

33. Differential forms, partitions of unity

Now we want to develop an integration theory on Lie groups. First we need to recall the basics about integration on manifolds.

33.1. Locally compact spaces. A Hausdorff topological space X is called **locally compact** if every point has a neighborhood whose closure is compact. For example, \mathbb{R}^n and thus every manifold is locally compact.

Lemma 33.1. *If X is a locally compact topological space with a countable base then it can be represented as a nested union of compact subsets: $X = \cup_{n \in \mathbb{N}} K_n$, $K_i \subset K_{i+1}$, such that every point $x \in X$ has a neighborhood U_x contained in some K_n .*

Proof. For each $x \in X$ fix a neighborhood U_x of x such that $\overline{U_x}$ is compact. By Lemma 1.4 the open cover $\{U_x\}$ of X has a countable subcover $\{W_i, i \in \mathbb{N}\}$. Then the sets $K_n = \cup_{i=0}^n \overline{W_i}$ form a desired nested sequence of compact subsets of X . \square

An open cover of a topological space X is said to be **locally finite** if every point of X has a neighborhood intersecting only finitely many members of this cover.

Lemma 33.2. *Let X be a locally compact topological space with a countable base. Then every base of X has a countable, locally finite subcover.*

Proof. Use Lemma 33.1 to write X as a nested union of compact sets K_n such that every point is contained in some K_n together with its neighborhood. We construct the required subcover inductively as follows. Choose finitely many sets U_1, \dots, U_{N_0} of the base covering K_0 , and remove all other members of the base which meet K_0 . The remaining collection of open sets is no longer a base but still an open cover of X . So add finitely many new sets $U_{N_0+1}, \dots, U_{N_1}$ from this cover (all necessarily disjoint from K_0) to our list so that it now covers K_1 , and remove all other members that meet K_1 , and so on. The remaining sequence U_1, U_2, \dots has only finitely many members which meets every K_n , so every point of X has a neighborhood meeting only finitely many U_i . \square

33.2. Reminder on differential forms. Let M be a real smooth n -dimensional manifold. Recall that a differential k -form on M is a smooth section of the vector bundle $\wedge^k T^*M$, i.e., a skew-symmetric $(n, 0)$ -tensor field (see Subsection 5.3). Thus, for example, a 1-form is a section of T^*M . If x_1, \dots, x_n are local coordinates on M near some

point $p \in M$ then the differentials dx_1, \dots, dx_n form a basis in fibers of T^*M near this point, so a general 1-form in these coordinates has the form

$$\omega = \sum_{i=1}^n f_i(x_1, \dots, x_n) dx_i.$$

If we change the coordinates x_1, \dots, x_n to y_1, \dots, y_n then x_i are smooth functions of y_1, \dots, y_n and in the new coordinates ω looks like

$$\omega = \sum_{i,j=1}^n f_i(x_1, \dots, x_n) \frac{\partial x_i}{\partial y_j} dy_j.$$

Similarly, a differential k -form in the coordinates x_i looks like

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} f_{i_1, \dots, i_k}(x_1, \dots, x_n) dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

where f_{i_1, \dots, i_k} are smooth functions, and in the coordinates y_j it looks like

$$\omega = \sum_{1 \leq i_1 < \dots < i_k \leq n} \sum_{1 \leq j_1 < \dots < j_k \leq n} f_{i_1, \dots, i_k}(x_1, \dots, x_n) \det \left(\frac{\partial x_{i_r}}{\partial y_{j_s}} \right) dy_{j_1} \wedge \dots \wedge dy_{j_k}.$$

The space of differential k -forms on M is denoted $\Omega^k(M)$. For instance, $\Omega^0(M) = C^\infty(M)$ and $\Omega^k(M) = 0$ for $k > n$. Consider now the extremal case $k = n$. The bundle $\wedge^n T^*M$ is a line bundle (a vector bundle of rank 1), so locally any differential n -form in coordinates x_i has the form

$$\omega = f(x_1, \dots, x_n) dx_1 \wedge \dots \wedge dx_n,$$

which in coordinates y_j takes the form

$$\omega = f(x_1, \dots, x_n) \det \left(\frac{\partial x_i}{\partial y_j} \right) dy_1 \wedge \dots \wedge dy_n.$$

We have a canonical differentiation operator $d : \Omega^0(M) \rightarrow \Omega^1(M)$ given in local coordinates by

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i.$$

It is easy to check that this operator does not depend on the choice of coordinates (this becomes obvious if you define it without coordinates, $df(v) = \partial_v f$ for $v \in T_p M$). Also $\Omega^\bullet(M) := \bigoplus_{k=0}^n \Omega^k(M)$ is a graded algebra under wedge product, and d naturally extends to a degree 1 derivation $d : \Omega^\bullet(M) \rightarrow \Omega^\bullet(M)$ defined in coordinates by

$$d(f dx_{i_1} \wedge \dots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

Namely, this is independent on choices and gives rise to a derivation in the “graded” sense:

$$d(a \wedge b) = da \wedge b + (-1)^{\deg a} a \wedge db.$$

A form ω is **closed** if $d\omega = 0$ and **exact** if $\omega = d\eta$ for some η . It is easy to check that $d^2 = 0$, so any exact form is closed. However, not every closed form is exact: on the circle $S^1 = \mathbb{R}/\mathbb{Z}$ the form dx is closed but the function x is defined only up to adding integers, so dx is not exact. The space $\Omega_{\text{closed}}^k(M)/\Omega_{\text{exact}}^k(M)$ is called the k -th **de Rham cohomology** of M , denoted $H^k(M)$.

If $f : M \rightarrow N$ is a differentiable mapping then for a differential form $\omega \in \Omega^k(N)$ we can define the pullback $f^*\omega \in \Omega^k(M)$, given by $(f^*\omega)(v_1, \dots, v_k) = \omega(f_*v_1, \dots, f_*v_k)$ for $v_1, \dots, v_k \in T_pM$. This operation commutes with wedge product and the differential, and $(f \circ g)^* = g^* \circ f^*$.

33.3. Partitions of unity. Let M be a manifold and $\{U_i, i \in I\}$ be an open cover of M .

Definition 33.3. A smooth **partition of unity** subordinate to $\{U_i, i \in I\}$ is a collection $\{f_s, s \in S\}$ of smooth nonnegative functions on M such that

- (i) for all s the support of f_s is contained in U_i for some $i = i(s)$;
- (ii) Any $y \in M$ has a neighborhood in which all but finitely many f_s are zero;
- (iii) $\sum_s f_s = 1$.

Note that the sum in (iii) makes sense because of condition (ii).

Note also that given any partition of unity $\{f_s\}$ subordinate to $\{U_i\}$, we can define

$$F_i := \sum_{s:i(s)=i} f_s,$$

and this is a new partition of unity subordinate to the same cover now labeled by the set I , with the support of F_i contained in U_i .

Finally, note that in every partition of unity on M , the set of s such that f_s is not identically zero is countable, and moreover finite if M is compact. This follows from the fact that by Lemma 1.4, any open cover of a manifold M has a countable subcover, and moreover a finite one if M is compact (applied to the neighborhoods from condition (ii)).

Proposition 33.4. *Any open cover $\{U_i, i \in I\}$ of a manifold M admits a partition of unity subordinate to this cover.*

Proof. Define a function $h : [0, \infty] \rightarrow \mathbb{R}$ given by $h(t) = 0$ for $t \geq 1$ and $h(t) = \exp(\frac{1}{t-1})$ for $t < 1$. It is easy to check that h is smooth.

Thus we can define the smooth **hat function** $H(x) := h(|x|^2)$ on \mathbb{R}^n , supported on the closed unit ball $\overline{B(0,1)}$.

If $\phi : \overline{B(0,1)} \rightarrow M$ is a C^∞ -map which is a diffeomorphism onto the image, we will say that the image of ϕ is a **closed ball** in M . Thus given a closed ball \overline{B} on M (equipped with a diffeomorphism $\phi : \overline{B(0,1)} \rightarrow \overline{B}$), we have a hat function $H_B(y) := H(\phi^{-1}(y))$ on \overline{B} , which we extend by zero to a smooth function on M whose support is \overline{B} and which is strictly positive in its interior $B \subset \overline{B}$.

Now let $\{\overline{B}_s, s \in J\}$ be the collection of all closed balls in M such that their interiors B_s are contained in some U_i . Then $\{B_s, s \in J\}$ is clearly a base for M . Thus by Lemma 33.2, this base has a countable, locally finite subcover $\{B_s, s \in S\}$. Picking diffeomorphisms $\phi_s : \overline{B(0,1)} \rightarrow \overline{B}_s, s \in S$, we can define the smooth function $F(y) := \sum_{s \in S} H_{B_s}(y)$, which is strictly positive on M since B_s cover M (this makes sense by the local finiteness). Now define the smooth functions $f_s(y) := \frac{H_{B_s}(y)}{F(y)}$. This collection is a partition of unity subordinate to the cover $\{U_i\}$, as desired. \square

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