

30. Fundamental and minuscule weights

30.1. Minuscule weights. Let \mathfrak{g} be a simple complex Lie algebra. Minuscule weights for \mathfrak{g} are highest weights for which irreducible representations are especially simple.

Definition 30.1. A dominant integral weight ω for \mathfrak{g} is called **minuscule** if $(\omega, \beta) \leq 1$ for all positive coroots β .

Equivalently, $|(\omega, \beta)| \leq 1$ for any coroot β .

Obviously, $\omega = 0$ is minuscule, but there may exist other minuscule weights. For example, for $\mathfrak{g} = \mathfrak{sl}_n$, all fundamental weights are minuscule, since $(\omega_i, \mathbf{e}_j - \mathbf{e}_k) = 0$ if $j, k \leq i$ or $j, k > i$ and $(\omega_i, \mathbf{e}_j - \mathbf{e}_k) = 1$ if $j \leq i < k$.

It is easy to see that any minuscule weight $\omega \neq 0$ is fundamental. Indeed, we can have $(\omega, \alpha_i^\vee) = 1$ only for one i , and for all other simple coroots this inner product must be zero. Otherwise we will have $(\omega, \theta^\vee) \geq 2$, where θ^\vee is the maximal coroot (the maximal root of the dual root system R^\vee).¹³

On the other hand, not all fundamental weights are minuscule. In fact, we will see that the simple Lie algebras of types G_2 , F_4 and E_8 do not have any nonzero minuscule weights. To formulate a criterion for a fundamental weight to be minuscule, recall that $\theta^\vee = \sum_i m_i \alpha_i^\vee$, where $m_i = (\omega_i, \theta^\vee)$ are strictly positive integers.

Lemma 30.2. *A fundamental weight ω_i is minuscule if and only if $m_i = 1$.*

Proof. The definition of minuscule means that $m_i \leq 1$. On the other hand, if $m_i = 1$ then given a positive coroot $\beta = \sum_j n_j \alpha_j^\vee$, we have $n_j \leq m_j$, in particular $n_i \leq 1$, so ω_i is minuscule. \square

Lemma 30.3. *Let $\omega \in Q$ and $|(\omega, \beta)| \leq 1$ for all coroots β . Then $\omega = 0$.*

Proof. Assume the contrary. Choose a counterexample $\omega = \sum_i m_i \alpha_i$ so that $\sum_i |m_i|$ is minimal possible. We have

$$(\omega, \omega) = \sum_i m_i (\omega, \alpha_i) > 0.$$

¹³The maximal coroot θ^\vee should not be confused with the coroot $\tilde{\theta}^\vee$ corresponding to the maximal root θ (highest weight of the adjoint representation) under a W -invariant identification $\mathfrak{h}^* \cong \mathfrak{h}$. In the non-simply-laced case they are not even proportional: e.g., for the root system B_2 , $\theta^\vee = (1, 1)$ while $\tilde{\theta}^\vee = (2, 0)$. This may be confusing since according to the general coroot notation, $\tilde{\theta}^\vee$ should be denoted by θ^\vee .

So there exists j such that m_j and (ω, α_j^\vee) are nonzero and have the same sign. Replacing ω with $-\omega$ if needed, we may assume that both are positive, then $(\omega, \alpha_j^\vee) = 1$. Then $s_j\omega = \omega - \alpha_j = \sum_i m'_i \alpha_i$ where $m'_j = m_j - 1$ and $m'_i = m_i$ for all $i \neq j$ is another counterexample. But we have $\sum_i |m'_i| = \sum_i |m_i| - 1$, a contradiction. \square

Why are minuscule weights interesting? It is because of the following result.

Proposition 30.4. *The following conditions on a dominant integral weight ω are equivalent:*

- (1) ω is minuscule;
- (2) all weights of the representation L_ω belong to the orbit $W\omega$;
- (3) if λ is a dominant integral weight such that $\omega - \lambda \in Q_+$ then $\lambda = \omega$.

Proof. Let us prove that (1) implies (3). If $\omega = 0$, there is nothing to prove, since then $-\lambda \in Q_+$, so $(\lambda, \rho) \leq 0$, hence $\lambda = 0$. So suppose that $\omega = \omega_i$ is minuscule. We have $\omega_i - \lambda = \sum_k m_k \alpha_k$ with $m_k \geq 0$. If $m_k = 0$ for some $k \neq i$ then the problem reduces to smaller rank by deleting the vertex k from the Dynkin diagram. So we may assume $m_k > 0$ for all $k \neq i$. Let β be a positive coroot. Then

$$(\omega_i - \lambda, \beta) = (\omega_i, \beta) - (\lambda, \beta) \leq (\omega_i, \beta) \leq 1$$

and if α_i^\vee does not occur in β then it is ≤ 0 . So in particular we have $(\omega_i - \lambda, \alpha_j^\vee) \leq 0$ if $j \neq i$. If also $(\omega_i - \lambda, \alpha_i^\vee) \leq 0$ then $(\omega_i - \lambda, \omega_i - \lambda) \leq 0$, so $\omega_i = \lambda$, as claimed. Thus we may assume that $(\omega_i - \lambda, \alpha_i^\vee) = 1$, i.e., $m_i > 0$, so $m_j > 0$ for all j . Thus, $(\omega_i - \lambda, \theta^\vee) \geq 1$ (as θ^\vee is a dominant coweight). Hence $(\lambda, \theta^\vee) \leq 0$, i.e., $\lambda = 0$, as θ^\vee contains all α_j^\vee with positive coefficients. Thus $\omega_i \in Q$. But this is impossible by Lemma 30.3.

To see that (3) implies (2), note that if μ is any weight of L_ω then for some $w \in W$ the weight $\lambda = w\mu$ is dominant and $\omega - \lambda \in Q_+$, so $\lambda = \omega$ and $\mu = w^{-1}\omega$.

Finally, we show that (2) implies (1). Assume (2) holds. If ω is not minuscule then there is a positive root α such that $(\omega, \alpha^\vee) > 1$, hence $2(\omega, \alpha) > (\alpha, \alpha)$. Then $\omega - \alpha$ is a weight of L_ω (the weight of the nonzero vector $f_\alpha v_\omega$), and it is not W -conjugate to ω , as

$$(\omega - \alpha, \omega - \alpha) = (\omega, \omega) - 2(\omega, \alpha) + (\alpha, \alpha) < (\omega, \omega).$$

\square

This immediately implies

Corollary 30.5. *The character of L_ω with minuscule ω is*

$$\chi_\omega = \sum_{\gamma \in W\omega} e^\gamma.$$

Proposition 30.6. *$\omega \in P_+$ is minuscule if and only if the restriction of L_ω to any root \mathfrak{sl}_2 -subalgebra of \mathfrak{g} is the direct sum of 1-dimensional and 2-dimensional representations.*

Proof. Let ω be minuscule and $v \in L_\omega$ be a weight vector which is a highest weight vector for $(\mathfrak{sl}_2)_\alpha$. Then $h_\alpha v = (w\omega, \alpha^\vee)v = (\omega, w^{-1}\alpha^\vee)v$ for some $w \in W$. Thus $h_\alpha v = 0$ or $h_\alpha v = v$, as claimed.

On the other hand, if ω is not minuscule then there is a positive root α such that $(\omega, \alpha^\vee) = m > 1$. So $h_\alpha v_\omega = m v_\omega$ and v_ω generates the irreducible $m + 1$ -dimensional representation of $(\mathfrak{sl}_2)_\alpha$. \square

30.2. Tensor product with a minuscule representation.

Corollary 30.7. *If ω is minuscule then for any dominant integral weight λ of \mathfrak{g} we have*

$$L_\omega \otimes L_\lambda = \bigoplus_{\gamma \in W\omega} L_{\lambda+\gamma},$$

where if $\lambda + \gamma$ is not dominant then we agree that $L_{\lambda+\gamma} = 0$.

Proof. By the Weyl character formula and Corollary 30.5, the character of $L_\omega \otimes L_\lambda$ is

$$\begin{aligned} \chi_{L_\omega \otimes L_\lambda} &= \frac{\sum_{\mu \in W\omega} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)+\mu}}{\prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2})} = \\ &= \frac{\sum_{\gamma \in W\omega} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\gamma+\rho)}}{\prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2})}. \end{aligned}$$

If $\lambda + \gamma \notin P_+$ then for some i we have $(\lambda + \gamma, \alpha_i^\vee) < 0$. But $(\gamma, \alpha_i^\vee) \geq -1$. So $(\lambda + \gamma, \alpha_i^\vee) = -1$ and thus $(\lambda + \gamma + \rho, \alpha_i^\vee) = 0$. So for such γ , for any $w \in W$ the summand for w cancels with the summand for ws_i . Thus we get

$$\chi_{L_\omega \otimes L_\lambda} = \frac{\sum_{\gamma \in W\omega: \lambda+\gamma \in P_+} \sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\gamma+\rho)}}{\prod_{\alpha \in R_+} (e^{\alpha/2} - e^{-\alpha/2})} = \sum_{\gamma \in W\omega: \lambda+\gamma \in P_+} \chi_{L_{\lambda+\gamma}}.$$

\square

Example 30.8. 1. Let V be the vector representation of GL_n . Then for a partition λ , $V \otimes L_\lambda = \bigoplus_{\mu \in \lambda + \square} L_\mu$, where μ runs over all partitions obtained by adding one **addable** box to the Young diagram of λ , i.e., such that it remains a Young diagram. For example,

$$V \otimes S^{(3,3,2,1)}V = S^{(4,3,2,1)}V \oplus S^{(3,3,3,1)}V \oplus S^{(3,3,2,2)}V \oplus S^{(3,3,2,1,1)}V.$$

2. More generally, $\wedge^m V \otimes L_\lambda = \bigoplus_{\mu \in \lambda + m\Box} L_\mu$, where we sum over partitions obtained by adding m addable boxes to different rows of the Young diagram of λ (going from top to bottom), i.e. a collection of m boxes in different rows after adding which we still have a Young diagram. This follows immediately from Corollary 30.7. For example,

$$\wedge^2 V \otimes S^{(3,1)} V = S^{(4,2)} V \oplus S^{(4,1,1)} V \oplus S^{(3,2,1)} V \oplus S^{(3,1,1,1)} V.$$

Proposition 30.9. (i) *Let λ be a partition of N . Then we have*

$$\mathbb{C}S_{N+1} \otimes_{\mathbb{C}S_N} \pi_\lambda = \bigoplus_{\mu \in \lambda + \Box} \pi_\mu.$$

(ii) *Let μ be a partition of $N + 1$. Then we have*

$$\pi_\mu|_{S_N} = \bigoplus_{\lambda \in \mu - \Box} \pi_\lambda.$$

Here in (ii) we sum over all ways to delete a **removable box** from the Young diagram of μ , i.e., such that the remaining collection of boxes is still a Young diagram.

Proof. (i) Let V be a vector space of sufficiently large dimension. Using Frobenius reciprocity and Schur-Weyl duality, we have

$$\mathrm{Hom}_{S_{N+1}}(\mathbb{C}S_{N+1} \otimes_{\mathbb{C}S_N} \pi_\lambda, V^{\otimes N+1}) = \mathrm{Hom}_{S_N}(\pi_\lambda, V \otimes V^{\otimes N}) = V \otimes S^\lambda V.$$

On the other hand, again by the Schur-Weyl duality,

$$\mathrm{Hom}_{S_{N+1}}\left(\bigoplus_{\mu \in \lambda + \Box} \pi_\mu, V^{\otimes N+1}\right) = \bigoplus_{\mu \in \lambda + \Box} S^\mu V.$$

So the statement follows from Example 30.8(1).

(ii) follows from (i) and Frobenius reciprocity. \square

Let λ^\dagger be the **conjugate partition** to λ , which consists of the boxes (j, i) where $(i, j) \in \lambda$. In other words, the Young diagram of λ^\dagger is obtained by transposing the Young diagram of λ . For example, $(3, 3, 2, 1)^\dagger = (4, 3, 2)$.

Corollary 30.10. *Let \mathbb{C}_- be the sign representation of S_N . Then*

$$\pi_\lambda \otimes \mathbb{C}_- \cong \pi_{\lambda^\dagger}.$$

Proof. We argue by induction in $N = |\lambda|$, with obvious base $N = 1$. Suppose the statement is known for N and let us prove it for $N + 1$. Given a partition ν of $N + 1$, let λ be obtained from ν by deleting a removable box (i, j) . Note that we have a natural isomorphism

$$\xi : (\mathbb{C}S_{N+1} \otimes_{\mathbb{C}S_N} \pi_\lambda) \otimes \mathbb{C}_- \rightarrow \mathbb{C}S_{N+1} \otimes_{\mathbb{C}S_N} (\pi_\lambda \otimes \mathbb{C}_-) = \mathbb{C}S_{N+1} \otimes_{\mathbb{C}S_N} \pi_{\lambda^\dagger}.$$

This can be written as an isomorphism

$$\bigoplus_{\mu \in \lambda + \square} \pi_{\mu} \otimes \mathbb{C}_{-} \cong \bigoplus_{\eta \in \lambda^{\dagger} + \square} \pi_{\eta}.$$

Suppose $\pi_{\nu} \otimes \mathbb{C}_{-} = \pi_{\bar{\nu}}$. Then $\bar{\nu} \in \lambda^{\dagger} + \square$. But by Exercise 27.9, π_{ν} is the eigenspace of the Jucys-Murphy element $\mathbf{c} \in \mathbb{C}S_{N+1}$ in $\mathbb{C}S_{N+1} \otimes_{\mathbb{C}S_N} \pi_{\lambda}$ with eigenvalue $c(\nu)$ (as $c(\mu)$ are all distinct for $\mu \in \lambda + \square$). Hence the eigenvalue of \mathbf{c} on $\pi_{\bar{\nu}}$ is $-c(\nu)$. This implies that $\bar{\nu} = \nu^{\dagger}$, which justifies the induction step. \square

Proposition 30.11. (Skew Howe duality) *Let V, W be complex vector spaces. Show that*

$$\wedge^n(V \otimes W) \cong \bigoplus_{\lambda: |\lambda|=n} S^{\lambda}V \otimes S^{\lambda^{\dagger}}W$$

as $GL(V) \times GL(W)$ -modules.

Exercise 30.12. Prove Proposition 30.11.

Hint: Repeat the proof of the usual Howe duality (Subsection 29.2), using Corollary 30.10.

Exercise 30.13. Compute characters and dimensions of irreducible representations $L_{a+b,b,0}$ of $SL_3(\mathbb{C})$, where $a, b \geq 0$. Compute the weight multiplicities and draw the weights on the hexagonal lattice for $a + b \leq 3$, indicating the multiplicities. What are the special features of the case $b = 0$?

Hint. The best way to do this exercise is to compute the characters recursively, using that $V \otimes L_{a+b,b,0} = L_{a+b+1,b,0} \oplus L_{a+b,b+1,0} \oplus L_{a+b-1,b-1,0}$ (if $a = 0$, the second summand drops out and if $b = 0$ then the third one drops out), by the “addable boxes” rule. This allows one to express the characters for $b + 1$ in terms of the characters for b and $b - 1$. And we know the characters of $L_{a,0,0}$ - they are the complete symmetric functions h_a .

Exercise 30.14. Compute the decomposition of $\wedge^m V \otimes S^k V$, $\wedge^m V \otimes \wedge^k V$, $S^2(\wedge^m V)$, $\wedge^2(\wedge^m V)$ into irreducible representations of $GL(V)$.

Exercise 30.15. Let \mathfrak{g} be a finite dimensional simple complex Lie algebra, and V a finite dimensional representation of \mathfrak{g} . Given a homomorphism $\Phi : L_{\lambda} \rightarrow V \otimes L_{\mu}$, let $\langle \Phi \rangle := (\text{Id} \otimes v_{\mu}^*, \Phi v_{\lambda}) \in V$, where v_{λ} is a highest weight vector of L_{λ} and v_{μ}^* the lowest weight vector of L_{μ}^* . In other words, we have

$$\Phi v_{\lambda} = \langle \Phi \rangle \otimes v_{\mu} + \text{lower terms}$$

where the lower terms have lower weight than μ in the second component.

(i) Show that $\langle \Phi \rangle$ has weight $\lambda - \mu$.

(ii) Show that $f_i^{(\lambda, \alpha_i^\vee)+1} \langle \Phi \rangle = 0$ for all i .

(iii) Let $V[\nu]_\lambda$ be the subspace of vectors $v \in V[\nu]$ of weight ν which satisfy the equalities $f_i^{(\lambda, \alpha_i^\vee)+1} v = 0$ for all i . Show that the map $\Phi \mapsto \langle \Phi \rangle$ defines an isomorphism of vector spaces $\text{Hom}_{\mathfrak{g}}(L_\lambda, V \otimes L_\mu) \cong V[\lambda - \mu]_\lambda$.

Hint. Let M_λ be the Verma module with highest weight λ , and $\overline{M}_{-\mu}$ be the **lowest weight** Verma module with lowest weight $-\mu$, i.e., generated by a vector $v_{-\mu}$ with defining relations $h v_{-\mu} = -\mu(h) v_{-\mu}$ for $h \in \mathfrak{h}$ and $f_i v_{-\mu} = 0$. Show first that the map $\Phi \mapsto \langle \Phi \rangle$ defines an isomorphism $\text{Hom}_{\mathfrak{g}}(M_\lambda, V \otimes \overline{M}_{-\mu}^*) \cong V[\lambda - \mu]$. Next, show that $\Phi \in \text{Hom}_{\mathfrak{g}}(M_\lambda, V \otimes \overline{M}_{-\mu}^*)$ factors through L_λ iff $\langle \Phi \rangle \in V[\lambda - \mu]_\lambda$, i.e., $f_i^{(\lambda, \alpha_i^\vee)+1} \langle \Phi \rangle = 0$ (for this, use that $e_j f_i^{(\lambda, \alpha_i^\vee)+1} v_\lambda = 0$, and that the kernel of $M_\lambda \rightarrow L_\lambda$ is generated by the vectors $f_i^{(\lambda, \alpha_i^\vee)+1} v_\lambda$). This implies that the above map defines an isomorphism $\text{Hom}_{\mathfrak{g}}(L_\lambda, V \otimes \overline{M}_{-\mu}^*) \cong V[\lambda - \mu]_\lambda$. Finally, show that every homomorphism $L_\lambda \rightarrow V \otimes \overline{M}_{-\mu}^*$ in fact lands in $V \otimes L_\mu \subset V \otimes \overline{M}_{-\mu}^*$.

(iv) Let V be the vector representation of $SL_n(\mathbb{C})$. Determine the weight subspaces of $S^m V$, and compute the decomposition of $S^m V \otimes L_\mu$ into irreducibles for all μ (use (iii)).

(v) For any \mathfrak{g} , compute the decomposition of $\mathfrak{g} \otimes L_\mu$, where \mathfrak{g} is the adjoint representation of \mathfrak{g} (again use (iii)).

In both (iv) and (v) you should express the answer in terms of the numbers k_i such that $\mu = \sum_i k_i \omega_i$ and the Cartan matrix entries.

Proposition 30.16. *Every coset in P/Q contains a unique minuscule weight. This gives a bijection between P/Q and minuscule weights. So the number of minuscule weights equals $\det A$, where A is the Cartan matrix.*

Proof. Let $C := a + Q \in P/Q$ be a coset, and consider the intersection $C \cap P_+$. Let $\omega \in C \cap P_+$ be an element with smallest (ω, ρ^\vee) . If λ is a dominant weight of L_ω then $\lambda \in C \cap P_+$, so $(\lambda, \rho^\vee) \geq (\omega, \rho^\vee)$, hence $(\omega - \lambda, \rho^\vee) \leq 0$. But $\omega - \lambda \in Q_+$, so $\lambda = \omega$. Thus ω is minuscule. On the other hand, if $\omega_1, \omega_2 \in C$ are minuscule and distinct then $\omega_1 - \omega_2 \in Q$, so by Lemma 30.3, there is a coroot β such that $(\omega_1 - \omega_2, \beta) \geq 2$. So $(\omega_1, \beta) = 1$ and $(\omega_2, \beta) = -1$. The first identity implies $\beta > 0$ and the second one $\beta < 0$, a contradiction. \square

30.3. Fundamental weights of classical Lie algebras. Let us now determine the fundamental weights of classical Lie algebras of types B_n, C_n, D_n .

Type C_n . Then $\mathfrak{g} = \mathfrak{sp}_{2n}$. The positive roots are $\mathbf{e}_i \pm \mathbf{e}_j, 2\mathbf{e}_i$, the simple roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \dots, \alpha_n = 2\mathbf{e}_n$, so $\alpha_i^\vee = \alpha_i$ for $i \neq n$ and $\alpha_n^\vee = \mathbf{e}_n$. So $\omega_i = (1, \dots, 1, 0, \dots, 0)$ (i ones) for $1 \leq i \leq n$.

Type B_n . Then $\mathfrak{g} = \mathfrak{so}_{2n+1}$, so we have the same story as for C_n except $\alpha_n = \mathbf{e}_n$ and $\alpha_n^\vee = 2\mathbf{e}_n$, so we have the same ω_i for $i < n$ but $\omega_n = (\frac{1}{2}, \dots, \frac{1}{2})$.

Type D_n . Then $\mathfrak{g} = \mathfrak{so}_{2n}$, so the positive roots are $\mathbf{e}_i \pm \mathbf{e}_j$, the simple roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \dots, \alpha_{n-2} = \mathbf{e}_{n-2} - \mathbf{e}_{n-1}, \alpha_{n-1} = \mathbf{e}_{n-1} - \mathbf{e}_n, \alpha_n = \mathbf{e}_{n-1} + \mathbf{e}_n$. So $\omega_i = (1, \dots, 1, 0, \dots, 0)$ (i ones) for $i = 1, \dots, n-2$, but $\omega_{n-1} = (\frac{1}{2}, \dots, \frac{1}{2}, \frac{1}{2}), \omega_n = (\frac{1}{2}, \dots, \frac{1}{2}, -\frac{1}{2})$.

30.4. Minuscule weights outside type A . Proposition 30.16 immediately tells us how many minuscule weights we have. For type A we saw that all fundamental weights are minuscule. For G_2, F_4, E_8 , $\det A = 1$, so the only minuscule weight is 0. For type B_n we have $\det A = 2$, so we should have one nonzero minuscule weight, and this is the weight $(\frac{1}{2}, \dots, \frac{1}{2})$. The corresponding representation has weights $(\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$, so it has dimension 2^n . It is called the **spin representation**, denoted S .

For C_n we also have $\det A = 2$, so we again have a unique nonzero minuscule weight. Namely, it is the weight $(1, 0, \dots, 0)$ (so the minuscule representation is the tautological representation of \mathfrak{sp}_{2n} , of dimension $2n$). For D_n we have $\det A = 4$, so we have three nontrivial minuscule representations, with highest weights $\omega_1, \omega_{n-1}, \omega_n$, of dimensions $2n, 2^{n-1}, 2^{n-1}$. The first one is the tautological representation and the remaining two are the **spin representations** S_+, S_- , whose weights are $(\pm\frac{1}{2}, \dots, \pm\frac{1}{2})$ with even, respectively odd number of minuses.

For E_6 there are two nontrivial minuscule representations V, V^* of dimension 27. For E_7 there is just one of dimension 56. These dimensions are computed easily by counting elements in the corresponding Weyl group orbits.

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