31. **Fundamental representations of classical Lie algebras**

31.1. **Type C_n.** Since the fundamental weights for $\mathfrak{g} = \mathfrak{sp}_{2n}$ are $\omega_i = (1, ..., 1, 0, .., 0)$ (i ones), same as for $\mathfrak{gl}_n$, one may think that the fundamental representations are also "the same", i.e. $\wedge^i V$, where V is the $2n$-dimensional vector representation. Indeed, a Cartan subalgebra in $\mathfrak{g}$ is the space of matrices $\mathrm{diag}(a_1, ..., a_n, -a_1, ..., -a_n)$, so $L_{\omega_1} = V$, with highest weight vector e_1. However, the representation $\wedge^2 V$ is not irreducible, even though it has the correct highest weight ω_2. Indeed, we have $\wedge^2 V = \wedge_0^2 V \oplus \mathbb{C}$, where $\mathbb{C}$ is the trivial representation spanned by the inverse $B^{-1} = \sum_i e_{i+n} \wedge e_i$ of the invariant nondegenerate skew-symmetric form $B = \sum_i e_i^* \wedge e_{i+n}^* \in \wedge^2 V^*$ preserved by $\mathfrak{g}$, and $\wedge_0^2 V$ is the orthogonal complement of B.

It turns out that $\wedge_0^2 V$ is irreducible. (You can show it directly or using the Weyl dimension formula). Thus we have $L_{\omega_2} = \wedge_0^2 V$ (if $n \geq 2$).

So what happens for L_{ω_j} with any $j \geq 2$? To determine this, note that we have a homomorphism of representations $\iota_B : \wedge^{i+1} V \to \wedge^{i-1} V$, which is just the contraction with B (we agree that $\wedge^j V = 0$ for $j < 0$). So we may consider the subrepresentation $\wedge_0^i V = \mathrm{Ker}(\iota_B|_{\wedge^i V}) \subset \wedge^i V$.

Exercise 31.1. (i) Let $m_B : \wedge^{i-1} V \to \wedge^{i+1} V$ be the operator defined by $m_B(u) := B^{-1} \wedge u$. Show that the operators m_B, ι_B generate a representation of the Lie algebra $\mathfrak{sl}_2$ on $\wedge V := \oplus_{i=0}^{2n} \wedge^i V$ where they are proportional to the operators e, f, such that h acts on $\wedge^i V$ by multiplication by $i - n$.

(ii) Show that ι_B is injective when $i \geq n$ and surjective when $i \leq n$ (so an isomorphism for $i = n$).

(iii) Show that $\mathrm{Ker}(\iota_B|_{\wedge^j V})$ is irreducible for $j \leq n$, and is isomorphic to L_{ω_j}, where we agree that $\omega_0 = 0$. Deduce that

$$\wedge V = \oplus_{i=0}^n L_{\omega_i} \otimes L_{n-j}$$

as a representation of $\mathfrak{sp}_{2n} \oplus \mathfrak{sl}_2$, where L_m is the $m + 1$-dimensional irreducible representation of $\mathfrak{sl}_2$ of highest weight m.

(iv) Show that every irreducible representation of $\mathfrak{sp}_{2n}$ occurs in $V^{\otimes N}$ for some N.

Thus we see another instance of the double centralizer property.

31.2. **Type B_n.** We have $\mathfrak{g} = \mathfrak{so}_{2n+1}$, preserving the quadratic form $Q = \sum_{i=1}^n x_i x_{i+n} + x_{2n+1}^2$. A Cartan subalgebra consists of matrices $\mathrm{diag}(a_1, ..., a_n, -a_1, ..., -a_n, 0)$. So the representations $\wedge^i V$, $1 \leq i \leq n$, where V is the $2n + 1$-dimensional vector representation, have highest weight $(1, ..., 1, 0, ...0)$ (i ones), which is ω_i if $i \leq n - 1$.

Exercise 31.2. Show that the representation $\wedge^i V$ is irreducible for $0 \le i \le n$.

Thus for $1 \le i \le n-1$ we have $\wedge^i V = L_{\omega_i}$. On the other hand, the representation $\wedge^n V$, even though irreducible, is not fundamental. Indeed, its highest weight is $(1, ..., 1) = 2\omega_n$, as $\omega_n = (\frac{1}{2}, ..., \frac{1}{2})$. In fact, we see that the representation L_{ω_n} does not occur in $V^{\otimes N}$ for any N, since coordinates of its highest weight are not integer. As mentioned above, this representation is called the **spin representation** S. Vectors in S are called **spinors**. The weights of S are Weyl group translates of ω_n, so they are $(\pm\frac{1}{2}, ..., \pm\frac{1}{2})$ for any choices of signs, so $\dim S = 2^n$, and the character of S is given by the formula

$$\chi_S(x_1, ..., x_n) = (x_1^{\frac{1}{2}} + x_1^{-\frac{1}{2}})...(x_n^{\frac{1}{2}} + x_n^{-\frac{1}{2}}).$$

This is supposed to be the trace of $\operatorname{diag}(x_1, ..., x_n, x_1^{-1}, ..., x_n^{-1}, 1) \in SO_{2n+1}(\mathbb{C})$, which does not make sense since the square roots on the right hand side are defined only up to sign. This shows that the spin representation S **does not lift** to the group $SO_{2n+1}(\mathbb{C})$. Namely, the group $SO_{2n+1}(\mathbb{C})$ is not simply connected, and the representation S only lifts to the universal covering group $\widetilde{SO}_{2n+1}(\mathbb{C})$, which is called the **spin group**, and is denoted $\mathrm{Spin}_{2n+1}(\mathbb{C})$.

Example 31.3. Let $n = 1$. Then $\mathfrak{g} = \mathfrak{so}_3(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C})$ and S is the 2-dimensional irreducible representation. We know that this representation does not lift to $SO_3(\mathbb{C})$ but only to its double cover $SL_2(\mathbb{C})$, which is simply connected (so $\pi_1(SO_3(\mathbb{C})) = \mathbb{Z}/2$, demonstrated by the famous **belt trick**). So we have $\mathrm{Spin}_3(\mathbb{C}) = SL_2(\mathbb{C})$. This is related to the spin phenomenon in quantum mechanics which we will discuss later. This explains the terminology.

Proposition 31.4. *For $n \ge 3$ we have $\pi_1(SO_n(\mathbb{C})) = \mathbb{Z}/2$.*

Proof.

Lemma 31.5. *Let X_n be the hypersurface in $\mathbb{C}^n$ given by the equation $z_1^2 + ... + z_n^2 = 1$. Then for any $1 \le k \le n-2$ we have $\pi_k(X_n) = 0$, i.e., every continuous map $S^k \to X_n$ contacts to a point. E.g., X_n is connected for $n \ge 2$, simply connected for $n \ge 3$, doubly connected for $n \ge 4$, etc.*

Proof. The surface X_n is the complexification of the $n-1$-sphere, $X_n^{\mathbb{R}} := X_n \cap \mathbb{R}^n = S^{n-1}$. We will define a continuous family of maps $f_t : X_n \to X_n$ such that $f_1 = \mathrm{Id}$ and f_0 lands in $X_n^{\mathbb{R}}$, with $f_t|_{X_n^{\mathbb{R}}} = \mathrm{Id}$. This will show that $X_n^{\mathbb{R}}$ is a retract of X_n, so X_n has the required properties since so does $X_n^{\mathbb{R}}$ (indeed, any map $\gamma = f_1 \circ \gamma : S^k \to X_n$ is homotopic to the map $f_0 \circ \gamma$ in $X_n^{\mathbb{R}}$, the homotopy being $f_t \circ \gamma$).

Let $z = x + iy \in X_n$, where $x, y \in \mathbb{R}^n$. Then $z^2 = 1$, so we have $x^2 - y^2 = 1, xy = 0$. Hence

$$(x + tiy)^2 = x^2 - t^2y^2 = 1 + (1 - t^2)y^2 \geq 1.$$

So we may define

$$f_t(z) := \frac{x + tiy}{\sqrt{x^2 - t^2y^2}}.$$

Then $f_t(z)^2 = 1$, $f_1(z) = z$, and $f_0(z) = \frac{x}{|x|}$ lands in the sphere S^{n-1}, as needed. $\square$

In particular, for $n = 4$, changing coordinates, we see that the surface $ad - bc = 1$ is doubly connected, i.e., $SL_2(\mathbb{C})$ is doubly connected and thus $\pi_1(SO_3(\mathbb{C})) = \mathbb{Z}/2$ (which we already knew).

Now, the group $SO_n(\mathbb{C})$ acts on X_n transitively with stabilizer $SO_{n-1}(\mathbb{C})$, so we have a fibration $SO_n \to X_n$ with fiber SO_{n-1}. Therefore, we have an exact sequence

$$\pi_2(X_n) \to \pi_1(SO_{n-1}(\mathbb{C})) \to \pi_1(SO_n(\mathbb{C})) \to \pi_1(X_n)$$

(a portion of the long exact sequence of homotopy groups). By Lemma 31.5, the first and the last group in this sequence are trivial for $n \geq 4$ which implies that in this case $\pi_1(SO_{n-1}(\mathbb{C})) \cong \pi_1(SO_n(\mathbb{C}))$, so we conclude by induction that $\pi_1(SO_n(\mathbb{C})) = \mathbb{Z}/2$ for all $n \geq 3$ (using the case $n = 3$ as the base). $\square$

Corollary 31.6. *For $n \geq 1$ the simply connected group* $\mathrm{Spin}_{2n+1}(\mathbb{C})$ *is a double cover of $SO_{2n+1}(\mathbb{C})$.*

Exercise 31.7. (i) Use a similar argument to show that the groups $SL_{n+1}(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$ are simply connected for $n \geq 1$ (consider their action on nonzero vectors in the vector representation and compute the stabilizer).

(ii) Generalize this argument to show that for any $k \geq 1$ the higher homotopy group π_k for the classical groups $SL_{n+1}(\mathbb{C})$, $SO_n(\mathbb{C})$, $Sp_{2n}(\mathbb{C})$ stabilizes (i.e., becomes independent on n) when n is large enough. How large does n have to be for that?

31.3. **Type D_n.** We have $\mathfrak{g} = \mathfrak{so}_{2n}$, preserving the quadratic form

$$Q = \sum_{i=1}^{n} x_i x_{i+n}.$$

A Cartan subalgebra consists of matrices $\mathrm{diag}(a_1, ..., a_n, -a_1, ..., -a_n)$. So the representation $\wedge^i V$, $1 \leq i \leq n$, where V is the $2n$-dimensional vector representation, have highest weight $(1, ..., 1, 0, ...0)$ (i ones), which is ω_i if $i \leq n-2$.

Exercise 31.8. Show that the representation $\wedge^i V$ is irreducible for $0 \leq i \leq n-1$.

Thus $L_{\omega_i} = \wedge^i V$ for $i \leq n-2$. On the other hand, while the representation $L_{(1,...,1,0)}$ is irreducible, it is not fundamental, as $(1, ..., 1, 0) = \omega_{n-1} + \omega_n$, where $\omega_{n-1} = (\frac{1}{2}, ..., \frac{1}{2}, \frac{1}{2})$ and $\omega_n = (\frac{1}{2}, ..., \frac{1}{2}, -\frac{1}{2})$. The fundamental representations $L_{\omega_{n-1}}, L_{\omega_n}$ are called the **spin representations** and denoted S_+, S_-; their elements are called **spinors**. Similarly to the odd dimensional case, they have dimensions 2^{n-1} and characters

$$\chi_{S_\pm} = \left((x_1^{\frac{1}{2}} + x_1^{-\frac{1}{2}})...(x_n^{\frac{1}{2}} + x_n^{-\frac{1}{2}})\right)_\pm$$

where the subscript $\pm$ means that we take the monomials with odd (for –), respectively even (for +) number of minuses. This shows that, similarly to the odd dimensional case, S_+, S_- don't occur in $V^{\otimes N}$ and don't lift to $SO_{2n}(\mathbb{C})$ but require the universal covering $\mathrm{Spin}_{2n}(\mathbb{C}) = \widetilde{SO}_{2n}(\mathbb{C})$, called the **spin group**. Proposition 31.4 implies

Corollary 31.9. *For $n \geq 2$ the group $\mathrm{Spin}_{2n}(\mathbb{C})$ is a double cover of $SO_{2n}(\mathbb{C})$.*

Example 31.10. Consider the spin groups and representations for small dimensions. We have seen that $\mathrm{Spin}_3 = SL_2$, $S = \mathbb{C}^2$. We also have $\mathrm{Spin}_4 = SL_2 \times SL_2$, with S_+, S_- being the 2-dimensional representations of the factors. We have $\mathrm{Spin}_5 = \mathrm{Sp}_4$, with S being the 4-dimensional vector representation. So $SO_5 = \mathrm{Sp}_4/(\pm 1)$. Finally, $\mathrm{Spin}_6 = SL_4$, with S_+, S_- being the 4-dimensional representation V and its dual V^*. Thus $SO_6 = SL_4/(\pm 1)$.

Exercise 31.11. Let V be a finite dimensional vector space with a nondegenerate inner product. Consider the algebra SV of polynomial functions on V^*. Let $x_1, ..., x_n$ be an orthonormal basis of V, so that $SV \cong \mathbb{C}[x_1, ..., x_n]$, and let $R^2 := \sum_{i=1}^n x_i^2 \in S^2V$ be the "squared radius". Also let $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ be the **Laplace operator.** Note that the Lie algebra $\mathfrak{so}(V)$ acts on SV by automorphisms and R^2 and Δ are $\mathfrak{so}(V)$-invariant. A polynomial $P \in SV$ is called **harmonic** if $\Delta P = 0$.

(i) Show that the operator of multiplication by R^2 and the Laplace operator Δ define an action of $\mathfrak{sl}_2$ on SV which commutes with $\mathfrak{so}(V)$. Namely, they are proportional to f, e respectively. Compute the operator h (it will be a first order differential operator in x_i).

(ii) Let $H_m \subset S^mV$ be the space of harmonic polynomials of degree m (a representation of $\mathfrak{so}(V)$). Show that as an $\mathfrak{so}(V) \oplus \mathfrak{sl}_2$-module,

SV decomposes as

$$SV = \oplus_{m=0}^{\infty} H_m \otimes W_m,$$

where W_m are irreducible (infinite dimensional) representations of $\mathfrak{sl}_2$. Find the dimensions of H_m.

(iii) Show that H_m is irreducible, in fact $H_m = L_{m\omega_1}$. Decompose $S^m V$ into a direct sum of irreducible representations of $\mathfrak{so}(V)$.

(iv) Show that W_m are Verma modules and compute their highest weights.

(v) For $s \in \mathbb{C}$ consider the algebra

$$A_s := \mathbb{C}[x_1, ..., x_n]/(x_1^2 + ... + x_n^2 - s),$$

the algebra of polynomial functions on the hypersurface $x_1^2 + ... + x_n^2 = s$ (here (f) denotes the principal ideal generated by f). This algebra has a natural action of $\mathfrak{so}(V)$. Decompose A into a direct sum of irreducible representations of $\mathfrak{so}(V)$.

31.4. **The Clifford algebra.** It is important to be able to realize the spin representations explicitly. The reason it is somewhat tricky is that these representations don't occur in tensor powers of V (as they have half-integer weights). However, the tensor product of a spin representation with its dual, $S \otimes S^*$, has integer weights and does express in terms of V. So we need to extract "the square root" from this representation, in the sense that "the space of vectors of size n is the square root of the space of square matrices of size n". This is the idea behind the Clifford algebra construction.

Definition 31.12. Let V be a finite dimensional vector space over an algebraically closed field $\mathbf{k}$ of characteristic $\neq 2$ with a nondegenerate symmetric inner product $(,)$. The **Clifford algebra** $\mathrm{Cl}(V)$ is the algebra generated by vectors $v \in V$ with defining relations

$$v^2 = \tfrac{1}{2}(v, v), v \in V.$$

Thus for $a, b \in V$ we have

$$ab + ba = (a + b)^2 - a^2 - b^2 = \tfrac{1}{2}((a + b, a + b) - (a, a) - (b, b)) = (a, b).$$

This is a deformation of the exterior algebra $\wedge V$ which is defined in the same way but $v^2 = 0$. More precisely, $\mathrm{Cl}(V)$ has a filtration (defined by setting $\deg(v) = 1$, $v \in V$) such that the associated graded algebra receives a surjective map $\phi : \wedge V \to \mathrm{grCl}(V)$. We will show that this is a nice ("flat") deformation, in the sense that $\dim \mathrm{Cl}(V) = \dim \wedge V = 2^{\dim V}$, so that ϕ is an isomorphism. This is a kind of Poincaré-Birkhoff-Witt theorem (namely, it is similar to the PBW theorem for Lie algebras, and in fact a special case of one if you pass from Lie algebras to more general Lie superalgebras). Namely, we have the following theorem.

Theorem 31.13. *The algebra* $\mathrm{Cl}(V)$ *is isomorphic to* $\mathrm{Mat}_{2^n}(\mathbf{k})$ *if* $\dim V = 2n$ *and to* $\mathrm{Mat}_{2^n}(\mathbf{k}) \oplus \mathrm{Mat}_{2^n}(\mathbf{k})$ *if* $\dim V = 2n+1$.

Proof. Let us start with the even case. Pick a basis $a_1, ..., a_n, b_1, ..., b_n$ of V so that the inner product is given by

$$(a_i, a_j) = (b_i, b_j) = 0, \ (a_i, b_j) = \delta_{ij}.$$

We have $a_ia_j + a_ja_i = 0$, $b_ib_j + b_jb_i = 0$, $b_ia_j + a_jb_i = 1$. Define the $\mathrm{Cl}(V)$-module $M = \wedge(a_1, ..., a_n)$ with the action of $\mathrm{Cl}(V)$ defined by

$$\rho(a_i)w = a_iw, \ \rho(b_i)w = \tfrac{\partial w}{\partial a_i},$$

where

$$\tfrac{\partial}{\partial a_i} a_{k_1}...a_{k_r} = (-1)^{j-1} a_{k_1}...\widehat{a_{k_j}}...a_{k_r}$$

if $i = k_j$ for some j (where hat means that the term is omitted), and otherwise the result is zero. It is easy to check that this is indeed a representation.

Now for $I = (i_1 < ... < i_k)$, $J = (j_1 < ... < j_m)$ consider the elements $c_{IJ} = a_{i1}...a_{i_k}b_{j_1}...b_{j_m} \in \mathrm{Cl}(V)$. It is easy to see that these elements span $\mathrm{Cl}(V)$. Also it is not hard to do the following exercise.

Exercise 31.14. Show that the operators $\rho(c_{IJ})$ are linearly independent.

Thus $\rho : \mathrm{Cl}(V) \to \mathrm{End}M$ is an isomorphism, which proves the proposition in even dimensions.

Now, if $\dim V = 2n+1$, we pick a basis as above plus an additional element z such that $(z, a_i) = (z, b_i) = 0$, $(z, z) = 2$. So we have

$$za_i + a_iz = 0, \ zb_i + b_iz = 0, \ z^2 = 1.$$

Now we can define the module $M_\pm$ on which a_i, b_i act as before and $zw = \pm(-1)^{\deg w}w$. It is easy to see as before that the map

$$\rho_+ \oplus \rho_- : \mathrm{Cl}(V) \to \mathrm{End}M_+ \oplus \mathrm{End}M_-.$$

is an isomorphism. This takes care of the odd case. □

We will now construct an inclusion of the Lie algebra $\mathfrak{so}(V)$ into the Clifford algebra. This will allow us to regard representations of the Clifford algebra as representations of $\mathfrak{so}(V)$, which will give us a construction of the spin representations.

Consider the linear map $\xi : \wedge^2 V = \mathfrak{so}(V) \to \mathrm{Cl}(V)$ given by the formula

$$\xi(a \wedge b) = \tfrac{1}{2}(ab - ba) = ab - \tfrac{1}{2}(a, b).$$

Then

$$[\xi(a \wedge b), \xi(c \wedge d)] = [ab, cd] = abcd - cdab = (b, c)ad - acbd - cdab =$$
$$(b, c)ad - (b, d)ac + acdb - cdab =$$
$$(b, c)ad - (b, d)ac + (a, c)db - cadb - cdab =$$
$$(b, c)ad - (b, d)ac + (a, c)db - (a, d)cb =$$
$$(b, c)\xi(a \wedge d) - (b, d)\xi(a \wedge c) + (a, c)\xi(d \wedge b) - (a, d)\xi(c \wedge b) = \xi([a \wedge b, c \wedge d]).$$

Thus ξ is a homomorphism of Lie algebras and we can define the representations $\xi^* M$ for even $\dim V$ and $\xi^* M_\pm$ for odd $\dim V$ by $\rho_{\xi^* M}(a) := \rho_M(\xi(a))$.

The representation $\xi^* M$ is reducible, namely

$$\xi^* M = (\xi^* M)_0 \oplus (\xi^* M)_1,$$

where subscripts 0 and 1 indicate the even and odd degree parts.

Exercise 31.15. (i) Show that for even $\dim V$, the representations $(\xi^* M)_0, (\xi^* M)_1$ are isomorphic to S_+, S_- respectively.

(ii) Show that for odd $\dim V$, the representations $\xi^* M_+$ and $\xi^* M_-$ are both isomorphic to S.

Hint. Find the highest weight vector for each of these representations and compute the weight of this vector. Then compare dimensions.