36. **Proof of the Peter-Weyl theorem**

36.1. **Compact operators and the Hilbert-Schmidt theorem.** To prove the Peter-Weyl theorem, we will use the Hilbert-Schmidt theorem – the spectral theorem for compact self-adjoint operators in a Hilbert space.

Recall that a **bounded** operator $A : H \to H$ on a Hilbert space H is a linear operator such that for some $C \geq 0$ we have $||A\mathbf{v}|| \leq C||\mathbf{v}||$, $\mathbf{v} \in H$. The smallest constant C with this property is called the **norm** of A and denoted $||A||$. Recall also that A is **compact** if there is a sequence of finite rank operators $A_n : H \to H$ such that $||A_n - A|| \to 0$ as $n \to \infty$. In other words, the space $K(H)$ of compact operators on H is the closure of the space $K_f(H)$ of finite rank operators under the norm $A \mapsto ||A||$ on the space of bounded operators $B(H)$.

Lemma 36.1. *If A is compact then it maps bounded sets to pre-compact sets (i.e., ones whose closure is compact). In other words, for every bounded sequence $\mathbf{v}_n \in H$, the sequence $A\mathbf{v}_n$ has a convergent subsequence.*[14]

Proof. Let $\mathbf{v}_n \in H$, $||\mathbf{v}_n|| \leq 1$. Pick a sequence of finite rank operators A_n such that $||A_n - A|| < \frac{1}{n}$. Let $\mathbf{v}_n^1$ be a subsequence of $\mathbf{v}_n$ such that $A_1\mathbf{v}_n^1$ is convergent. Let $\mathbf{v}_n^2$ be a subsequence of $\mathbf{v}_n^1$ such that $A_2\mathbf{v}_n^2$ is convergent, and so on. Finally, let $\mathbf{w}_n = \mathbf{v}_n^n$. Note that

$$\begin{aligned} ||A\mathbf{v}_i^k - A\mathbf{v}_j^k|| &\leq ||A_k\mathbf{v}_i^k - A_k\mathbf{v}_j^k|| + ||A - A_k|| \cdot ||\mathbf{v}_i^k - \mathbf{v}_j^k|| \\ &\leq ||A_k\mathbf{v}_i^k - A_k\mathbf{v}_j^k|| + \tfrac{2}{k} - \varepsilon_k. \end{aligned}$$

for some $\varepsilon_k > 0$. Since $A_k\mathbf{v}_i^k, i \geq 1$ is convergent, it is a Cauchy sequence, so there is M_k such that for $i, j \geq M_k$, $||A_k\mathbf{v}_i^k - A_k\mathbf{v}_j^k|| < \varepsilon_k$, hence

$$||A\mathbf{v}_i^k - A\mathbf{v}_j^k|| < \tfrac{2}{k}.$$

But $\mathbf{w}_n$ is a subsequence of $\mathbf{v}_n^k$ starting from the k-th term. So there is N_k such that

$$||A\mathbf{w}_i - A\mathbf{w}_j|| < \tfrac{2}{k},\ i, j \geq N_k.$$

In other words, the sequence $A\mathbf{w}_n$ is Cauchy. Hence it is convergent, as desired. ∎

Proposition 36.2. *Let M be a compact manifold with positive smooth probability measure $d\mathbf{x}$ and $K(\mathbf{x}, \mathbf{y})$ a continuous function on $M \times M$. Then the operator*

$$(A\psi)(\mathbf{y}) := \int_M K(\mathbf{x}, \mathbf{y})\psi(\mathbf{x})d\mathbf{x}.$$

[14] The converse statement also holds, but we will not need it.

on $L^2(M)$ is compact.

Proof. By using a partition of unity, the problem can be reduced to the case when M is replaced by the hypercube $[0,1]^n$. Let us split it in m^n pixels of sidelength $\frac{1}{m}$ and approximate $K(\mathbf{x},\mathbf{y})$ by its maximal value on each of the m^{2n} pixels in $[0,1]^{2n}$. Denote the corresponding approximation by $K_m(\mathbf{x},\mathbf{y})$ and the corresponding operator by A_m; it has rank $\le m^n$. Let $\varepsilon_m := \sup|K-K_m|$, then $||A-A_m|| \le \varepsilon_m$. Finally, by Cantor's theorem,[15] K is uniformly continuous, which implies that $\varepsilon_m \to 0$ as $m \to \infty$, hence the statement. $\square$

Recall that a bounded operator A is **self-adjoint** if $(A\mathbf{v},\mathbf{w}) = (\mathbf{v}, A\mathbf{w})$ for $\mathbf{v},\mathbf{w} \in H$.

Theorem 36.3. *(Hilbert-Schmidt) Let $A: H \to H$ be a compact self-adjoint operator. Then there is an orthogonal decomposition*

$$H = \mathrm{Ker}A \oplus \widehat{\bigoplus_\lambda} H_\lambda,$$

where λ runs over non-zero eigenvalues of A, and $A|_{H_\lambda} = \lambda \cdot \mathrm{Id}$. Moreover, the spaces H_λ are finite dimensional and the eigenvalues λ are real and either form a finite set or a sequence going to 0.

Note that for finite rank operators, this obviously reduces to the standard theorem in linear algebra: a self-adjoint (Hermitian) operator on a finite dimensional space V with a positive Hermitian form has an orthogonal eigenbasis, and its eigenvalues are real.

Proof. We first prove the theorem for the operator A^2. Let $\beta := ||A||^2 = \sup_{||\mathbf{v}||=1}(A^2\mathbf{v},\mathbf{v}) \ge 0$. We may assume without loss of generality that $\beta \ne 0$. Let A_n be a sequence of self-adjoint finite rank operators converging to A, and let $\beta_n = ||A_n||^2$, which is also the maximal eigenvalue of A_n^2. We have $\beta_n \to \beta$. Let $\mathbf{v}_n$ be a sequence of unit vectors in H such that $A_n^2\mathbf{v}_n = \beta_n\mathbf{v}_n$. By Lemma 36.1, the sequence $A^2\mathbf{v}_n$ has a convergent subsequence, so passing to this subsequence we may assume that $A^2\mathbf{v}_n$ is convergent to some $\mathbf{w} \in H$. Hence $A_n^2\mathbf{v}_n \to \mathbf{w}$, so $\mathbf{v}_n \to \beta^{-1}\mathbf{w}$. Thus $A^2\mathbf{w} = \beta\mathbf{w}$. We can now replace H with the orthogonal complement of $\mathbf{w}$ and iterate this procedure.

As a result we'll get a sequence of numbers $\beta_1 > \beta_2 > .... > 0$, which is either finite (in which case the theorem is obvious) or tends to 0 (by compactness of A^2), and the corresponding sequence of finite dimensional orthogonal eigenspaces H_{β_k} (also by compactness of A^2). Let $\mathbf{v}$ be a vector orthogonal to all H_{β_k}. Then $||A\mathbf{v}||^2 \le \beta_k||\mathbf{v}||^2$ for all

[15] Cantor's theorem says that any continuous function on a compact set X is uniformly continuous.

k, so if β_k is an infinite sequence going to 0, it follows that $A\mathbf{v} = 0$, as desired.

Now, we have $H = \mathrm{Ker}A^2 \oplus \widehat{\bigoplus}_n H_{\beta_n}$, and A preserves this decomposition, acting by 0 on $\mathrm{Ker}A^2$ and with eigenvalues $\pm\sqrt{\beta_n}$ on H_{β_n}. This implies the theorem. $\square$

36.2. **Proof of the Peter-Weyl theorem.** Let G be a compact Lie group and h_N a delta-like sequence around 1 on G. By replacing $h_N(x)$ with $\frac{1}{2}(h_N(x) + h_N(x^{-1}))$, we may assume that h_N is invariant under inversion. Define the **convolution operators** B_N on $L^2(G)$ by

$$(B_N\psi)(y) = \int_G h_N(x)\psi(x^{-1}y)dx = \int_G h_N(yz^{-1})\psi(z)dz.$$

By Proposition 36.2, these operators are compact (as the kernel $K(y, z) := h_N(yz^{-1})$ is continuous). Moreover, they are clearly self-adjoint (as $h_N(x) = h_N(x^{-1})$ and h_N is real) and commute with right translations by G. So by the Hilbert-Schmidt theorem, we have the corresponding spectral decomposition

$$L^2(G) = \mathrm{Ker}B_N \oplus \widehat{\bigoplus}_\lambda H_{N,\lambda}$$

invariant under right translations. Since $H_{N,\lambda}$ are finite dimensional and invariant under right translations, they are contained in $L^2_{\mathrm{alg}}(G)$ (this is the key step of the proof). Thus the closure $\overline{L^2_{\mathrm{alg}}(G)}$ contains the image of B_N. So for any $\psi \in L^2(G)$ we can find $\psi_N \in L^2_{\mathrm{alg}}(G)$ such that $||B_N\psi - \psi_N|| < \frac{1}{N}$.

Now let $\psi \in C(G)$. By Cantor's theorem, ψ is uniformly continuous. It follows that $B_N\psi$ uniformly converges to ψ as $N \to \infty$ (check it!). Thus

$$||\psi - \psi_N|| \le ||\psi - B_N\psi|| + ||B_N\psi - \psi_N|| < ||\psi - B_N\psi|| + \tfrac{1}{N} \to 0$$

as $N \to \infty$. So $\overline{L^2_{\mathrm{alg}}(G)}$ contains $C(G)$. But $C(G)$ is dense in $L^2(G)$ (namely, by using a partition of unity this reduces to the case of a box in $\mathbb{R}^n$, where it is well known). Thus $\overline{L^2_{\mathrm{alg}}(G)} = L^2(G)$. This completes the proof of the Peter-Weyl theorem.

36.3. **Existence of faithful representations.**

Lemma 36.4. *Let G be a compact Lie group and $G = G_0 \supset G_1 \supset ...$ be a nested sequence of closed subgroups without repetitions. Then this sequence is finite.*

Proof. Assume the contrary, i.e. that it is infinite. The dimensions must stabilize, so we may assume that $\dim G_n$ are all the same. Then $K = G_n^\circ$ is independent on n, and we have a nested sequence

$$G_0/K \supset G_1/K \supset ...$$

of finite groups, without repetitions. But such a sequence can't have length bigger than $|G_0/K|$, contradiction. $\square$

Corollary 36.5. *Any compact Lie group has a faithful finite dimensional representation, so it is isomorphic to a closed subgroup of the unitary group $U(n)$.*

Proof. Pick a nontrivial finite dimensional representation V_1 of $G = G_0$, and let G_1 be the kernel of this representation. Now pick another representation V_2 of G which is nontrivial as a G_1-representation, and let G_2 be the kernel of V_2 in G_1, and so on. By Lemma 36.4, at some point we will have a subgroup $G_k \subset G$ such that every finite dimensional representation of G is trivial when restricted to G_k. But then by the Peter-Weyl theorem, G_k acts trivially on $L^2(G)$, so $G_k = 1$. Thus $V_1 \oplus ... \oplus V_k$ is a faithful G-representation. $\square$

Remark 36.6. Conversely, any closed subgroup of $U(n)$ is a compact Lie group, see Exercise 36.13 below.

Remark 36.7. Corollary 36.5 is false for non-compact Lie groups, even for connected ones. For example, let G be the universal cover of $SL_2(\mathbb{R})$ (it has fiber $\mathbb{Z} = \pi_1(SL_2(\mathbb{R}))$). Indeed, any finite dimensional continuous representation V of G is smooth, so gives a finite dimensional representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, hence of $\mathfrak{sl}_2(\mathbb{C})$, which is therefore a direct sum of L_n. So V exponentiates to $SL_2(\mathbb{C})$, and thus its restriction to $\mathfrak{sl}_2(\mathbb{R})$ exponentiates to $SL_2(\mathbb{R})$, so is not faithful for G.

Exercise 36.8. Show that any compact Lie group admits a structure of a metric space such that the metric is invariant under left and right translations.

36.4. **Density in continuous functions.** In fact, we can now prove an even stronger version of the Peter-Weyl theorem. For this note that $L^2_{\rm alg}(G)$ is a unital algebra.

Theorem 36.9. *The algebra $L^2_{\rm alg}(G)$ is dense in the algebra of continuous functions $C(G)$ in the supremum norm*

$$||f|| = \max_{g\in G} |f(g)|.$$

Proof. Consider the closure $\mathcal{A}$ of $L^2_{\text{alg}}(G)$ inside $C(G)$ (under the supremum norm). Then $\mathcal{A}$ is a closed subalgebra invariant under complex conjugation, and by Corollary 36.5 it separates points on G. Therefore, by the Stone-Weierstrass theorem, $\mathcal{A} = C(G)$. $\square$

Remark 36.10. If $G = S^1$, this is the usual theorem of uniform approximation of continuous functions on the circle by trigonometric polynomials. If we restrict to even functions, this will be just the usual Weierstrass theorem on approximation of continuous functions on an interval by polynomials.

Corollary 36.11. *Let $A \subset L^2_{\text{alg}}(G)$ be a left-invariant subalgebra stable under complex conjugation and separating points on G. Then $A = L^2_{\text{alg}}(G)$.*

Proof. By the Stone-Weierstrass theorem, A is dense in $C(G)$ in uniform metric, hence in $L^2(G)$ in the Hilbert norm. Thus for every irreducible representation V of G, $\text{Hom}_G(V, A)$ must be dense in the space $\text{Hom}_G(V, L^2(G)_{\text{left}}) = V^*$. So $\text{Hom}_G(V, A) = V^*$, hence $A = L^2_{\text{alg}}(G)$. $\square$

Let us call a finite dimensional representation V of a group G **unimodular** if $\wedge^{\dim V} V \cong \mathbb{C}$ is the trivial representation.

Proposition 36.12. *Let V be a faithful finite dimensional representation of a compact Lie group G. Then:*

(i) If V is unimodular then the subalgebra $A \subset C(G)$ generated by matrix coefficients $f(\rho_V(g)v)$, $v \in V$, $f \in V^*$, coincides with $L^2_{\text{alg}}(G)$.

(ii) If Y an irreducible finite dimensional representation of G, then for some n, m, the representation Y is contained as a direct summand in $V^{\otimes n} \otimes V^{*\otimes m}$. Moreover, if V is unimodular then one may take $m = 0$.

Proof. (i) Let $d := \dim(V)$. It is clear that $A \subset L^2_{\text{alg}}(G)$ is G-invariant and A separates points on G, since V is faithful. Also G is a closed subgroup of $SU(V) \subset V \otimes V^*$, and for a unitary matrix with determinant 1 one has $g^\dagger = g^{-1} = \wedge^{d-1} g$. Thus A is invariant under complex conjugation. So by Corollary 36.11 $A = L^2_{\text{alg}}(G)$.

(ii) It suffices to establish the unimodular case since in general we may replace V with the unimodular representation $V \oplus V^*$. But then by (i), $L^2_{\text{alg}}(G)$ is a quotient of $S(V \otimes V^*)$, which implies the statement. $\square$

Exercise 36.13. In this exercise you will show that a closed subgroup of a Lie group G is a closed Lie subgroup (Theorem 3.13).

Clearly, it suffices to assume that G is connected. Let $\mathfrak{g} = \text{Lie} G$ and $H \subset G$ be a closed subgroup.

(i) Let $\mathfrak{h}$ be the set of vectors $a \in \mathfrak{g}$ such that there is a sequence $h_n \in H$, $h_n \to 1$, and nonzero real numbers c_n such that

$$c_n \log h_n \to a, \ n \to \infty.$$

This is clearly a subset of $\mathfrak{g}$ invariant under scalar multiplication (since we can rescale c_n). Show that $\mathfrak{h}$ consists of all $a \in \mathfrak{g}$ for which the 1-parameter subgroup $\exp(ta)$ is contained in H. (Consider the elements $h_n^{[c_n]}$, where $[c]$ is the floor of c).

(ii) Show that $\mathfrak{h}$ is a subspace of $\mathfrak{g}$. (For $a, b \in \mathfrak{h}$ consider the elements $h_N := \exp(\frac{a}{N})\exp(\frac{b}{N})$ to show that $a + b \in \mathfrak{h}$).

(iii) Show that $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$. (For $a, b \in \mathfrak{h}$ consider the elements

$$h_N := \exp(\tfrac{a}{N})\exp(\tfrac{b}{N})\exp(-\tfrac{a}{N})\exp(-\tfrac{b}{N})$$

to show that $[a, b] \in \mathfrak{h}$).

(iv) Let $H_0 \subset G$ be the connected Lie subgroup with Lie algebra $\mathfrak{h}$. Given a sequence $h_N \in H$, $h_N \to 1$, show that $h_N \in H_0$ for $N \gg 1$. To this end, pick a transverse slice $S \subset G$ to H_0 near 1, and write $h_N = s_N h_{N,0}$, where $h_{N,0} \in H_0$, $s_N \in S$. Look at the asymptotics of $\log s_N$ as $N \to \infty$, and deduce that $s_N = 1$ for large enough N.

(v) Conclude that G/H is a manifold, and S defines a local chart on this manifold near 1. Deduce that H is a closed Lie subgroup of G, and $H_0 = H^\circ$.