

### 37. Representations of compact topological groups

**37.1. Existence of the Haar measure.** One can generalize integration theory to arbitrary compact and even to locally compact topological groups. For simplicity we will describe this generalization in the case of compact topological groups with a countable base.

Namely, let  $X$  be a compact Hausdorff topological space with a countable base. For compact Hausdorff spaces this is equivalent to being metrizable. Let  $C(X, \mathbb{R})$  be the space of continuous real-valued functions on  $X$ . This is a real Banach space with norm

$$\|f\| = \max_{x \in X} |f(x)|.$$

Recall that by the **Riesz-Markov-Kakutani representation theorem**, a finite Borel measure  $\mu$  on  $X$  is the same thing as a positive continuous linear functional  $I : C(X, \mathbb{R}) \rightarrow \mathbb{R}$  (i.e., such that  $I(f) \geq 0$  for  $f \geq 0$ ), namely,

$$I(f) = \int_X f d\mu.$$

Moreover,  $\mu$  is a probability measure if and only if  $I(1) = 1$ , and any  $\mu \neq 0$  has positive volume and so can be normalized to be a probability measure.

Now let  $G$  be a compact topological group with a countable base. It acts on  $C(G, \mathbb{R})$  by left and right translations, so acts on nonnegative probability measures of  $G$ .

**Theorem 37.1.** (*Haar, von Neumann*)  $G$  admits a unique left-invariant probability measure.

This measure is also automatically right-invariant (since it is unique) and is called the **Haar measure** on  $G$ .

**Remark 37.2.** A unique up to scaling left-invariant regular Haar measure (albeit of infinite volume and not always right-invariant in the non-compact case) exists more generally for any locally compact group  $G$  (not necessarily having a countable base).<sup>16</sup> We will not prove this here, but we remark that Haar measures on Lie groups that we have constructed using top differential forms are a special case of this.

*Proof.* Let  $g_i, i \geq 1$  be a dense sequence in  $G$  (it exists since  $G$  has a countable base, hence is separable, as you can pick a point in every open set of this base). Let  $p_i$  be a sequence of positive numbers

---

<sup>16</sup>Note that a finite Borel measure on a compact Hausdorff space with a countable base is necessarily regular.

such that  $\sum_i p_i = 1$ . To this data attach the **averaging operator**  $A : C(G, \mathbb{R}) \rightarrow C(G, \mathbb{R})$  given by

$$(Af)(x) = \sum_i p_i f(xg_i).$$

This operator can be interpreted as follows: we have a Markov chain with states being points of  $G$  and the transition probability from  $x$  to  $xg_i$  equal to  $p_i$ , then  $(Af)(x)$  is the expected value of  $f$  after one transition starting from  $x$ . It is clear that  $A$  is a left-invariant bounded operator (of norm 1). Moreover,  $A$  acts by the identity on the line  $L \subset C(G, \mathbb{R})$  of constant functions.

For  $f \in C(G, \mathbb{R})$  denote by  $\nu(f)$  the distance from  $f$  to  $L$ , i.e.,

$$\nu(f) = \frac{1}{2}(\max f - \min f).$$

Then  $\nu(Af) < \nu(f)$  unless  $f \in L$ . Indeed, if  $f$  is not constant and  $x \in G$ , pick  $j$  such that  $f(xg_j) < \max f$  (exists since the sequence  $xg_i$  is dense in  $G$ ), then

$$(Af)(x) = \sum_i p_i f(xg_i) \leq (1 - p_j) \max f + p_j f(xg_j) < \max f.$$

So  $\max(Af) < \max f$ . Similarly,  $\min(Af) > \min f$ .

Now fix  $f \in C(G, \mathbb{R})$  and consider the sequence  $f_n := A^n f$ ,  $n \geq 0$ . This means that we let our Markov chain run for  $n$  steps. We know that for finite Markov chains there is an asymptotic distribution, and we'll show that this is also the case in the situation at hand, giving rise to a construction of the invariant integral.

Obviously, the sequence  $f_n$  is uniformly bounded by  $\max |f|$ . Also it is **equicontinuous**: for any  $\varepsilon > 0$  there exists a neighborhood  $U \subset G$  such that for any  $x \in G$  and  $u \in U$ ,

$$|f_n(x) - f_n(ux)| < \varepsilon.$$

Indeed, it suffices to show that  $f$  is uniformly continuous, i.e., for any  $\varepsilon$  find  $U$  such that for all  $x \in G, u \in U$  we have  $|f(x) - f(ux)| < \varepsilon$ ; this  $U$  will then work for all  $f_n$ . But this is guaranteed by Cantor's theorem. Namely, assume the contrary, that there is no such  $U$ . Then there are two sequences  $x_i, u_i \in G$ ,  $u_i \rightarrow 1$ , with  $|f(x_i) - f(u_i x_i)| \geq \varepsilon$ . The sequence  $x_i$  has a convergent subsequence, so we may assume without loss of generality that  $x_i \rightarrow x \in G$ . Then taking the limit  $i \rightarrow \infty$ , we get that  $\varepsilon \leq 0$ , a contradiction.

Therefore, by the **Ascoli-Arzelà theorem** the sequence  $f_n$  has a convergent subsequence. Let us remind the proof of this theorem. We construct subsequences  $f_n^k$  of  $f_n$  inductively by picking  $f_n^k$  from  $f_n^{k-1}$  so that  $f_n^k(g_k)$  converges (with  $f_n^0 = f_n$ ), which can be done by the

boundedness assumption, and then set  $h_m := f_m^m = f_{n(m)}$ . Then  $h_m(g_i)$  converges, hence Cauchy, for all  $i$ , which by equicontinuity implies that  $h_m(x)$  is a Cauchy sequence in  $C(G, \mathbb{R})$ , hence converges to some  $h \in C(G, \mathbb{R})$ .

We claim that  $h \in L$ . Indeed, we have

$$\nu(f_{n(m)}) \geq \nu(f_{n(m)+1}) = \nu(Af_{n(m)}) \geq \nu(f_{n(m+1)}),$$

so taking the limit when  $m \rightarrow \infty$ , we get

$$\nu(h) \geq \nu(Ah) \geq \nu(h),$$

i.e.,  $\nu(Ah) = \nu(h)$ . The assignment  $f \mapsto h$  is therefore a continuous left-invariant positive linear functional  $I : C(G, \mathbb{R}) \rightarrow L = \mathbb{R}$ , and  $I(1) = 1$ , as claimed.

Similarly, we may construct a right-invariant integral

$$I_* : C(G, \mathbb{R}) \rightarrow L = \mathbb{R}$$

with  $I_*(1) = 1$ , and by construction for any left invariant integral  $J$  we have  $J(f) = J(I_*(f))$ . Thus for every left invariant integral  $J$  with  $J(1) = 1$  we have  $J(f) = I_*(f)$ ; in particular  $I(f) = I_*(f)$ . This shows that  $I$  is unique, invariant on both sides and independent on the choice of  $g_i, p_i$ , and hence that  $A^n f \rightarrow I(f)$  as  $n \rightarrow \infty$ .  $\square$

**Example 37.3.** A basic example of a compact topological group with countable base which is, in general, not a Lie group, is a **profinite group**. Namely, let  $G_1, G_2, \dots$  be finite groups and  $\phi_i : G_{i+1} \rightarrow G_i$  be surjective homomorphisms. Then the **inverse limit**  $G := \varprojlim G_n$  is the group consisting of sequences  $g_1 \in G_1, g_2 \in G_2, \dots$  where  $\phi_i(g_{i+1}) = g_i$ . This group  $G$  has projections  $p_n : G \rightarrow G_n$  and a natural topology, for which a base of neighborhoods of 1 consists of  $\text{Ker}(p_n)$ . (This topology can be defined by a bi-invariant metric:  $d(\mathbf{a}, \mathbf{b}) = C^{n(\mathbf{a}, \mathbf{b})}$ , where  $n(\mathbf{a}, \mathbf{b})$  is the first position at which  $\mathbf{a}, \mathbf{b}$  differ, and  $0 < C < 1$ ). A sequence  $\mathbf{a}^n$  converges to  $\mathbf{a}$  in this topology if for each  $k$ ,  $a_k^n$  eventually stabilizes to  $a_k$ . It is easy to show that  $G$  is compact.

Profinite groups are ubiquitous in mathematics. For example, the  **$p$ -adic integers**  $\mathbb{Z}_p$  for a prime  $p$  form a profinite group, namely the inverse limit of  $\mathbb{Z}/p^n\mathbb{Z}$ ; in fact, it is a profinite ring. The multiplicative group of this ring  $\mathbb{Z}_p^\times$  is also a profinite group. One may also consider non-abelian profinite groups  $GL_n(\mathbb{Z}_p), O_n(\mathbb{Z}_p), Sp_{2n}(\mathbb{Z}_p)$ , etc. Finally, absolute Galois groups, such as  $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ , are (very complicated) profinite groups.

Note that infinite profinite groups are uncountable and **totally disconnected**, i.e.,  $G^\circ = 1$ .

More generally, the inverse limit makes sense if  $G_i$  are compact Lie groups. In this case  $G$  is equipped with the product topology, so also compact (by Tychonoff's theorem). For example, consider the sequence of Lie groups  $G_n = \mathbb{R}/\mathbb{Z}$  and maps  $\phi_i : G_{i+1} \rightarrow G_i$  given by  $\phi_i(x) = px$  for a prime  $p$ . We can realize  $G_n$  as  $\mathbb{R}/p^n\mathbb{Z}$ , then  $\phi_i(y) = y \bmod p^i$ . Let  $G := \varprojlim G_n$ . We have projections  $p_n : G \rightarrow G_n$ , and an element  $a \in \text{Ker}(p_1)$  is a sequence of elements  $a_n \in \mathbb{Z}/p^n$  such that  $a_{n+1}$  projects to  $a_n$ , i.e.,  $\text{Ker}(p_1) = \mathbb{Z}_p$ . Thus we have a short exact sequence of compact topological groups

$$0 \rightarrow \mathbb{Z}_p \rightarrow G \rightarrow \mathbb{R}/\mathbb{Z} \rightarrow 0$$

(non-split, as  $G$  is connected). In fact, we can obtain  $G$  as a quotient  $(\mathbb{R} \times \mathbb{Z}_p)/\mathbb{Z}$  where  $\mathbb{Z}$  is embedded diagonally.

**Corollary 37.4.** *Finite dimensional (continuous) representations of a compact topological group  $G$  with a countable base are unitary and completely reducible.*

The proof is the same as for Lie groups, once we have the integration theory, which we now do.

### 37.2. The Peter-Weyl theorem for compact topological groups.

**Theorem 37.5.** (i) *(Peter-Weyl theorem) Let  $G$  be a compact topological group with a countable base. Then the set  $\text{Irrep}G$  is countable, and*

$$L^2(G) = \widehat{\bigoplus}_{V \in \text{Irrep}(G)} V \otimes V^*$$

as a  $G \times G$ -module.

(ii) *The subspace  $L^2_{\text{alg}}(G) = \bigoplus_{V \in \text{Irrep}(G)} V \otimes V^*$  is dense in  $C(G)$  in the supremum norm.*

Again, the proof is analogous to Lie groups, using a delta-like sequence of continuous hat functions. Namely, we may take

$$h_N(x) = c_N \max\left(\frac{1}{N} - d(x, 1), 0\right),$$

where  $d$  is some metric defining the topology of  $G$ , and  $c_N > 0$  are normalization constants such that  $\int_G h_N(x) dx = 1$ .

**Remark 37.6.** If  $G$  is profinite then finite dimensional representations of  $G$  are just representations of  $G_n$  for various  $n$ :

$$\text{Irrep}G = \bigcup_{n \geq 1} \text{Irrep}G_n$$

(nested union).

**Corollary 37.7.** Any compact topological group with countable base is an inverse limit of a sequence of compact Lie groups  $\dots \rightarrow G_1 \rightarrow G_0$ , where the maps  $G_{i+1} \rightarrow G_i$  are surjective.

*Proof.* Let  $V_1, V_2, \dots$  be the irreducible representations of  $G$ . Let  $K_m = \text{Ker}(\rho_{V_1} \oplus \dots \oplus \rho_{V_m}) \subset G$ , a closed normal subgroup. Then  $G/K_m \subset U(V_1 \oplus \dots \oplus V_m)$  is a compact Lie group, and  $\bigcap_m K_m = 1$ , so  $G$  is the inverse limit of  $G/K_m$ .  $\square$

**Exercise 37.8.** (i) Let  $\mathbb{Q}_p = \mathbb{Z}_p[1/p]$  be the field of  $p$ -adic numbers, i.e., the field of fractions of  $\mathbb{Z}_p$ . Construct the Haar measure  $|dx|$  on the additive group of  $\mathbb{Q}_p$  in which the volume of  $\mathbb{Z}_p$  is 1 using the Haar measure on  $\mathbb{Z}_p$ .

(ii) Show that  $\mathbb{Q} \subset \mathbb{Q}_p$  and  $\mathbb{Q}_p = \mathbb{Q} + \mathbb{Z}_p$ , and use this to define an embedding  $\mathbb{Q}_p/\mathbb{Z}_p \rightarrow \mathbb{Q}/\mathbb{Z}$ . Show that  $\mathbb{Q}/\mathbb{Z} = \bigoplus_{p \text{ prime}} \mathbb{Q}_p/\mathbb{Z}_p$ .

(iii) Define the additive character  $\psi : \mathbb{Q}_p \rightarrow U(1) \subset \mathbb{C}^\times$  by  $\psi(x) := \exp(2\pi i \bar{x})$ , where  $\bar{x}$  is the image of  $x$  in  $\mathbb{Q}/\mathbb{Z}$ . Use  $\psi$  to label the characters (=irreducible representations) of  $\mathbb{Z}_p$  by  $\mathbb{Q}_p/\mathbb{Z}_p$ .

(iv) Let  $|x|$  be the  $p$ -adic norm of  $x \in \mathbb{Q}_p$  ( $|x| = p^{-n}$  if  $x \in p^n \mathbb{Z}_p$  but  $x \notin p^{n+1} \mathbb{Z}_p$ , and  $|0| = 0$ ). For which  $s \in \mathbb{C}$  is the function  $|x|^s$  in  $L^2(\mathbb{Z}_p)$ ?

(v) The Peter-Weyl theorem in particular implies that any  $L^2$  function  $f$  on a compact abelian group  $G$  with a countable base can be expanded in a Fourier series

$$f(x) = \sum_j c_j \psi_j(x),$$

where  $\psi_j$  are the characters of  $G$ . Write the Fourier expansion of  $|x|^s$  when it is in  $L^2(\mathbb{Z}_p)$ .

(vi) Show that  $\frac{|dx|}{|x|}$  is a Haar measure on the multiplicative group  $\mathbb{Q}_p^\times = GL_1(\mathbb{Q}_p)$ . More generally, show that  $|dX| := \frac{\prod_{1 \leq i, j \leq n} |dx_{ij}|}{|\det(X)|^n}$  is a Haar measure on  $GL_n(\mathbb{Q}_p)$  (where  $X = (x_{ij})$ ).

(vii) Classify characters of  $\mathbb{Z}_p^\times$ .

(viii) Let  $S$  be the space of locally constant functions on  $\mathbb{Q}_p$  with compact support (i.e., linear combinations of indicator functions of sets of the form  $a + p^n \mathbb{Z}_p$ ,  $a \in \mathbb{Q}_p$ ). Show that the Fourier transform operator

$$\mathcal{F}(f) = \int_{\mathbb{Q}_p} \psi(xy) f(y) |dy|$$

maps  $S$  to itself, and  $(\mathcal{F}^2 f)(x) = f(-x)$ . Show that  $\mathcal{F}$  preserves the integration pairing on  $S$ ,  $(f, g) = \int_{\mathbb{Q}_p} f(x) \overline{g(x)} |dx|$ , and therefore extends to a unitary operator  $L^2(\mathbb{Q}_p) \rightarrow L^2(\mathbb{Q}_p)$ .

MIT OpenCourseWare  
<https://ocw.mit.edu>

18.755 Lie Groups and Lie Algebras II  
Spring 2024

For information about citing these materials or our Terms of Use, visit: <https://ocw.mit.edu/terms>.