40. **Forms of semisimple Lie algebras over an arbitrary field**

40.1. **Automorophisms of semisimple Lie algebras.**

We showed in Corollary 17.10 that for a complex semisimple $\mathfrak{g}$, the group $\mathrm{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathfrak{g}$. We also showed in Theorem 20.10 that its connected component of the identity $\mathrm{Aut}(\mathfrak{g})^\circ$ acts transitively on the set of Cartan subalgebras in $\mathfrak{g}$. This group is called the **adjoint group** attached to $\mathfrak{g}$, and we will denote it by G_{ad}.

Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and $H \subset G_{\mathrm{ad}}$ be the corresponding connected Lie subgroup. This subgroup can be viewed as the group of linear operators $\mathfrak{g} \to \mathfrak{g}$ which act by 1 on $\mathfrak{h}$ and by $e^{\alpha(x)}$, $x \in \mathfrak{h}$, on each $\mathfrak{g}_\alpha$. Thus the exponential map $\mathfrak{h} \to H$ defines an isomorphism $\mathfrak{h}/2\pi i P^\vee \cong H$. The group H is called the **maximal torus** of G_{ad} corresponding to $\mathfrak{h}$.

Proposition 40.1. *The normalizer $N(H)$ of H in G_{ad} coincides with the stabilizer of $\mathfrak{h}$ and contains H as a normal subgroup, so that $N(H)/H$ is naturally isomorphic to the Weyl group W.*

Proof. First note that since $SL_2(\mathbb{C})$ is simply connected, for any simple root α_i we have a homomorphism $\eta_i : SL_2(\mathbb{C}) \to G_{\mathrm{ad}}$ which identifies $\mathrm{Lie}(SL_2(\mathbb{C}))$ with the $\mathfrak{sl}_2$-subalgebra of $\mathfrak{g}$ corresponding to this simple root. Let

$$S_i := \eta_i\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right). \tag{40.1}$$

Given $w \in W$, pick a decomposition $w = s_{i_1}...s_{i_n}$, and let $\widetilde{w} := S_{i_1}...S_{i_n} \in G_{\mathrm{ad}}$.[19] Note that $\widetilde{w}$ acts on $\mathfrak{h}$ by w. So if $w = w_1 w_2 \in W$ then $\widetilde{w} = \widetilde{w}_1 \widetilde{w}_2 h$, where h preserves the root decomposition and acts trivially on $\mathfrak{h}$. Thus if $h|_{\mathfrak{g}_{\alpha_j}} = \exp(b_j)$ then $h = \exp(\sum_j b_j \omega_j^\vee) \in H$. So the elements $\widetilde{w}$ and H generate a subgroup $N \subset N(H)$ of G_{ad} such that $N/H \cong W$.

It remains to show that $N(H) = N$. To this end, for $x \in N(H)$, let $\alpha_i' = x(\alpha_i)$. Then α_i' form a system of simple roots, so there exists $w \in W$ such that $w(\alpha_i') = \alpha_{p(i)}$, where p is some permutation. Then $\widetilde{w}x(\alpha_i) = \alpha_{p(i)}$. So $\widetilde{w}x$ defines a Dynkin diagram automorphism of $\mathfrak{g}$. Since this automorphism is defined by an element of G_{ad}, it stabilizes all fundamental representations, so $p = \mathrm{id}$, hence $\widetilde{w}x \in H$, as claimed. □

In particular, we see that H is a maximal commutative subgroup of G_{ad}, hence the terminology "maximal torus".

[19]The element $\widetilde{w}$ in general depends on the decomposition of w as a product of simple reflections. One can show it does not if we take only reduced decompositions, but we will not need this.

Remark 40.2. Note that in general $N(H)$ is **not** isomorphic to $W \ltimes H$: it can be a non-split extension of W by H.

Another obvious subgroup of $\mathrm{Aut}(\mathfrak{g})$ is the finite group $\mathrm{Aut}(D)$ of automorphisms of the Dynkin diagram of $\mathfrak{g}$, which just permutes the generators e_i, f_i, h_i in the Serre presentation. Thus we have a natural homomorphism

$$\xi : \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}} \to \mathrm{Aut}(\mathfrak{g}),$$

which is the identity map on the connected components of 1. This homomorphism is clearly injective, since the center of G_{ad} is trivial and any nontrivial element of $\mathrm{Aut}(D)$ nontrivially permutes fundamental representations of $\mathfrak{g}$.

Proposition 40.3. *ξ is an isomorphism.*

Proof. Our job is to show that ξ is surjective, i.e. for $a \in \mathrm{Aut}(\mathfrak{g})$ show that $a \in \mathrm{Im}\xi$. By Theorem 20.10, we may assume without loss of generality that a preserves a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (indeed, this can be arranged by multiplying by an element of G_{ad}, since G_{ad} acts transitively on Cartan subalgebras of $\mathfrak{g}$). Then by multiplying by an element of $\mathrm{Aut}(D) \cdot N(H)$ we can make sure that a acts trivially on $\mathfrak{h}$ and $\mathfrak{g}_{\alpha_i}$. Then $a = 1$, which implies the proposition. □

40.2. **Forms of semisimple Lie algebras.** We have classified semisimple Lie algebras over $\mathbb{C}$, but what about other fields (say of characteristic zero), notably $\mathbb{R}$ (the case relevant to the theory of Lie groups)?

To address this question, note that the Serre presentation of a semisimple Lie algebra is defined over $\mathbb{Q}$, so it defines a Lie algebra of the same dimension over any such field, by imposing the same generators and relations. Such a Lie algebra is called **split**. So for example, over an algebraically closed field of characteristic zero, any semisimple Lie algebra is automatically split.

Now let $\mathfrak{g}$ be a semisimple Lie algebra over a field K of characteristic zero which splits over a Galois extension L of K, i.e., $\mathfrak{g} \otimes_K L = \mathfrak{g}_L$ is split (corresponds to a Dynkin diagram via Serre's presentation). Can we classify such $\mathfrak{g}$?

To this end, let $\Gamma = \mathrm{Gal}(L/K)$ be the Galois group of L over K and observe that we can recover $\mathfrak{g}$ as the subalgebra of invariants $\mathfrak{g}_L^\Gamma$. So $\mathfrak{g}$ is determined by the action of Γ on the split semisimple Lie algebra $\mathfrak{g}_L$. Note that this action is **twisted-linear**, i.e., additive and $g(\lambda x) = g(\lambda)g(x)$ for $x \in \mathfrak{g}_L$, $\lambda \in L$, $g \in \Gamma$. The simplest example of such an action is the action $\rho_0(g)$ which preserves all the generators e_i, f_i, h_i and just acts on the scalars, which corresponds to the split form of $\mathfrak{g}$.

So any twisted-linear action ρ can be written as

$$\rho(g) = \eta(g)\rho_0(g)$$

for some map

$$\eta : \Gamma \to \mathrm{Aut}(\mathfrak{g}_L).$$

In order that ρ be a homomorphism, we need

$$\eta(gh)\rho_0(gh) = \eta(g)\rho_0(g)\eta(h)\rho_0(h),$$

which is equivalent to

$$\eta(gh) = \eta(g) \cdot g(\eta(h)).$$

In other words, η is a **1-cocycle**. We will denote the Lie algebra attached to such cocycle η by $\mathfrak{g}_\eta$.

It remains to determine when $\mathfrak{g}_{\eta_1}$ is isomorphic to $\mathfrak{g}_{\eta_2}$. This will happen exactly when the corresponding representations ρ_1 and ρ_2 are isomorphic, i.e., there is $a \in \mathrm{Aut}(\mathfrak{g}_L)$ such that $\rho_1(g)a = a\rho_2(g)$, i.e.,

$$\eta_1(g)\rho_0(g)a = a\eta_2(g)\rho_0(g),$$

or

$$\eta_1(g) = a\eta_2(g)g(a)^{-1}.$$

Two 1-cocycles related in this way are called **cohomologous** (obviously, an equivalence relation), and the set of equivalence classes of cohomologous cocycles is called the **first Galois cohomology** of Γ with coefficients in $\mathrm{Aut}(\mathfrak{g}_L)$ and denoted by $H^1(\Gamma, \mathrm{Aut}(\mathfrak{g}_L))$. Note that this is cohomology with coefficients in a nonabelian group, so it is just a set and not a group.

So we obtain

Proposition 40.4. *Semisimple Lie algebras $\mathfrak{g}$ over K with fixed $\mathfrak{g}_L$ are classified by the first Galois cohomology $H^1(\Gamma, \mathrm{Aut}(\mathfrak{g}_L))$.*

Remark 40.5. There is nothing special about semisimplicity or about Lie algebras here – this works for any kind of linear algebraic structures, such as associative algebras, algebraic varieties, schemes, etc.

40.3. **Real forms of a semisimple Lie algebra.** Let us now make this classification more concrete in the case $K = \mathbb{R}$, $L = \mathbb{C}$, which is relevant to classification of real semisimple Lie groups. In this case, $\Gamma = \mathbb{Z}/2$ generated by complex conjugation and, as we have shown, $\mathrm{Aut}(\mathfrak{g}_L) = \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}}$, where D is the Dynkin diagram of $\mathfrak{g}$ and G_{ad} is the corresponding connected adjoint complex Lie group. Also since we always have $\eta(1) = 1$, the cocycle η is determined by the element $s = \eta(-1) \in \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}}$. Moreover, s must satisfy the cocycle condition

$$s\overline{s} = 1$$

and the corresponding real Lie algebra, up to isomorphism, depends only on the cohomology class of s, which is the equivalence class modulo transformations $s \mapsto as\overline{a}^{-1}$. We thus obtain the following theorem.

Theorem 40.6. *Real semisimple Lie algebras whose complexification is $\mathfrak{g}$ (i.e.,* **real forms** *of $\mathfrak{g}$) are classified by $s \in \mathrm{Aut}(D) \ltimes G_{\mathrm{ad}}$ such that $s\overline{s} = 1$ modulo equivalence $s \mapsto as\overline{a}^{-1}$, $a \in \mathrm{Aut}(\mathfrak{g})$, where complex conjugation acts trivially on* $\mathrm{Aut}(D)$*.*

We denote the real form of $\mathfrak{g}$ corresponding to s by $\mathfrak{g}_{(s)}$. Namely, $\mathfrak{g}_{(s)} = \{x \in \mathfrak{g} : \overline{x} = s(x)\}$. For example, $\mathfrak{g}_{(1)}$ is the split form, consisting of real $x \in \mathfrak{g}$, i.e., such that $\overline{x} = x$.

Alternatively, one may define the **antilinear involution** $\sigma_s(x) = \overline{s(x)}$, and $\mathfrak{g}_{(s)}$ is the set of fixed points of σ_s in $\mathfrak{g}$.

In particular, such s defines an element $s_0 \in \mathrm{Aut}(D)$ such that $s_0^2 = 1$. Note that the conjugacy class of s_0 is invariant under equivalences. The element s_0 permutes connected components of D, preserving some and matching others into pairs. Thus every semisimple real Lie algebra is a direct sum of simple ones, and each simple one either has a connected Dynkin diagram D (i.e., the complexified Lie algebra $\mathfrak{g}$ is still simple) or consists of two identical components (i.e., the complexified Lie algebra is $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}$ for some simple complex $\mathfrak{a}$). In the latter case $s = (g, \overline{g}^{-1})s_0$ where s_0 is the transposition and $g \in \mathrm{Aut}(\mathfrak{a})$, so s is cohomologous to s_0 by taking $a = (g, 1)$. Thus in this case $\mathfrak{g}_{(s)} = \mathfrak{g}_{(s_0)} = \mathfrak{a}$, a complex simple Lie algebra regarded as a real Lie algebra.

It remains to consider the case when D is connected, i.e., $\mathfrak{g}$ is simple.

Definition 40.7. (i) A real form $\mathfrak{g}_{(s)}$ of a complex simple Lie algebra $\mathfrak{g}$ is said to be **inner** to $\mathfrak{g}_{(s')}$ if $s' = gs$ up to equivalence, where $g \in \mathrm{G}_{\mathrm{ad}}$ (i.e., s and s' differ by an inner automorphism). The **inner class** of $\mathfrak{g}_{(s)}$ is the collection of all real forms inner to $\mathfrak{g}_{(s)}$. In particular, an **inner form** is a form inner to the split form.

(ii) $\mathfrak{g}_{(s)}$ is called **quasi-split** if $s = s_0 \in \mathrm{Aut}(D)$ (modulo equivalence).

So in particular any real form is inner to a unique quasi-split form, and a real form that is both inner and quasi-split is split.

Exercise 40.8. Let $\mathfrak{g}_{\mathbb{R}}$ be a real semisimple Lie algebra and $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ a Cartan subalgebra (the centralizer of a regular semisimple element of $\mathfrak{g}_{\mathbb{R}}$). Let $\mathfrak{h} \subset \mathfrak{g}$ be their complexifications, and $H \subset G_{\mathrm{ad}}$ the corresponding complex Lie groups. Let $\mathbf{K}$ be the kernel of the natural map of Galois cohomology sets $H^1(\mathbb{Z}/2, N(H)) \to H^1(\mathbb{Z}/2, G_{\mathrm{ad}})$ (i.e., the preimage of the unit element), where $\mathbb{Z}/2$ acts on G_{ad} by complex conjugation associated to the real form $\mathfrak{g}_{\mathbb{R}}$ of $\mathfrak{g}$.

(i) Show that conjugacy classes of Cartan subalgebras in $\mathfrak{g}_{\mathbb{R}}$ are bijectively labeled by elements of $\mathbf{K}$, with the unit element corresponding to $\mathfrak{h}_{\mathbb{R}}$.

(ii) Show that $\mathbf{K}$ is a finite set.[20]

[20]For classical Lie algebras the set $\mathbf{K}$ will be computed explicitly in Exercise 44.18. The explicit answer is known for exceptional Lie algebras as well, but we will not discuss it here.

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