\vee$ with positive coefficients. Thus $\omega_i \in Q$. But this is impossible by Lemma 30.3.

To see that (3) implies (2), note that if μ is any weight of L_ω then for some $w \in W$ the weight $\lambda = w\mu$ is dominant and $\omega - \lambda \in Q_+$, so $\lambda = \omega$ and $\mu = w^{-1}\omega$.

Finally, we show that (2) implies (1). Assume (2) holds. If ω is not minuscule then there is a positive root α such that $(\omega, \alpha^\vee) > 1$, hence $2(\omega, \alpha) > (\alpha, \alpha)$. Then $\omega - \alpha$ is a weight of L_ω (the weight of the nonzero vector $f_\alpha v_\omega$), and it is not W-conjugate to ω, as

$$(\omega - \alpha, \omega - \alpha) = (\omega, \omega) - 2(\omega, \alpha) + (\alpha, \alpha) < (\omega, \omega).$$

□

This immediately implies

Corollary 30.5. *The character of L_ω with minuscule ω is*

$$\chi_\omega = \sum_{\gamma \in W\omega} e^\gamma.$$

Proposition 30.6. *$\omega \in P_+$ is minuscule if and only if the restriction of L_ω to any root $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ is the direct sum of 1-dimensional and 2-dimensional representations.*

Proof. Let ω be minuscule and $v \in L_\omega$ be a weight vector which is a highest weight vector for $(\mathfrak{sl}_2)_\alpha$. Then $h_\alpha v = (w\omega, \alpha^\vee)v = (\omega, w^{-1}\alpha^\vee)v$ for some $w \in W$. Thus $h_\alpha v = 0$ or $h_\alpha v = v$, as claimed.

On the other hand, if ω is not minuscule then there is a positive root α such that $(\omega, \alpha^\vee) = m > 1$. So $h_\alpha v_\omega = m v_\omega$ and v_ω generates the irreducible $m+1$-dimensional representation of $(\mathfrak{sl}_2)_\alpha$. □

30.2. **Tensor product with a minuscule representation.**

Corollary 30.7. *If ω is minuscule then for any dominant integral weight λ of $\mathfrak{g}$ we have*

$$L_\omega \otimes L_\lambda = \oplus_{\gamma \in W\omega} L_{\lambda+\gamma},$$

where if $\lambda + \gamma$ is not dominant then we agree that $L_{\lambda+\gamma} = 0$.

Proof. By the Weyl character formula and Corollary 30.5, the character of $L_\omega \otimes L_\lambda$ is

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\mu \in W\omega} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)+\mu}}{\prod_{\alpha \in R_+}(e^{\alpha/2} - e^{-\alpha/2})} =$$

$$\frac{\sum_{\gamma \in W\omega} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\gamma+\rho)}}{\prod_{\alpha \in R_+}(e^{\alpha/2} - e^{-\alpha/2})}.$$

If $\lambda+\gamma \notin P_+$ then for some i we have $(\lambda+\gamma, \alpha_i^\vee) < 0$. But $(\gamma, \alpha_i^\vee) \geq -1$. So $(\lambda+\gamma, \alpha_i^\vee) = -1$ and thus $(\lambda+\gamma+\rho, \alpha_i^\vee) = 0$. So for such γ, for any $w \in W$ the summand for w cancels with the summand for ws_i. Thus we get

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\gamma \in W\omega : \lambda+\gamma \in P_+} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\gamma+\rho)}}{\prod_{\alpha \in R_+}(e^{\alpha/2} - e^{-\alpha/2})} = \sum_{\gamma \in W\omega : \lambda+\gamma \in P_+} \chi_{L_{\lambda+\gamma}}.$$

□

Example 30.8. 1. Let V be the vector representation of GL_n. Then for a partition λ, $V \otimes L_\lambda = \bigoplus_{\mu \in \lambda + \square} L_\mu$, where μ runs over all partitions obtained by adding one **addable** box to the Young diagram of λ, i.e., such that it remains a Young diagram. For example,

$$V \otimes S^{(3,3,2,1)}V = S^{(4,3,2,1)}V \oplus S^{(3,3,3,1)}V \oplus S^{(3,3,2,2)}V \oplus S^{(3,3,2,1,1)}V.$$

2. More generally, $\wedge^m V\otimes L_\lambda=\bigoplus_{\mu\in\lambda+m\square}L_\mu$, where we sum over partitions obtained by adding m addable boxes to different rows of the Young diagram of λ (going from top to bottom), i.e. a collection of m boxes in different rows after adding which we still have a Young diagram. This follows immediately from Corollary 30.7. For example,

$$\wedge^2V\otimes S^{(3,1)}V=S^{(4,2)}V\oplus S^{(4,1,1)}V\oplus S^{(3,2,1)}V\oplus S^{(3,1,1,1)}V.$$

Proposition 30.9. *(i) Let λ be a partition of N. Then we have*

$$\mathbb{C}S_{N+1}\otimes_{\mathbb{C}S_N}\pi_\lambda=\bigoplus_{\mu\in\lambda+\square}\pi_\mu.$$

(ii) Let μ be a partition of $N+1$. Then we have

$$\pi_\mu|_{S_N}=\bigoplus_{\lambda\in\mu-\square}\pi_\mu.$$

Here in (ii) we sum over all ways to delete a **removable box** from the Young diagram of μ, i.e., such that the remaining collection of boxes is still a Young diagram.

Proof. (i) Let V be a vector space of sufficiently large dimension. Using Frobenius reciprocity and Schur-Weyl duality, we have

$$\mathrm{Hom}_{S_{N+1}}(\mathbb{C}S_{N+1}\otimes_{\mathbb{C}S_N}\pi_\lambda,V^{\otimes N+1})=\mathrm{Hom}_{S_N}(\pi_\lambda,V\otimes V^{\otimes N})=V\otimes S^\lambda V.$$

On the other hand, again by the Schur-Weyl duality,

$$\mathrm{Hom}_{S_{N+1}}(\bigoplus_{\mu\in\lambda+\square}\pi_\mu,V^{\otimes N+1})=\bigoplus_{\mu\in\lambda+\square}S^\mu V.$$

So the statement follows from Example 30.8(1).

(ii) follows from (i) and Frobenius reciprocity. $\square$

Let $\lambda^\dagger$ be the **conjugate partition** to λ, which consists of the boxes (j,i) where $(i,j)\in\lambda$. In other words, the Young diagram of $\lambda^\dagger$ is obtained by transposing the Young diagram of λ. For example, $(3,3,2,1)^\dagger=(4,3,2)$.

Corollary 30.10. *Let $\mathbb{C}_-$ be the sign representation of S_N. Then*

$$\pi_\lambda\otimes\mathbb{C}_-\cong\pi_{\lambda^\dagger}.$$

Proof. We argue by induction in $N=|\lambda|$, with obvious base $N=1$. Suppose the statement is known for N and let us prove it for $N+1$. Given a partition ν of $N+1$, let λ be obtained from ν by deleting a removable box (i,j). Note that we have a natural isomorphism

$$\xi:(\mathbb{C}S_{N+1}\otimes_{\mathbb{C}S_N}\pi_\lambda)\otimes\mathbb{C}_-\to\mathbb{C}S_{N+1}\otimes_{\mathbb{C}S_N}(\pi_\lambda\otimes\mathbb{C}_-)=\mathbb{C}S_{N+1}\otimes_{\mathbb{C}S_N}\pi_{\lambda^\dagger}.$$

This can be written as an isomorphism

$$\bigoplus_{\mu\in\lambda+\square}\pi_\mu\otimes\mathbb{C}_-\cong\bigoplus_{\eta\in\lambda^\dagger+\square}\pi_\eta.$$

Suppose $\pi_\nu\otimes\mathbb{C}_-=\pi_{\overline{\nu}}$. Then $\overline{\nu}\in\lambda^\dagger+\square$. But by Exercise 27.9, π_ν is the eigenspace of the Jucys-Murphy element $\mathbf{c}\in\mathbb{C}S_{N+1}$ in $\mathbb{C}S_{N+1}\otimes_{\mathbb{C}S_N}\pi_\lambda$ with eigenvalue $c(\nu)$ (as $c(\mu)$ are all distinct for $\mu\in\lambda+\square$). Hence the eigenvalue of $\mathbf{c}$ on $\pi_{\bar\nu}$ is $-c(\nu)$. This implies that $\bar\nu=\nu^\dagger$, which justifies the induction step. $\square$

Proposition 30.11. *(***Skew Howe duality***) Let V,W be complex vector spaces. Show that*

$$\wedge^n(V\otimes W)\cong\bigoplus_{\lambda:|\lambda|=n}S^\lambda V\otimes S^{\lambda^\dagger}W$$

as $GL(V)\times GL(W)$-modules.

Exercise 30.12. Prove Proposition 30.11.

Hint: Repeat the proof of the usual Howe duality (Subsection 29.2), using Corollary 30.10.

Exercise 30.13. Compute characters and dimensions of irreducible representations $L_{a+b,b,0}$ of $SL_3(\mathbb{C})$, where $a,b\geq 0$. Compute the weight multiplicities and draw the weights on the hexagonal lattice for $a+b\leq 3$, indicating the multiplicities. What are the special features of the case $b=0$?

Hint. The best way to do this exercise is to compute the characters recursively, using that $V\otimes L_{a+b,b,0}=L_{a+b+1,b,0}\oplus L_{a+b,b+1,0}\oplus L_{a+b-1,b-1,0}$ (if $a=0$, the second summand drops out and if $b=0$ then the third one drops out), by the "addable boxes" rule. This allows one to express the characters for $b+1$ in terms of the characters for b and $b-1$. And we know the characters of $L_{a,0,0}$ - they are the complete symmetric functions h_a.

Exercise 30.14. Compute the decomposition of $\wedge^m V\otimes S^k V$, $\wedge^m V\otimes\wedge^k V$, $S^2(\wedge^m V)$, $\wedge^2(\wedge^m V)$ into irreducible representations of $GL(V)$.

Exercise 30.15. Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra, and V a finite dimensional representation of $\mathfrak{g}$. Given a homomorphism $\Phi:L_\lambda\to V\otimes L_\mu$, let $\langle\Phi\rangle:=(\mathrm{Id}\otimes v_\mu^*,\Phi v_\lambda)\in V$, where v_λ is a highest weight vector of L_λ and v_μ^* the lowest weight vector of L_μ^*. In other words, we have

$$\Phi v_\lambda=\langle\Phi\rangle\otimes v_\mu+\text{lower terms}$$

where the lower terms have lower weight than μ in the second component.

(i) Show that $\langle\Phi\rangle$ has weight $\lambda-\mu$.

(ii) Show that $f_i^{(\lambda,\alpha_i^\vee)+1}\langle\Phi\rangle=0$ for all i.

(iii) Let $V[\nu]_\lambda$ be the subspace of vectors $v\in V[\nu]$ of weight ν which satisfy the equalities $f_i^{(\lambda,\alpha_i^\vee)+1}v=0$ for all i. Show that the map $\Phi\mapsto\langle\Phi\rangle$ defines an isomorphism of vector spaces $\mathrm{Hom}_{\mathfrak g}(L_\lambda,V\otimes L_\mu)\cong V[\lambda-\mu]_\lambda$.

Hint. Let M_λ be the Verma module with highest weight λ, and $\overline{M}_{-\mu}$ be the **lowest weight** Verma module with lowest weight $-\mu$, i.e., generated by a vector $v_{-\mu}$ with defining relations $hv_{-\mu}=-\mu(h)v_{-\mu}$ for $h\in\mathfrak h$ and $f_iv_{-\mu}=0$. Show first that the map $\Phi\mapsto\langle\Phi\rangle$ defines an isomorphism $\mathrm{Hom}_{\mathfrak g}(M_\lambda,V\otimes\overline{M}^*_{-\mu})\cong V[\lambda-\mu]$. Next, show that $\Phi\in\mathrm{Hom}_{\mathfrak g}(M_\lambda,V\otimes\overline{M}^*_{-\mu})$ factors through L_λ iff $\langle\Phi\rangle\in V[\lambda-\mu]_\lambda$, i.e., $f_i^{(\lambda,\alpha_i^\vee)+1}\langle\Phi\rangle=0$ (for this, use that $e_jf_i^{(\lambda,\alpha_i^\vee)+1}v_\lambda=0$, and that the kernel of $M_\lambda\to L_\lambda$ is generated by the vectors $f_i^{(\lambda,\alpha_i^\vee)+1}v_\lambda$). This implies that the above map defines an isomorphism $\mathrm{Hom}_{\mathfrak g}(L_\lambda,V\otimes\overline{M}^*_{-\mu})\cong V[\lambda-\mu]_\lambda$. Finally, show that every homomorphism $L_\lambda\to V\otimes\overline{M}^*_{-\mu}$ in fact lands in $V\otimes L_\mu\subset V\otimes\overline{M}^*_{-\mu}$.

(iv) Let V be the vector representation of $SL_n(\mathbb{C})$. Determine the weight subspaces of S^mV, and compute the decomposition of $S^mV\otimes L_\mu$ into irreducibles for all μ (use (iii)).

(v) For any $\mathfrak g$, compute the decomposition of $\mathfrak g\otimes L_\mu$, where $\mathfrak g$ is the adjoint representation of $\mathfrak g$ (again use (iii)).

In both (iv) and (v) you should express the answer in terms of the numbers k_i such that $\mu=\sum_i k_i\omega_i$ and the Cartan matrix entries.

Proposition 30.16. *Every coset in P/Q contains a unique minuscule weight. This gives a bijection between P/Q and minuscule weights. So the number of minuscule weights equals* $\det A$, *where A is the Cartan matrix.*

Proof. Let $C:=a+Q\in P/Q$ be a coset, and consider the intersection $C\cap P_+$. Let $\omega\in C\cap P_+$ be an element with smallest $(\omega,\rho^\vee)$. If λ is a dominant weight of L_ω then $\lambda\in C\cap P_+$, so $(\lambda,\rho^\vee)\ge(\omega,\rho^\vee)$, hence $(\omega-\lambda,\rho^\vee)\le 0$. But $\omega-\lambda\in Q_+$, so $\lambda=\omega$. Thus ω is minuscule. On the other hand, if $\omega_1,\omega_2\in C$ are minuscule and distinct then $\omega_1-\omega_2\in Q$, so by Lemma 30.3, there is a coroot β such that $(\omega_1-\omega_2,\beta)\ge 2$. So $(\omega_1,\beta)=1$ and $(\omega_2,\beta)=-1$. The first identity implies $\beta>0$ and the second one $\beta<0$, a contradiction. $\square$

30.3. **Fundamental weights of classical Lie algebras.** Let us now determine the fundamental weights of classical Lie algebras of types B_n, C_n, D_n.

Type C_n. Then $\mathfrak{g} = \mathfrak{sp}_{2n}$. The positive roots are $\mathbf{e}_i \pm \mathbf{e}_j, 2\mathbf{e}_i$, the simple roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ..., \alpha_n = 2\mathbf{e}_n$, so $\alpha_i^\vee = \alpha_i$ for $i \neq n$ and $\alpha_n^\vee = \mathbf{e}_n$. So $\omega_i = (1, ..., 1, 0, ..., 0)$ (i ones) for $1 \leq i \leq n$.

Type B_n. Then $\mathfrak{g} = \mathfrak{so}_{2n+1}$, so we have the same story as for C_n except $\alpha_n = \mathbf{e}_n$ and $\alpha_n^\vee = 2\mathbf{e}_n$, so we have the same ω_i for $i < n$ but $\omega_n = (\frac{1}{2}, ..., \frac{1}{2})$.

Type D_n. Then $\mathfrak{g} = \mathfrak{so}_{2n}$, so the positive roots are $\mathbf{e}_i \pm \mathbf{e}_j$, the simple roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, ..., \alpha_{n-2} = \mathbf{e}_{n-2} - \mathbf{e}_{n-1}$, $\alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n$, $\alpha_n = \mathbf{e}_{n-1} + \mathbf{e}_n$. So $\omega_i = (1, ..., 1, 0, ..., 0)$ (i ones) for $i = 1, ..., n-2$, but $\omega_{n-1} = (\frac{1}{2}, ..., \frac{1}{2}, \frac{1}{2})$, $\omega_n = (\frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2})$.

30.4. **Minuscule weights outside type A.** Proposition 30.16 immediately tells us how many minuscule weights we have. For type A we saw that all fundamental weights are minuscule. For G_2, F_4, E_8, $\det A = 1$, so the only minuscule weight is 0. For type B_n we have $\det A = 2$, so we should have one nonzero minuscule weight, and this is the weight $(\frac{1}{2}, ..., \frac{1}{2})$. The corresponding representation has weights $(\pm\frac{1}{2}, ..., \pm\frac{1}{2})$, so it has dimension 2^n. It is called the **spin representation**, denoted S.

For C_n we also have $\det A = 2$, so we again have a unique nonzero minuscule weight. Namely, it is the weight $(1, 0, ..., 0)$ (so the minuscule representation is the tautological representation of $\mathfrak{sp}_{2n}$, of dimension $2n$). For D_n we have $\det A = 4$, so we have three nontrivial minuscule representations, with highest weights $\omega_1, \omega_{n-1}, \omega_n$, of dimensions $2n, 2^{n-1}, 2^{n-1}$. The first one is the tautological representation and the remaining two are the **spin representations** S_+, S_-, whose weights are $(\pm\frac{1}{2}, ..., \pm\frac{1}{2})$ with even, respectively odd number of minuses.

For E_6 there are two nontrivial minuscule representations V, V^* of dimension 27. For E_7 there is just one of dimension 56. These dimensions are computed easily by counting elements in the corresponding Weyl group orbits.