42. **Real forms of exceptional Lie algebras**

42.1. **Equivalence of Vogan diagrams.** For exceptional Lie algebras, it is convenient to make a more systematic use of Vogan diagrams (we could do this also for classical Lie algebras, but there we can also do everything explicitly using linear algebra). Recall that any real form comes from a certain Vogan diagram, but different Vogan diagrams may be equivalent, i.e., define the same real form. So our job is to describe this equivalence relation.

First consider the case of the compact inner class. In this case the Vogan diagram is just the Dynkin diagram with black and white vertices (i.e., no matched vertices). Moreover, the case of all white vertices corresponds to the compact form, while the case when there are black vertices to noncompact forms. So let us focus on the latter case. Thus we have an element $\theta \in H \subset G_{\mathrm{ad}}$ such that $\theta \neq 1$ but $\theta^2 = 1$, but we are allowed to conjugate θ by elements of $N(H)$, i.e., transform it by elements of the Weyl group W. So how do simple reflections s_i act on θ (in terms of its Vogan diagram)?

The Vogan diagram of θ is determined by the numbers $\alpha_j(\theta) = \pm 1$: if this number is 1 then j is white, and if it is -1 then j is black. Now, we have

$$\alpha_j(s_i(\theta)) = (s_i\alpha_j)(\theta) = (\alpha_j - a_{ij}\alpha_i)(\theta) = \alpha_j(\theta)\alpha_i(\theta)^{-a_{ij}}.$$

This equals $\alpha_j(\theta)$ unless $\alpha_i(\theta) = -1$ and a_{ij} is odd. Thus we obtain the following lemma.

Lemma 42.1. *Suppose the Vogan diagram of θ contains a black vertex i. Then changing the colors of all neighbors j of i such that a_{ij} is odd gives an equivalent Vogan diagram.*

The same lemma holds, with the same proof, in the case of another inner class (which for exceptional Lie algebras is possible only for E_6), except we should ignore the vertices matched into pairs (so i and j should be θ-stable vertices).

42.2. **Classification of real forms.** We are now ready to classify real forms of exceptional Lie algebras.

1. Type G_2. We have two color configurations up to equivalence: ○○ and (●○, ○●, ●●). The first corresponds to the compact form G_2^c and the second to the split form G_2^{spl}. It is easy to check that in the second case $\mathfrak{k} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ (indeed, it has dimension 6 and rank 2). So we don't have other real forms.

2. Type F_4. Let α_1, α_2 be short roots and α_3, α_4 long roots. Then all nonzero off-diagonal a_{ij} are odd except $a_{23} = -2$. So we may change the colors of the neighbors of any black vertex, except that if the black vertex is 2 then we should not change the color of 3. By such changes, we can bring the colors at $3, 4$ into the form $\circ\circ$ or $\circ\bullet$, and then bring the colors at $1, 2$ to the form $\circ\circ$ or $\bullet\circ$. So we are down to four configurations:

$$\circ\circ\circ\circ,\ \bullet\circ\circ\circ,\ \circ\circ\circ\bullet,\ \bullet\circ\circ\bullet$$

Moreover, the fourth case, $\bullet\circ\circ\bullet$, is actually equivalent to the third one, $\circ\circ\circ\bullet$. This is seen from the chain of equivalences

$$\circ\circ\circ\bullet = \circ\circ\bullet\bullet = \circ\bullet\bullet\circ = \bullet\circ\bullet\circ = \bullet\bullet\bullet\bullet = \bullet\bullet\circ\bullet = \bullet\circ\circ\bullet$$

Thus we are left with three variants,

$$\circ\circ\circ\circ, \bullet\circ\circ\circ, \circ\circ\circ\bullet\ .$$

The first configuration, $\circ\circ\circ\circ$, corresponds to the compact form F_4^c.

In the second case, $\bullet\circ\circ\circ$, $\alpha(\theta) = -1$ exactly when the root α has half-integer coordinates (recall that there are 16 such roots, see Subsection 23.3). Thus the Lie algebra $\mathfrak{k}$ is comprised by the root subspaces for roots with integer coordinates and the Cartan subagebra, i.e., $\mathfrak{k} = \mathfrak{so}_9$ (type B_4). Also in this case $\mathfrak{p} = S$, the spin representation of $\mathfrak{so}_9$. This is not the split form, since for the split form $\dim \mathfrak{k}$ should be 24 and here it is 36. Let us denote this form F_4^1.

Thus, the third case, $\circ\circ\circ\bullet$, must be the split form, $F_4^{\rm spl}$. We see that $\mathfrak{k}$ contains the 21-dimensional Lie algebra $\mathfrak{sp}_6 = C_3$ (generated by the simple roots $\alpha_1, \alpha_2, \alpha_3$), so given that $\mathfrak{k}$ has rank 4 and dimension 24, we have $\mathfrak{k} = \mathfrak{sp}_6 \oplus \mathfrak{sl}_2$.

3. Type E_6, split inner class. In this case in the Vogan diagram two pairs of vertices are connected, so we can only color the two remaining vertices. So we have two equivalence classes of colorings – $\circ\circ$ and $(\bullet\bullet, \bullet\circ, \circ\bullet)$. Let us show that they correspond to two different real forms. Consider first the $\circ\circ$ case. In this case θ is simply the diagram automorphism, so we have $\mathfrak{k} = F_4$, as the Dynkin diagram of F_4 is obtained by folding the Dynkin diagram of E_6 (check it!). This is not the split form since $\dim \mathfrak{k} = 52$, but for the split form it is 36; denote this form by E_6^1. So the split form $E_6^{\rm spl}$ corresponds to the second equivalence class $(\bullet\bullet, \bullet\circ, \circ\bullet)$. One can show that in this case $\mathfrak{k} = \mathfrak{sp}_8$, i.e., type C_4 (check it!).

4. E_6, E_7, E_8, compact inner class. In this case the Vogan diagram has no arrows and just is the usual Dynkin diagram with vertices colored black and white. One option is that all vertices are white, this corresponds to the compact forms E_6^c, E_7^c, E_8^c ($\theta = 1$). If there is at least one black vertex, then by using equivalence transformations we can make sure that the nodal vertex is black. Then flipping the color of its neighbors if needed, we can make sure that the vertex on the shortest leg is also black. This allows us to change the color of the nodal vertex whenever we want (as long as the vertex on the shortest leg remains black).

We now want to unify the coloring of the long leg. We can bring the long leg to the following normal forms:

E_6: $\circ\circ, \bullet\circ = \bullet\bullet = \circ\bullet$. But by flipping the colors on the neighbors of the nodal vertex, we see that $\bullet\circ$ and $\circ\circ$ are equivalent, so all patterns are equivalent to $\bullet\bullet$.

E_7: $\circ\ \circ\ \circ, \bullet\ \circ\ \circ = \bullet\ \bullet\ \circ = \circ\ \bullet\ \bullet = \circ\ \circ\ \bullet, \bullet\ \circ\ \bullet = \bullet\ \bullet\ \bullet = \circ\ \bullet\ \circ$. But by flipping the colors on the neighbors of the nodal vertex, we see that all patterns are equivalent to $\bullet\ \bullet\ \bullet$.

E_8: $\circ\ \circ\ \circ\circ, \bullet\ \circ\ \circ\circ = \bullet\ \bullet\ \circ\circ = \circ\ \bullet\ \bullet\circ = \circ\ \circ\ \bullet\bullet = \circ\ \circ\ \circ\bullet$,

$$\bullet\ \circ\ \circ\bullet = \bullet\ \circ\ \bullet\bullet = \bullet\ \bullet\ \bullet\circ = \begin{cases} \bullet\ \circ\ \bullet\circ \\ \circ\ \bullet\ \circ\circ \end{cases}$$

$$= \bullet\ \bullet\ \bullet\bullet = \bullet\ \bullet\ \circ\bullet = \circ\ \bullet\ \bullet\bullet = \begin{cases} \circ\ \bullet\ \circ\bullet \\ \circ\ \circ\ \bullet\circ \end{cases} .$$

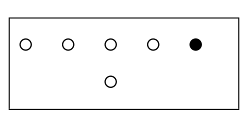
But by flipping the colors on the neighbors of the nodal vertex, we see that all patterns are equivalent to $\bullet\ \bullet\ \bullet\bullet$.

Thus we can always arrange all vertices on the long leg except possibly the neighbor of the node to be black, while the short leg and the node also remain black. In addition, as seen from the pictures above, in the cases E_6 and E_8 these two configurations are equivalent by transformations inside the leg.

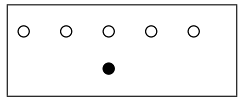
Now we can consider the configurations on the remaining leg (of length 2). The equivalence classes are $\circ\circ$ and $\bullet\circ = \circ\bullet = \bullet\bullet$.

So in the case of E_6 and E_8 we get just two cases. It turns out that both for E_6 and E_8 these give two different real forms, one of which is split in the case of E_8.

Consider first the E_6 case. One option is to take the Vogan diagram with just one black vertex, at the end of the long leg:

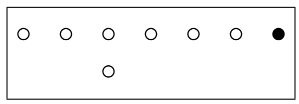


Then $\mathfrak{k} = \mathfrak{so}_{10} \oplus \mathfrak{so}_2$ (as the black vertex corresponds to a minuscule weight). We denote this real form by E_6^2. On the other hand, if there is only one black vertex on the short leg,

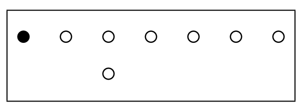


then $\mathfrak{k}$ contains $\mathfrak{sl}_6$, so this real form is different (as $\mathfrak{sl}_6$ is not a Lie subalgebra of $\mathfrak{so}_{10}$). It's not difficult to show that in this case $\mathfrak{k} = \mathfrak{sl}_6 \oplus \mathfrak{sl}_2$. We denote this real form by E_6^3.

Now consider the E_8 case. Again one option is the Vogan diagram with just one black vertex, at the end of the long leg:

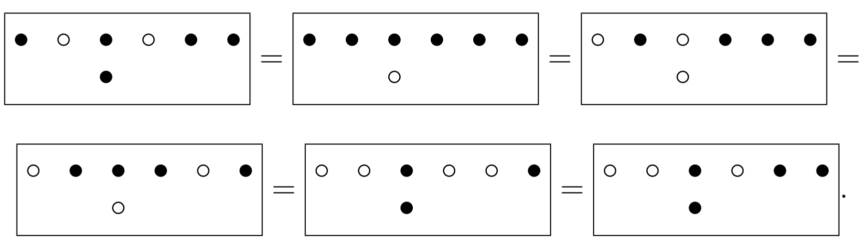


Then $\mathfrak{k}$ contains E_7, so this is not the split form since $\dim \mathfrak{k} \geq 133$ but for the split form it should be 120. In fact, it is not hard to see that $\mathfrak{k} = E_7 \oplus \mathfrak{sl}_2$. We denote this real form by E_8^1. The second form is the split one, E_8^{spl}. It can, for example, be obtained if we color black only one vertex, at the end of the middle leg:

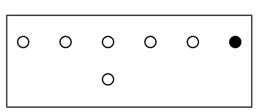


In fact, it's not hard to show that the algebra $\mathfrak{k}$ in this case is $\mathfrak{so}_{16}$.

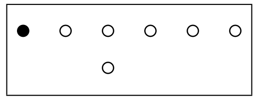
Finally, consider the E_7 case. In this case we have four options, but two of them end up being equivalent. Namely, we have



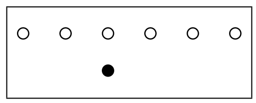
So we are left with three cases, which all turn out different. The first one is just one black vertex at the end of the long leg:



In this case $\mathfrak{k}$ contains E_6, so this is not the split form, as $\dim \mathfrak{k} \geq 78$ but for the split form it is 63. It is easy to see that $\mathfrak{k} = E_6 \oplus \mathfrak{so}_2$ in this case (the black vertex corresponds to the minuscule weight). We denote this real form by E_7^1. The second option is a black vertex at the end of the middle leg:



Then $\mathfrak{k}$ contains $\mathfrak{so}_{12}$, of dimension 66, so again not the split form. One can show that for this form $\mathfrak{k} = \mathfrak{so}_{12} \oplus \mathfrak{sl}_2$. We denote it by E_7^2. Finally, the split form E_7^{spl} is obtained when one colors black just the end of the short leg:



Then $\mathfrak{k}$ contains $\mathfrak{sl}_7$ and one can show that $\mathfrak{k} = \mathfrak{sl}_8$.

Exercise 42.2. Work out the details of computation of $\mathfrak{k}$ for real forms of exceptional Lie algebras.

Exercise 42.3. Let $\mathfrak{g}$ be the complex Lie algebra of type G_2, and G the corresponding Lie group. Let $\mathfrak{sl}_3 \subset \mathfrak{g}$ be the Lie subalgebra generated by long root elements and $SU(3) \subset G^c$ be the corresponding subgroup. Show that $G^c/SU(3) \cong S^6$. Use this to construct embeddings $G^c \hookrightarrow SO(7)$ and $G^c \hookrightarrow \mathrm{Spin}(7)$.

Hint. Consider the 7-dimensional irreducible representation of G^c. Show that it is of real type (obtained by complexifying a real representation V) and then consider the action of G^c on the set of unit vectors in V under a positive invariant inner product. Then compute the Lie algebra of the stabilizer and use that the sphere is simply connected.

Exercise 42.4. Keep the notation of Exercise 42.3. Show that one has $\mathrm{Spin}(7)/G^c = S^7$ and $SO(7)/G^c = \mathbb{RP}^7$.

Hint. Let S be the spin representation of $\mathrm{Spin}(7)$. Use that it is of real type (this can be deduced from Proposition 32.14) and then consider the action of $\mathrm{Spin}(7)$ on vectors of norm 1 in $S_{\mathbb{R}}$. Compute the Lie algebra of the stabilizer and use that the sphere is simply connected.

Remark 42.5. More generally, one can classify automorphisms of a simple complex Lie algebra $\mathfrak{g}$ of arbitrary finite order. This was done by V. Kac using diagrams now known as **Kac diagrams**, see [OV], Subsection 4.7. In particular, this approach can be applied to classify automorphisms of order 2 which correspond to real forms of $\mathfrak{g}$, see [OV], Subsection 5.5.

18.755 Lie Groups and Lie Algebras II
Spring 2024