43. **Classification of connected compact and complex reductive groups**

43.1. **Connected compact Lie groups.** We are now ready to classify connected compact Lie groups. We start with the following exercise.

Exercise 43.1. Show that if K^c is a compact Lie group then $\mathfrak{k} := \mathrm{Lie}(K^c)_{\mathbb{C}}$ is a reductive Lie algebra.

Hint. First use integration over K^c to show that $\mathfrak{k}$ has a K^c-invariant positive definite Hermitian form. Then show that if I is an ideal in $\mathfrak{k}$ then its orthogonal complement $I^\perp$ is also an ideal.

Now we can proceed. We already know many examples of compact connected Lie groups - namely tori $(S^1)^r$ and also groups G^c_{ad} where $G_{\mathrm{ad}} = \mathrm{Aut}(\mathfrak{g})^\circ$ for a semisimple Lie algebra $\mathfrak{g}$. We can also consider products $(S^1)^r \times G^c_{\mathrm{ad}}$. Exercise 43.1 shows that the Lie algebra of any compact Lie group is isomorphic to one of such a product, so this should be an exhaustive list up to taking coverings and quotients by finite central subgroups. It thus remains to understand the nature of these coverings, which reduces to understanding $\pi_1(G^c_{\mathrm{ad}})$. So our next task is to compute this group. In particular, we will show that it is finite.

So let $\mathfrak{g}$ be a semisimple complex Lie algebra and G the corresponding simply connected complex Lie group (the universal cover of G_{ad}). Let Z be the kernel of the covering map $G \to G_{\mathrm{ad}}$, which is also $\pi_1(G_{\mathrm{ad}})$ and the center of G. The finite dimensional representations of G are the same as those of $\mathfrak{g}$, so the irreducible ones are L_λ, $\lambda \in P_+$. The center Z acts by a certain character $\boldsymbol{\chi}_\lambda : Z \to \mathbb{C}^\times$ on each L_λ. Since $L_{\lambda+\mu}$ is contained in $L_\lambda \otimes L_\mu$, we have $\boldsymbol{\chi}_{\lambda+\mu} = \boldsymbol{\chi}_\lambda \boldsymbol{\chi}_\mu$, so $\boldsymbol{\chi}$ uniquely extends to a homomorphism $\boldsymbol{\chi} : P \to \mathrm{Hom}(Z, \mathbb{C}^\times)$. Also, by definition $\boldsymbol{\chi}_\theta = 1$ (since the maximal root θ is the highest weight of the adjoint representation on which Z acts trivially).

Now, by Exercise 30.15, if $\lambda(h_i)$ are sufficiently large then for every root α of $\mathfrak{g}$ we have $L_{\lambda+\alpha} \subset L_\lambda \otimes \mathfrak{g}$. Thus $\boldsymbol{\chi}_{\lambda+\alpha} = \boldsymbol{\chi}_\lambda$, hence $\boldsymbol{\chi}_\alpha = 1$. So $\boldsymbol{\chi}$ is trivial on the root lattice Q, i.e., defines a homomorphism $P/Q \to \mathrm{Hom}(Z, \mathbb{C}^\times)$, or, equivalently, $Z \to P^\vee/Q^\vee$.

Note that the same argument works for G^c_{ad}, its universal cover G^c, and its center Z^c instead of G_{ad}, G, Z.

Proposition 43.2. *A representation L_λ of $\mathfrak{g}$ of highest weight $\lambda \in P_+$ lifts to a representation of G_{ad} (or, equivalently, G^c_{ad}) if and only if $\lambda \in P_+ \cap Q$.*

Proof. We have just shown that if $\lambda \in P_+ \cap Q$ then L_λ lifts. The converse follows from Proposition 36.12 applied to $V = \mathfrak{g}$. $\square$

Now we can proceed with the classification of semisimple compact connected Lie groups. We begin with the following lemma from topology (see e.g. [M], Supplementary exercises to Chapter 13, p.500, Exercise 4).

Lemma 43.3. *If X is a connected compact manifold then the fundamental group $\pi_1(X)$ is finitely generated.*

Proof. (sketch) Cover X by small balls, pick a finite subcover, connect the centers. We get a finite graph whose fundamental group maps surjectively to $\pi_1(X)$. □

Theorem 43.4. *Let $\mathfrak{g}$ be a semisimple complex Lie algebra and $G^c_{\rm ad}$ the corresponding adjoint compact group. Then $\pi_1(G^c_{\rm ad}) = P^\vee/Q^\vee$. Thus the universal cover G^c of $G^c_{\rm ad}$ is a compact Lie group.*

Proof. Let G^c_* be a finite cover of $G^c_{\rm ad}$, and $Z_{G^c_*} \subset G^c_*$ be the kernel of the projection $G^c_* \to G^c_{\rm ad}$. Then finite dimensional irreducible representations of G^c_* are a subset of finite dimensional irreducible representations of $\mathfrak{g}$, labeled by a subset $P_+(G^c_*) \subset P_+$ containing $P_+ \cap Q$ (as by Proposition 43.2 these are highest weights of representations of $G^c_{\rm ad}$). Let $P(G^c_*) \subset P$ be generated by $P_+(G^c_*)$. Let $\boldsymbol{\chi}_\lambda$ be the character by which $Z_{G^c_*}$ acts on the irreducible representation L_λ of G^c_*. By Proposition 43.2, $\boldsymbol{\chi}$ defines an injective homomorphism $\xi : P(G^c_*)/Q \to Z^\vee_{G^c_*}$. Since G^c_* is compact, by the Peter-Weyl theorem this homomorphism is surjective, hence is an isomorphism.

It remains to show that $\pi_1(G^c_{\rm ad})$ is finite (then we can take G^c_* to be the universal cover of $G^c_{\rm ad}$, in which case $P(G^c_*) = P$, so we get $P/Q \cong Z^\vee$, hence $Z = \pi_1(G_{\rm ad}) \cong P^\vee/Q^\vee$). To this end, note that by Lemma 43.3, $\pi_1(G^c_{\rm ad})$ is a finitely generated abelian group. Take a subgroup of finite index N in $\pi_1(G^c_{\rm ad})$ and let G^c_* be the corresponding cover. As we have shown, then $N = |Z_{G^c_*}| \le |P(G^c_*)/Q| \le |P/Q|$. But for finitely generated abelian groups this implies that the group is finite. □

This immediately implies the following corollary.

Corollary 43.5. *(i) If $\mathfrak{g}$ is a simple complex Lie algebra then the simply connected Lie group G^c corresponding to the Lie algebra $\mathfrak{g}^c$ is compact, and its center is $P^\vee/Q^\vee$, which also equals $\pi_1(G^c_{\rm ad})$.*

(ii) Let $\Gamma \subset P^\vee/Q^\vee$ be a subgroup. Then the irreducible representations of G/Γ are L_λ such that λ defines the trivial character of Γ.

(iii) Let G^c_i be the simply connected compact Lie group corresponding to a simple summand $\mathfrak{g}_i$ of a semisimple Lie algebra $\mathfrak{g} = \oplus_{i=1}^n \mathfrak{g}_i$. Then any connected Lie group with Lie algebra $\mathfrak{g}^c$ is compact and has the

form $(\prod_{i=1}^n G_i^c)/Z$*, where* $Z = \pi_1(G^c)$ *is a subgroup of* $\prod_i Z_i$*, and* $Z_i = P_i^\vee/Q_i^\vee$ *are the centers of* G_i^c*. Moreover, every semisimple connected compact Lie group has this form.*

In particular, it follows that simply connected semisimple compact Lie groups are of the form $\prod_{i=1}^n G_i^c$, where G_i^c are simply connected and simple.[24]

Corollary 43.6. *Any connected compact Lie group is the quotient of* $T \times C$ *by a finite central subgroup, where* $T = (S^1)^m$ *is a torus and* C *is compact, semisimple and simply connected.*

Proof. Let L be such a group, $\mathfrak{l}$ its Lie algebra. It is reductive, so we can uniquely decompose $\mathfrak{l}$ as $\mathfrak{t} \oplus \mathfrak{c}$ where $\mathfrak{t}$ is the center and $\mathfrak{c}$ is semisimple. Let $T, C \subset L$ be the connected Lie subgroups corresponding to $\mathfrak{t}, \mathfrak{c}$. It is clear that $\mathrm{Lie}\overline{T} = \mathfrak{t} = \mathrm{Lie}T$, so T is closed, hence compact, hence a torus. Also C is compact, so also closed, with $\mathrm{Lie}C = \mathfrak{c}$. Thus we have a surjective homomorphism $T \times C \to L$ whose kernel is finite, as desired. □

43.2. **Polar decomposition.** Now let us study the structure of the Lie subgroup $G_{\mathrm{ad},\theta} \subset G_{\mathrm{ad}}$ corresponding to the real form $\mathfrak{g}_\theta \subset \mathfrak{g}$ of a semisimple complex Lie algebra $\mathfrak{g}$, namely, the group of fixed points of the antiholomorphic involution $\omega_\theta = \omega \circ \theta$ in G_{ad}. It is clear that this subgroup is closed ($\mathrm{Lie}\overline{G_{\mathrm{ad},\theta}} = \mathfrak{g}_\theta = \mathrm{Lie}G_{\mathrm{ad},\theta}$), but it may be disconnected: e.g. if $\mathfrak{g}_\theta = \mathfrak{sl}_2(\mathbb{R})$ then $G_{\mathrm{ad}} = PGL_2(\mathbb{C})$, so $G_{\mathrm{ad},\theta} = PGL_2(\mathbb{R})$, the quotient of $GL_2(\mathbb{R})$ by scalars, which has two components. However, the results below apply mutatis mutandis to the connected group $G_{\mathrm{ad},\theta}^\circ$.

Let $K^c \subset G_{\mathrm{ad},\theta}$ be the subgroup of elements acting on $\mathfrak{g}$ by unitary operators); namely, K^c is the set of fixed points of ω_θ on G_{ad}^c.[25] This a closed (possibly disconnected) subgroup of G_{ad}^c since $\mathrm{Lie}\overline{K^c} = \mathfrak{k}^c = \mathrm{Lie}K^c$, hence it is compact. Also let $P_\theta := \exp(\mathfrak{p}_\theta) \subset G_{\mathrm{ad},\theta}$ (note that it is not a subgroup!). Since $\mathfrak{p}_\theta$ acts on $\mathfrak{g}$ by Hermitian operators, the exponential map $\exp : \mathfrak{p}_\theta \to P_\theta$ is a diffeomorphism, so $P_\theta \subset G_{\mathrm{ad},\theta}$ is a closed embedded submanifold (the set of elements acting on $\mathfrak{g}$ by positive Hermitian operators).

[24] We say that a connected Lie group G is **simple** if so is its Lie algebra. Thus this does not quite mean that G is simple as an abstract group: it may have a finite center (e.g., $G = SU(2)$ or $SL_2(\mathbb{C})$). For this reason such "simple" groups are sometimes called **almost simple**. However, the corresponding adjoint group G_{ad} is indeed simple as an abstract group.

[25] Of course, the group K^c depends on θ, but for simplicity we will not indicate this dependence in the notation.

Theorem 43.7. *(Polar decomposition for $G_{\mathrm{ad},\theta}$) The multiplication map $\mu : K^c \times P_\theta \to G_{\mathrm{ad},\theta}$ is a diffeomorphism. Thus $G_{\mathrm{ad},\theta} \cong K^c \times \mathbb{R}^{\dim \mathfrak{p}}$ as a manifold (in particular, $G_{\mathrm{ad},\theta}$ is homotopy equivalent to K^c).*

Proof. Recall that every invertible complex matrix A can be uniquely written as a product $A = U_A R_A$, where $U = U_A$ is a unitary matrix and $R = R_A$ a positive Hermitian matrix, namely $R = (A^\dagger A)^{1/2}$, $U = A(A^\dagger A)^{-1/2}$ (the classical polar decomposition). Let us consider this decomposition for $g \in G_{\mathrm{ad},\theta} \subset \mathrm{Aut}(\mathfrak{g}) \subset GL(\mathfrak{g})$. Since $g^\dagger g$ is an automorphism of $\mathfrak{g}$ with positive eigenvalues, so is $(g^\dagger g)^{1/2} = R_g$, so $R_g \in P_\theta$ (a positive self-adjoint element in $G_{\mathrm{ad},\theta}$). Also since U_g is unitary, it belongs to K^c. Thus the regular map $g \mapsto (U_g, R_g)$ is the inverse to μ (using the uniqueness of the polar decomposition). □

In particular, applying Theorem 43.7 to complex Lie groups, we get

Corollary 43.8. *The multiplication map defines a diffeomorphism*

$$G_{\mathrm{ad}}^c \times \mathbf{P} \cong G_{\mathrm{ad}},$$

where $\mathbf{P}$ is the set of elements of G_{ad} acting on $\mathfrak{g}$ by positive Hermitian operators. In particular, $\pi_1(G_{\mathrm{ad}}) = \pi_1(G_{\mathrm{ad}}^c) = P^\vee/Q^\vee$.

Corollary 43.9. *If G is a semisimple complex Lie group then the center Z of G is contained in G^c, i.e., coincides with the center Z^c of G^c. Thus the restriction of finite dimensional representations from G to G^c is an equivalence of categories.*

This also implies that by taking coverings the polar decomposition applies verbatim to the real form $G_\theta = G^{\omega_\theta} \subset G$ of any connected complex semisimple Lie group G instead of G_{ad}. We note, however, that if G is simply connected, then G_θ° need not be. In fact, its fundamental group could be infinite. The simplest example is $G = SL_2(\mathbb{C})$, then for the split form $G_\theta = SL_2(\mathbb{R})$, which as we showed is homotopy equivalent to $SO(2) = S^1$, i.e. its fundamental group is $\mathbb{Z}$.

Example 43.10. 1. For $G_\theta = SL_n(\mathbb{C})$ we have $K^c = SU(n)$ and P_θ is the set of positive Hermitian matrices of determinant 1, so the polar decomposition in this case is the usual polar decomposition of complex matrices.

2. For $G_\theta = SL_n(\mathbb{R})$ we have $K^c = SO(n)$ and P_θ is the set of positive symmetric matrices of determinant 1, so the polar decomposition in this case is the usual polar decomposition of real matrices.

43.3. **Connected complex reductive groups.**

Definition 43.11. A connected complex Lie group G is **reductive** if it is of the form $((\mathbb{C}^\times)^r \times G_{\rm ss})/Z$ where $G_{\rm ss}$ is semisimple and Z is a finite central subgroup. A complex Lie group G is reductive if G° is reductive and G/G° is finite.

Example 43.12. $GL_n(\mathbb{C}) = (\mathbb{C}^\times \times SL_n(\mathbb{C}))/\mu_n$ is reductive.

It is clear that the Lie algebra Lie$G$ of any complex reductive Lie group G is reductive, and any complex reductive Lie algebra is the Lie algebra of a connected complex reductive Lie group. However, a simply connected complex Lie group with a reductive Lie algebra need not be reductive (e.g. $G = \mathbb{C}$).

If $G = ((\mathbb{C}^\times)^r \times G_{\rm ss})/Z$ is a connected complex reductive Lie group then by Corollary 43.9, $Z \subset (S^1)^r \times G_{\rm ss}^c \subset (\mathbb{C}^\times)^r \times G_{\rm ss}$, so we can define the compact subgroup $G^c \subset G$ by $G^c := ((S^1)^r \times G_{\rm ss}^c)/Z$. Then it is easy to see that restriction of finite dimensional representations from G to G^c is an equivalence, so representations of G are completely reducible. The irreducible representations are parametrized by collections $(n_1, ..., n_r, \lambda)$, $\lambda \in P_+(G_{\rm ss})$, $n_i \in \mathbb{Z}$, which define the trivial character of Z.

43.4. **Linear groups.** A connected Lie group G (real or complex) is called **linear** if it can be realized as a Lie subgroup of $GL_n(\mathbb{R})$, respectively $GL_n(\mathbb{C})$. We have seen that any complex semisimple group is linear. However, for real semisimple groups this is not so (e.g. the universal cover of $SL_2(\mathbb{R})$ is not linear, see Exercise 11.20). In fact, we see that we can characterize connected real semisimple linear groups as follows.

Proposition 43.13. *Suppose $\mathfrak{g}_\theta$ is a real form of a semisimple complex Lie algebra $\mathfrak{g}$, G a connected complex Lie group with Lie algebra $\mathfrak{g}$, and $G_\theta = G^{\omega_\theta}$. Then G_θ, G_θ° are linear groups. Moreover, every connected real semisimple linear Lie group is of the form G_θ°*

Exercise 43.14. Classify simply connected real semisimple **linear** Lie groups.

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Spring 2024