40. Forms of semisimple Lie algebras over an arbitrary field

40.1. Automorphisms of semisimple Lie algebras. We showed in Corollary 17.10 that for a complex semisimple $\mathfrak{g}$, the group $\text{Aut}(\mathfrak{g})$ is a Lie group with Lie algebra $\mathfrak{g}$. We also showed in Theorem 20.10 that its connected component of the identity $\text{Aut}(\mathfrak{g})^0$ acts transitively on the set of Cartan subalgebras in $\mathfrak{g}$. This group is called the adjoint group attached to $\mathfrak{g}$, and we will denote it by $\text{Ad}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and $H \subset \text{Ad}$ be the corresponding connected Lie subgroup. This subgroup can be viewed as the group of linear operators $\mathfrak{g} \to \mathfrak{g}$ which act by 1 on $\mathfrak{h}$ and by $e^{\alpha}(x)$, $x \in \mathfrak{h}$, on each $\mathfrak{g}_\alpha$. Thus the exponential map $\mathfrak{h} \to H$ defines an isomorphism $\mathfrak{h} / 2\pi i \mathfrak{P}^\vee \cong H$. The group $H$ is called the maximal torus of $\text{Ad}$ corresponding to $\mathfrak{h}$.

Proposition 40.1. The normalizer $N(H)$ of $H$ in $\text{Ad}$ coincides with the stabilizer of $\mathfrak{h}$ and contains $H$ as a normal subgroup, so that $N(H)/H$ is naturally isomorphic to the Weyl group $W$.

Proof. First note that since $SL_2(\mathbb{C})$ is simply connected, for any simple root $\alpha_i$ we have a homomorphism $\eta_i : SL_2(\mathbb{C}) \to \text{Ad}$ which identifies $\text{Lie}(SL_2(\mathbb{C}))$ with the $sl_2$-subalgebra of $\mathfrak{g}$ corresponding to this simple root. Let

\[ S_i := \eta_i \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right). \]

Given $w \in W$, pick a decomposition $w = s_{i_1} \cdots s_{i_n}$, and let $\tilde{w} := S_{i_1} \cdots S_{i_n} \in \text{Ad}$.\(^{19}\) Note that $\tilde{w}$ acts on $\mathfrak{h}$ by $w$. So if $w = w_1 w_2 \in W$ then $\tilde{w} = \tilde{w}_1 \tilde{w}_2 h$, where $h$ preserves the root decomposition and acts trivially on $\mathfrak{h}$. Thus if $h|_{\mathfrak{g}_{\alpha_j}} = \exp(b_j)$ then $h = \exp(\sum_j b_j \omega_j^\vee) \in H$. So the elements $\tilde{w}$ and $H$ generate a subgroup $N \subset N(H)$ of $\text{Ad}$ such that $N/H \cong W$.

It remains to show that $N(H) = N$. To this end, for $x \in N(H)$, let $\alpha'_i = x(\alpha_i)$. Then $\alpha'_i$ form a system of simple roots, so there exists $w \in W$ such that $w(\alpha'_i) = \alpha_{p(i)}$, where $p$ is some permutation. Then $\tilde{w}x(\alpha_i) = \alpha_{p(i)}$. So $\tilde{w}x$ defines a Dynkin diagram automorphism of $\mathfrak{g}$. Since this automorphism is defined by an element of $\text{Ad}$, it stabilizes all fundamental representations, so $p = \text{id}$, hence $\tilde{w}x \in H$, as claimed. \(\square\)

In particular, we see that $H$ is a maximal commutative subgroup of $\text{Ad}$, hence the terminology “maximal torus”.

\(^{19}\)The element $\tilde{w}$ in general depends on the decomposition of $w$ as a product of simple reflections. One can show it does not if we take only reduced decompositions, but we will not need this.
Remark 40.2. Note that in general $N(H)$ is not isomorphic to $W \ltimes H$: it can be a non-split extension of $W$ by $H$.

Another obvious subgroup of $\text{Aut}(\mathfrak{g})$ is the finite group $\text{Aut}(D)$ of automorphisms of the Dynkin diagram of $\mathfrak{g}$, which just permutes the generators $e_i, f_i, h_i$ in the Serre presentation. Thus we have a natural homomorphism

$$\xi : \text{Aut}(D) \ltimes G_{\text{ad}} \rightarrow \text{Aut}(\mathfrak{g}),$$

which is the identity map on the connected components of 1. This homomorphism is clearly injective, since the center of $G_{\text{ad}}$ is trivial and any nontrivial element of $\text{Aut}(D)$ nontrivially permutes fundamental representations of $\mathfrak{g}$.

Proposition 40.3. $\xi$ is an isomorphism.

Proof. Our job is to show that $\xi$ is surjective, i.e. for $a \in \text{Aut}(\mathfrak{g})$ show that $a \in \text{Im}\xi$. By Theorem 20.10, we may assume without loss of generality that $a$ preserves a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ (indeed, this can be arranged by multiplying by an element of $G_{\text{ad}}$, since $G_{\text{ad}}$ acts transitively on Cartan subalgebras of $\mathfrak{g}$). Then by multiplying by an element of $\text{Aut}(D) \cdot N(H)$ we can make sure that $a$ acts trivially on $\mathfrak{h}$ and $\mathfrak{g}_\alpha$. Then $a = 1$, which implies the proposition. \qed

40.2. Forms of semisimple Lie algebras. We have classified semisimple Lie algebras over $\mathbb{C}$, but what about other fields (say of characteristic zero), notably $\mathbb{R}$ (the case relevant to the theory of Lie groups)?

To address this question, note that the Serre presentation of a semisimple Lie algebra is defined over $\mathbb{Q}$, so it defines a Lie algebra of the same dimension over any such field, by imposing the same generators and relations. Such a Lie algebra is called split. So for example, over an algebraically closed field of characteristic zero, any semisimple Lie algebra is automatically split.

Now let $\mathfrak{g}$ be a semisimple Lie algebra over a field $K$ of characteristic zero which splits over a Galois extension $L$ of $K$, i.e., $\mathfrak{g} \otimes_K L = \mathfrak{g}_L$ is split (corresponds to a Dynkin diagram via Serre’s presentation). Can we classify such $\mathfrak{g}$?

To this end, let $\Gamma = \text{Gal}(L/K)$ be the Galois group of $L$ over $K$ and observe that we can recover $\mathfrak{g}$ as the subalgebra of invariants $\mathfrak{g}_L^\Gamma$. So $\mathfrak{g}$ is determined by the action of $\Gamma$ on the split semisimple Lie algebra $\mathfrak{g}_L$. Note that this action is twisted-linear, i.e., additive and $g(\lambda x) = g(\lambda)g(x)$ for $x \in \mathfrak{g}_L$, $\lambda \in L$, $g \in \Gamma$. The simplest example of such an action is the action $\rho_0(g)$ which preserves all the generators $e_i, f_i, h_i$ and just acts on the scalars, which corresponds to the split form of $\mathfrak{g}$. 213
So any twisted-linear action $\rho$ can be written as

$$\rho(g) = \eta(g) \rho_0(g)$$

for some map

$$\eta : \Gamma \to \text{Aut}(g_L).$$

In order that $\rho$ be a homomorphism, we need

$$\eta(gh) \rho_0(gh) = \eta(g) \rho_0(g) \eta(h) \rho_0(h),$$

which is equivalent to

$$\eta(gh) = \eta(g) \cdot g(\eta(h)).$$

In other words, $\eta$ is a 1-cocycle. We will denote the Lie algebra attached to such cocycle $\eta$ by $g_\eta$.

It remains to determine when $g_{\eta_1}$ is isomorphic to $g_{\eta_2}$. This will happen exactly when the corresponding representations $\rho_1$ and $\rho_2$ are isomorphic, i.e., there is $a \in \text{Aut}(g_L)$ such that $\rho_1(g)a = a\rho_2(g)$, i.e.,

$$\eta_1(g) \rho_0(g)a = a\eta_2(g) \rho_0(g),$$

or

$$\eta_1(g) = a\eta_2(g)g(a)^{-1}.$$ Two 1-cocycles related in this way are called cohomologous (obviously, an equivalence relation), and the set of equivalence classes of cohomologous cocycles is called the first Galois cohomology of $\Gamma$ with coefficients in $\text{Aut}(g_L)$ and denoted by $H^1(\Gamma, \text{Aut}(g_L))$. Note that this is cohomology with coefficients in a nonabelian group, so it is just a set and not a group.

So we obtain

**Proposition 40.4.** _Semisimple Lie algebras $g$ over $K$ with fixed $g_L$ are classified by the first Galois cohomology $H^1(\Gamma, \text{Aut}(g_L))$._

**Remark 40.5.** There is nothing special about semisimplicity or about Lie algebras here – this works for any kind of linear algebraic structures, such as associative algebras, algebraic varieties, schemes, etc.

40.3. **Real forms of a semisimple Lie algebra.** Let us now make this classification more concrete in the case $K = \mathbb{R}$, $L = \mathbb{C}$, which is relevant to classification of real semisimple Lie groups. In this case, $\Gamma = \mathbb{Z}/2$ generated by complex conjugation and, as we have shown, $\text{Aut}(g_L) = \text{Aut}(D) \rtimes G_{\text{ad}}$, where $D$ is the Dynkin diagram of $g$ and $G_{\text{ad}}$ is the corresponding connected adjoint complex Lie group. Also since we always have $\eta(1) = 1$, the cocycle $\eta$ is determined by the
element \( s = \eta(-1) \in \operatorname{Aut}(D) \ltimes G_{\text{ad}} \). Moreover, \( s \) must satisfy the cocycle condition
\[
s \overline{s} = 1
\]
and the corresponding real Lie algebra, up to isomorphism, depends only on the cohomology class of \( s \), which is the equivalence class modulo transformations \( s \mapsto as^{-1} \). We thus obtain the following theorem.

\textbf{Theorem 40.6.} Real semisimple Lie algebras whose complexification is \( \mathfrak{g} \) (i.e., real forms of \( \mathfrak{g} \)) are classified by \( s \in \operatorname{Aut}(D) \ltimes G_{\text{ad}} \) such that \( s \overline{s} = 1 \) modulo equivalence \( s \mapsto as^{-1} \), \( a \in \operatorname{Aut}(\mathfrak{g}) \), where complex conjugation acts trivially on \( \operatorname{Aut}(D) \).

We denote the real form of \( \mathfrak{g} \) corresponding to \( s \) by \( \mathfrak{g}(s) \). Namely, \( \mathfrak{g}(s) = \{ x \in \mathfrak{g} : \overline{x} = s(x) \} \). For example, \( \mathfrak{g}(1) \) is the split form, consisting of real \( x \in \mathfrak{g} \), i.e., such that \( \overline{x} = x \).

Alternatively, one may define the antilinear involution \( \sigma_s(x) = s(x) \), and \( \mathfrak{g}(s) \) is the set of fixed points of \( \sigma_s \) in \( \mathfrak{g} \).

In particular, such \( s \) defines an element \( s_0 \in \operatorname{Aut}(D) \) such that \( s_0^2 = 1 \). Note that the conjugacy class of \( s_0 \) is invariant under equivalences. The element \( s_0 \) permutes connected components of \( D \), preserving some and matching others into pairs. Thus every semisimple real Lie algebra is a direct sum of simple ones, and each simple one either has a connected Dynkin diagram \( D \) (i.e., the complexified Lie algebra \( \mathfrak{g} \) is still simple) or consists of two identical components (i.e., the complexified Lie algebra is \( \mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a} \) for some simple complex \( \mathfrak{a} \)). In the latter case \( s = (g, \overline{g}^{-1})s_0 \) where \( s_0 \) is the transposition and \( g \in \operatorname{Aut}(\mathfrak{a}) \), so \( s \) is cohomologous to \( s_0 \) by taking \( a = (g,1) \). Thus in this case \( \mathfrak{g}(s) = \mathfrak{g}(s_0) = \mathfrak{a} \), a complex simple Lie algebra regarded as a real Lie algebra.

It remains to consider the case when \( D \) is connected, i.e., \( \mathfrak{g} \) is simple.

\textbf{Definition 40.7.} (i) A real form \( \mathfrak{g}(s) \) of a complex simple Lie algebra \( \mathfrak{g} \) is said to be \textbf{inner} to \( \mathfrak{g}(s') \) if \( s' = gs \) up to equivalence, where \( g \in G_{\text{ad}} \) (i.e., \( s \) and \( s' \) differ by an inner automorphism). The \textbf{inner class} of \( \mathfrak{g}(s) \) is the collection of all real forms inner to \( \mathfrak{g}(s) \). In particular, an \textbf{inner form} is a form inner to the split form.

(ii) \( \mathfrak{g}(s) \) is called \textbf{quasi-split} if \( s = s_0 \in \operatorname{Aut}(D) \) (modulo equivalence).

So in particular any real form is inner to a unique quasi-split form, and a real form that is both inner and quasi-split is split.

\textbf{Exercise 40.8.} Let \( \mathfrak{g}_R \) be a real semisimple Lie algebra and \( \mathfrak{h}_R \subset \mathfrak{g}_R \) a Cartan subalgebra (the centralizer of a regular semisimple element
of $\mathfrak{g}_\mathbb{R}$). Let $\mathfrak{h} \subset \mathfrak{g}$ be their complexifications, and $H \subset G_{\text{ad}}$ the corresponding complex Lie groups. Let $K$ be the kernel of the natural map of Galois cohomology sets $H^1(\mathbb{Z}/2, N(H)) \to H^1(\mathbb{Z}/2, G_{\text{ad}})$ (i.e., the preimage of the unit element), where $\mathbb{Z}/2$ acts on $G_{\text{ad}}$ by complex conjugation associated to the real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$.

(i) Show that conjugacy classes of Cartan subalgebras in $\mathfrak{g}_\mathbb{R}$ are bijectively labeled by elements of $K$, with the unit element corresponding to $\mathfrak{h}_\mathbb{R}$.

(ii) Show that $K$ is a finite set.\(^{20}\)

\(^{20}\)For classical Lie algebras the set $K$ will be computed explicitly in Exercise 44.18. The explicit answer is known for exceptional Lie algebras as well, but we will not discuss it here.