37. Representations of compact topological groups

37.1. Existence of the Haar measure. One can generalize integration theory to arbitrary compact and even to locally compact topological groups. For simplicity we will describe this generalization in the case of compact topological groups with a countable base.

Namely, let $X$ be a compact Hausdorff topological space with a countable base. For compact Hausdorff spaces this is equivalent to being metrizable. Let $C(X, \mathbb{R})$ be the space of continuous real-valued functions on $X$. This is a real Banach space with norm $||f|| = \max_{x \in X} |f(x)|$.

Recall that by the Riesz-Markov-Kakutani representation theorem, a finite Borel measure $\mu$ on $X$ is the same thing as a positive continuous linear functional $I : C(X, \mathbb{R}) \to \mathbb{R}$ (i.e., such that $I(f) \geq 0$ for $f \geq 0$), namely,

$$I(f) = \int_X f d\mu.$$ 

Moreover, $\mu$ is a probability measure if and only if $I(1) = 1$, and any $\mu \neq 0$ has positive volume and so can be normalized to be a probability measure.

Now let $G$ be a compact topological group with a countable base. It acts on $C(G, \mathbb{R})$ by left and right translations, so acts on nonnegative probability measures of $G$.

**Theorem 37.1.** (Haar, von Neumann) $G$ admits a unique left-invariant probability measure.

This measure is also automatically right-invariant (since it is unique) and is called the Haar measure on $G$.

**Remark 37.2.** A unique up to scaling left-invariant regular Haar measure (albeit of infinite volume and not always right-invariant in the non-compact case) exists more generally for any locally compact group $G$ (not necessarily having a countable base).\(^\text{16}\) We will not prove this here, but we remark that Haar measures on Lie groups that we have constructed using top differential forms are a special case of this.

**Proof.** Let $g_i, i \geq 1$ be a dense sequence in $G$ (it exists since $G$ has a countable base, hence is separable, as you can pick a point in every open set of this base). Let $p_i$ be a sequence of positive numbers

\(^{16}\)Note that a finite Borel measure on a compact Hausdorff space with a countable base is necessarily regular.
such that $\sum_i p_i = 1$. To this data attach the **averaging operator** $A : C(G, \mathbb{R}) \to C(G, \mathbb{R})$ given by

$$(Af)(x) = \sum_i p_i f(xg_i).$$

This operator can be interpreted as follows: we have a Markov chain with states being points of $G$ and the transition probability from $x$ to $xg_i$ equal to $p_i$, then $(Af)(x)$ is the expected value of $f$ after one transition starting from $x$. It is clear that $A$ is a left-invariant bounded operator (of norm 1). Moreover, $A$ acts by the identity on the line $L \subset C(G, \mathbb{R})$ of constant functions.

For $f \in C(G, \mathbb{R})$ denote by $\nu(f)$ the distance from $f$ to $L$, i.e.,

$$\nu(f) = \frac{1}{2} (\max f - \min f).$$

Then $\nu(Af) < \nu(f)$ unless $f \in L$. Indeed, if $f$ is not constant and $x \in G$, pick $j$ such that $f(xg_j) < \max f$ (exists since the sequence $xg_i$ is dense in $G$), then

$$(Af)(x) = \sum_i p_i f(xg_i) \leq (1 - p_j) \max f + p_j f(xg_j) < \max f.$$  

So $\max(Af) < \max f$. Similarly, $\min(Af) > \min f$.

Now fix $f \in C(G, \mathbb{R})$ and consider the sequence $f_n := A^n f, n \geq 0$. This means that we let our Markov chain run for $n$ steps. We know that for finite Markov chains there is an asymptotic distribution, and we’ll show that this is also the case in the situation at hand, giving rise to a construction of the invariant integral.

Obviously, the sequence $f_n$ is uniformly bounded by $\max |f|$. Also it is **equicontinuous**: for any $\varepsilon > 0$ there exists a neighborhood $1 \in U \subset G$ such that for any $x \in G$ and $u \in U$,

$$|f_n(x) - f_n(ux)| < \varepsilon.$$  

Indeed, it suffices to show that $f$ is uniformly continuous, i.e., for any $\varepsilon$ find $U$ such that for all $x \in G, u \in U$ we have $|f(x) - f(ux)| < \varepsilon$; this $U$ will then work for all $f_n$. But this is guaranteed by Cantor’s theorem. Namely, assume the contrary, that there is no such $U$. Then there are two sequences $x_i, u_i \in G, u_i \to 1$, with $|f(x_i) - f(u_i x_i)| \geq \varepsilon$. The sequence $x_i$ has a convergent subsequence, so we may assume without loss of generality that $x_i \to x \in G$. Then taking the limit $i \to \infty$, we get that $\varepsilon \leq 0$, a contradiction.

Therefore, by the **Ascoli-Arzela theorem** the sequence $f_n$ has a convergent subsequence. Let us remind the proof of this theorem. We construct subsequences $f^k_n$ of $f_n$ inductively by picking $f^k_n$ from $f^{k-1}_n$ so that $f^k_n(g_k)$ converges (with $f^0_n = f_n$), which can be done by the
boundedness assumption, and then set \( h_m := f^m = f_{n(m)} \). Then \( h_m(g_i) \) converges, hence Cauchy, for all \( i \), which by equicontinuity implies that \( h_m(x) \) is a Cauchy sequence in \( C(G, \mathbb{R}) \), hence converges to some \( h \in C(G, \mathbb{R}) \).

We claim that \( h \in L \). Indeed, we have

\[
\nu(f_{n(m)}) \geq \nu(f_{n(m)}+1) = \nu(Af_{n(m)}) \geq \nu(f_{n(m+1)}),
\]

so taking the limit when \( m \to \infty \), we get

\[
\nu(h) \geq \nu(Ah) \geq \nu(h),
\]

i.e., \( \nu(Ah) = \nu(h) \). The assignment \( f \mapsto h \) is therefore a continuous left-invariant positive linear functional \( I : C(G, \mathbb{R}) \to L = \mathbb{R} \), and \( I(1) = 1 \), as claimed.

Similarly, we may construct a right-invariant integral

\[
I_* : C(G, \mathbb{R}) \to L = \mathbb{R}
\]

with \( I_*(1) = 1 \), and by construction for any left invariant integral \( J \) we have \( J(f) = J(I_*(f)) \). Thus for every left invariant integral \( J \) with \( J(1) = 1 \) we have \( J(f) = I_*(f) \); in particular \( I(f) = I_*(f) \). This shows that \( I \) is unique, invariant on both sides and independent on the choice of \( g_i, p_i \), and hence that \( A^n f \to I(f) \) as \( n \to \infty \). \( \square \)

**Example 37.3.** A basic example of a compact topological group with countable base which is, in general, not a Lie group, is a profinite group. Namely, let \( G_1, G_2, ... \) be finite groups and \( \phi_i : G_{i+1} \to G_i \) be surjective homomorphisms. Then the inverse limit \( G := \varprojlim G_n \) is the group consisting of sequences \( g_1 \in G_1, g_2 \in G_2, ... \) where \( \phi_i(g_{i+1}) = g_i \). This group \( G \) has projections \( p_n : G \to G_n \) and a natural topology, for which a base of neighborhoods of 1 consists of \( \text{Ker}(p_n) \). (This topology can be defined by a bi-invariant metric: \( d(a, b) = C^{n(a,b)} \), where \( n(a, b) \) is the first position at which \( a, b \) differ, and \( 0 < C < 1 \)). A sequence \( a^n \) converges to \( a \) in this topology if for each \( k \), \( a_k^n \) eventually stabilizes to \( a_k \). It is easy to show that \( G \) is compact.

Profinite groups are ubiquitous in mathematics. For example, the \( p \)-adic integers \( \mathbb{Z}_p \) for a prime \( p \) form a profinite group, namely the inverse limit of \( \mathbb{Z}/p^n\mathbb{Z} \); in fact, it is a profinite ring. The multiplicative group of this ring \( \mathbb{Z}_p^* \) is also a profinite group. One may also consider non-abelian profinite groups \( GL_n(\mathbb{Z}_p), O_n(\mathbb{Z}_p), Sp_{2n}(\mathbb{Z}_p), \) etc. Finally, absolute Galois groups, such as \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), are (very complicated) profinite groups.

Note that infinite profinite groups are uncountable and **totally disconnected**, i.e., \( G^o = 1 \).
More generally, the inverse limit makes sense if \( G_i \) are compact Lie groups. In this case \( G \) is equipped with the product topology, so also compact (by Tychonoff’s theorem). For example, consider the sequence of Lie groups \( G_n = \mathbb{R}/\mathbb{Z} \) and maps \( \phi_i : G_{i+1} \to G_i \) given by \( \phi_i(x) = px \) for a prime \( p \). We can realize \( G_n \) as \( \mathbb{R}/p^n\mathbb{Z} \), then \( \phi_i(y) = y \mod p^i \).

Let \( G = \lim \leftarrow G_n \). We have projections \( p_n : G \to G_n \), and an element \( a \in \text{Ker}(p_1) \) is a sequence of elements \( a_n \in \mathbb{Z}/p^n \) such that \( a_{n+1} \) projects to \( a_n \), i.e., \( \text{Ker}(p_1) = \mathbb{Z}_p \). Thus we have a short exact sequence of compact topological groups

\[
0 \to \mathbb{Z}_p \to G \to \mathbb{R}/\mathbb{Z} \to 0
\]

(non-split, as \( G \) is connected). In fact, we can obtain \( G \) as a quotient \( (\mathbb{R} \times \mathbb{Z}_p)/\mathbb{Z} \) where \( \mathbb{Z} \) is embedded diagonally.

**Corollary 37.4.** Finite dimensional (continuous) representations of a compact topological group \( G \) with a countable base are unitary and completely reducible.

The proof is the same as for Lie groups, once we have the integration theory, which we now do.

### 37.2. The Peter-Weyl theorem for compact topological groups.

**Theorem 37.5.** (i) (Peter-Weyl theorem) Let \( G \) be a compact topological group with a countable base. Then the set \( \text{Irrep}G \) is countable, and

\[
L^2(G) = \bigoplus_{V \in \text{Irrep}(G)} V \otimes V^*
\]

as a \( G \times G \)-module.

(ii) The subspace \( L^2_{\text{alg}}(G) = \bigoplus_{V \in \text{Irrep}(G)} V \otimes V^* \) is dense in \( C(G) \) in the supremum norm.

Again, the proof is analogous to Lie groups, using a delta-like sequence of continuous hat functions. Namely, we may take

\[
h_N(x) = c_N \max(\frac{1}{N} - d(x, 1), 0),
\]

where \( d \) is some metric defining the topology of \( G \), and \( c_N > 0 \) are normalization constants such that \( \int_G h_N(x) dx = 1 \).

**Remark 37.6.** If \( G \) is profinite then finite dimensional representations of \( G \) are just representations of \( G_n \) for various \( n \):

\[
\text{Irrep}G = \bigcup_{n \geq 1} \text{Irrep}G_n
\]

(nested union).
Corollary 37.7. Any compact topological group with countable base is an inverse limit of a sequence of compact Lie groups \( G \to G_1 \to G_0 \), where the maps \( G_{i+1} \to G_i \) are surjective.

Proof. Let \( V_1, V_2, \ldots \) be the irreducible representations of \( G \). Let \( K_m = \ker(\rho_{V_1} \oplus \ldots \oplus \rho_{V_m}) \subset G \), a closed normal subgroup. Then \( G/K_m \subset U(V_1 \oplus \ldots \oplus V_m) \) is a compact Lie group, and \( \cap_m K_m = 1 \), so \( G \) is the inverse limit of \( G/K_m \). \( \square \)

Exercise 37.8. (i) Let \( \mathbb{Q}_p = \mathbb{Z}_p[1/p] \) be the field of \( p \)-adic numbers, i.e., the field of fractions of \( \mathbb{Z}_p \). Construct the Haar measure \( |dx| \) on the additive group of \( \mathbb{Q}_p \) in which the volume of \( \mathbb{Z}_p \) is 1 using the Haar measure on \( \mathbb{Z}_p \).

(ii) Show that \( \mathbb{Q} \subset \mathbb{Q}_p \) and \( \mathbb{Q}_p = \mathbb{Q} + \mathbb{Z}_p \), and use this to define an embedding \( \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z} \). Show that \( \mathbb{Q}/\mathbb{Z} = \bigoplus_{p \text{ prime}} \mathbb{Q}_p/\mathbb{Z}_p \).

(iii) Define the additive character \( \psi : \mathbb{Q}_p \to U(1) \subset \mathbb{C} \) by \( \psi(x) = \exp(2\pi i x) \), where \( x \) is the image of \( x \) in \( \mathbb{Q}/\mathbb{Z} \). Use \( \psi \) to label the characters (=irreducible representations) of \( \mathbb{Z}_p \) by \( \mathbb{Q}_p/\mathbb{Z}_p \).

(iv) Let \( |x| \) be the \( p \)-adic norm of \( x \in \mathbb{Q}_p \) (\( |x| = p^{-n} \) if \( x \in p^n\mathbb{Z}_p \) but \( x \notin p^{n+1}\mathbb{Z}_p \), and \( |0| = 0 \)). For which \( s \in \mathbb{C} \) is the function \( |x|^s \) in \( L^2(\mathbb{Z}_p) \)?

(v) The Peter-Weyl theorem in particular implies that any \( L^2 \) function \( f \) on a compact abelian group \( G \) with a countable base can be expanded in a Fourier series

\[
f(x) = \sum_j c_j \psi_j(x),
\]

where \( \psi_j \) are the characters of \( G \). Write the Fourier expansion of \( |x|^s \) when it is in \( L^2(\mathbb{Z}_p) \).

(vi) Show that \( |dx|/|x| \) is a Haar measure on the multiplicative group \( \mathbb{Q}_p^\times = GL_1(\mathbb{Q}_p) \). More generally, show that \( |dX| := \prod_{1 \leq i < j \leq n} |dx_{ij}|/|\det(X)|^n \) is a Haar measure on \( GL_n(\mathbb{Q}_p) \) (where \( X = (x_{ij}) \)).

(vii) Classify characters of \( \mathbb{Z}_p^\times \).

(viii) Let \( S \) be the the space of locally constant functions on \( \mathbb{Q}_p \) with compact support (i.e., linear combinations of indicator functions of sets of the form \( a + p^n\mathbb{Z}_p, a \in \mathbb{Q}_p \)). Show that the Fourier transform operator

\[
F(f) = \int_{\mathbb{Q}_p} \psi(xy) f(y) |dy|
\]

maps \( S \) to itself, and \( (F^2 f)(x) = f(-x) \). Show that \( F \) preserves the integration pairing on \( S, (f, g) = \int_{\mathbb{Q}_p} f(x) \overline{g(x)} |dx| \), and therefore extends to a unitary operator \( L^2(\mathbb{Q}_p) \to L^2(\mathbb{Q}_p) \).