

#### 44. Maximal tori in compact groups, Cartan decomposition

44.1. **Maximal tori in connected compact Lie groups.** Let  $\mathfrak{g}$  be a complex semisimple Lie algebra,  $\mathfrak{g}^c$  its compact form,  $G$  a connected Lie group with Lie algebra  $\mathfrak{g}$ ,  $G^c \subset G$  its compact part (the connected Lie subgroup with Lie algebra  $\mathfrak{g}^c$ ), as above.

A **Cartan subalgebra**  $\mathfrak{h}^c \subset \mathfrak{g}^c$  is a maximal commutative Lie subalgebra (note that it automatically consists of semisimple elements since all elements of  $\mathfrak{g}^c$  are semisimple). In other words, it is a subspace such that  $\mathfrak{h}^c \otimes_{\mathbb{R}} \mathbb{C}$  is a Cartan subalgebra of  $\mathfrak{g}$ .

Recall that all Cartan subalgebras of  $\mathfrak{g}$  are conjugate, even if equipped with a system of simple roots (Theorem 20.10). Namely, given two such subalgebras  $(\mathfrak{h}, \Pi)$  and  $(\mathfrak{h}', \Pi')$ , there is  $g \in G$  such that  $\text{Ad}_g(\mathfrak{h}, \Pi) = (\mathfrak{h}', \Pi')$ . It turns out that the same result holds for  $\mathfrak{g}^c$ .

**Lemma 44.1.** *Any two Cartan subalgebras in  $\mathfrak{g}^c$  equipped with systems of simple roots are conjugate under  $G^c$ .*

*Proof.* Given  $(\mathfrak{h}^c, \Pi)$  and  $(\mathfrak{h}'^c, \Pi')$ , there is  $g \in G$  such that  $\text{Ad}_g(\mathfrak{h}^c, \Pi) = (\mathfrak{h}'^c, \Pi')$ . Then we also have  $\text{Ad}_{\bar{g}}(\mathfrak{h}^c, \Pi) = (\mathfrak{h}'^c, \Pi')$ , where  $\bar{g} := \omega(g)$ . So  $\bar{g}^{-1}g$  commutes with  $\mathfrak{h}^c$  and preserves  $\Pi$ , i.e.,  $\bar{g}h = g$ ,  $h \in H : \exp(\mathfrak{h}_{\mathbb{C}}^c)$ . Writing  $g = kp$ , where  $k \in G^c$ ,  $p \in \mathbf{P}$ , we have  $kp^{-1}h = kp$ , so  $h = p^2$ . Since  $p$  is positive,  $p = h^{1/2}$ , so it commutes with  $\mathfrak{h}^c$  and preserves  $\Pi$ , thus  $\text{Ad}_k(\mathfrak{h}^c, \Pi) = (\mathfrak{h}'^c, \Pi')$ , as claimed.  $\square$

Note that for every Cartan subalgebra  $\mathfrak{h}^c \subset \mathfrak{g}^c$ ,  $H^c = \exp(\mathfrak{h}^c) \subset G^c$  is a torus, which is clearly a **maximal torus** (as the complexified Lie algebra of a larger torus would be a larger commutative subalgebra than  $\mathfrak{h}^c$ ). Conversely, if  $H^c \subset G^c$  is a maximal torus then  $\text{Lie}(H^c)$  can be included in a Cartan subalgebra, hence it is itself a Cartan subalgebra. So we have a bijection between Cartan subalgebras in  $\mathfrak{g}^c$  and maximal tori in  $G^c$ . Similarly, there is a bijection between Cartan subalgebras in  $\mathfrak{g}$  and maximal tori in  $G$ .

This implies

**Corollary 44.2.** *Any two maximal tori in  $G$  or  $G^c$  equipped with systems of simple roots are conjugate.*

We also have

**Theorem 44.3.** *Every element of a connected compact Lie group  $K$  is contained in a maximal torus, and all maximal tori in  $K$  are conjugate (even when equipped with systems of simple roots).*

*Proof.* We may assume without loss of generality that  $K$  is semisimple, i.e.,  $K = G^c$  for a connected semisimple complex Lie group  $G$ , which

implies the second statement. To prove the first statement, let  $K' \subset K$  be the set of elements contained in a maximal torus. Fix a maximal torus  $T \subset K$  and consider the map  $f : K \times T \rightarrow K$  given by  $f(k, t) = ktk^{-1}$ , whose image is  $K'$ . This implies that  $K'$  is compact, hence closed, so  $K \setminus K'$  is open.

On the other hand, recall from Subsection 20.1 that a generic  $x \in \mathfrak{g}^c$  is **regular**, meaning that its centralizer  $\mathfrak{z}_x$  has dimension  $\leq \text{rank}(\mathfrak{g})$ , in which case it must have dimension exactly  $\text{rank}(\mathfrak{g})$  and be a Cartan subalgebra. It is clear that every regular element  $x$  is contained in a unique maximal torus, namely  $\exp(\mathfrak{z}_x)$ , so the elements of  $K \setminus K'$  are all non-regular. But the set of non-regular elements is defined by polynomial equations in  $\text{Ad}_x$  (the minors of  $\text{Ad}_x$  of codimension  $\text{rank}(\mathfrak{g})$  all vanish), so  $K \setminus K'$  must be empty (as it is an open set contained in the set of solutions of nontrivial polynomial equations in  $\text{Ad}_x$ ).  $\square$

**Corollary 44.4.** *The exponential map  $\exp : \mathfrak{g}^c \rightarrow G^c$  is surjective.*

**Exercise 44.5.** Is the exponential map surjective for the group  $SL_2(\mathbb{C})$ ?

**44.2. Semisimple and unipotent elements.** Let  $G$  be a connected reductive complex Lie group. An element  $g \in G$  is called **semisimple** if it acts in every finite dimensional representation of  $G$  by a semisimple (=diagonalizable) operator, and **unipotent** if it acts in every finite dimensional representation of  $G$  by a unipotent operator (all eigenvalues are 1).

**Exercise 44.6.** Let  $Y$  be a faithful finite dimensional representation of  $G$  (it exists by Corollary 36.5). Show that  $g \in G$  is semisimple if and only if it acts semisimply on  $Y$ , and unipotent if and only if it acts unipotently on  $Y$ .

**Hint:** Use Proposition 36.12.

**Exercise 44.7.** Show that if  $G$  is semisimple then the exponential map defines a homeomorphism between the set of nilpotent elements in  $\mathfrak{g} = \text{Lie}G$  and the set of unipotent elements in  $G$ .

**Exercise 44.8.** Let  $Z$  be the center of a connected complex reductive group  $G$ .

(i) Show that the homomorphism  $\pi : G \rightarrow G/Z$  defines a bijection between unipotent elements of  $G$  and  $G/Z$ .

(ii) Show that the set of semisimple elements of  $G$  is the preimage under  $\pi$  of the set of semisimple elements of  $G/Z$ .

**Proposition 44.9.** *(Jordan decomposition in  $G$ ). Every element  $g \in G$  has a unique factorization  $g = g_s g_u$ , where  $g_s \in G$  is semisimple,  $g_u \in G$  is unipotent and  $g_s g_u = g_u g_s$ .*

**Exercise 44.10.** Prove Proposition 44.9.

**Hint.** Use Exercise 44.8 to reduce to the case when  $G = G_{\text{ad}}$  is a semisimple adjoint group. In this case, write  $\text{Ad}_g$  as  $su$ , where  $s$  is a semisimple and  $u$  a unipotent operator with  $su = us$  (Jordan decomposition for matrices). Show that  $s = \text{Ad}_{g_s}$  and  $u = \text{Ad}_{g_u}$  for some commuting  $g_s, g_u \in G_{\text{ad}}$ . Then establish uniqueness using the uniqueness of Jordan decomposition of matrices.

**44.3. Maximal abelian subspaces of  $\mathfrak{p}_\theta$ .** Let  $G$  be a connected complex semisimple group,  $G_\theta \subset G$  a real form,  $\mathfrak{g}_\theta \subset \mathfrak{g}$  their Lie algebras. We have the polar decomposition  $G_\theta = K^c P_\theta$  and the additive version  $\mathfrak{g}_\theta = \mathfrak{k}^c \oplus \mathfrak{p}_\theta$ , with  $\mathfrak{p}_\theta = i\mathfrak{p}^c$ . Also  $\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c$ .

**Proposition 44.11.** (i) Let  $\mathfrak{a}$  be a maximal abelian subspace of  $\mathfrak{p}_\theta$ . Then the centralizer  $\mathfrak{z}$  of  $\mathfrak{a}$  in  $\mathfrak{g}^c$  has the form  $\mathfrak{m} \oplus \mathfrak{a}$ , where  $\mathfrak{m}$  is a reductive Lie algebra contained in  $\mathfrak{k}^c$ . Moreover, if  $\mathfrak{t}$  is a Cartan subalgebra of  $\mathfrak{m}$  then  $\mathfrak{t} \oplus i\mathfrak{a}$  is a Cartan subalgebra of  $\mathfrak{g}^c$  and  $\mathfrak{t} \oplus \mathfrak{a}$  is a Cartan subalgebra of  $\mathfrak{g}_\theta$ .

(ii) If  $a \in \mathfrak{a}$  is sufficiently generic then the centralizer of  $a$  in  $\mathfrak{p}_\theta$  is  $\mathfrak{a}$ .

(iii) For any  $p \in \mathfrak{p}_\theta$  there exists  $k \in K^c$  such that  $\text{Ad}_k(p) \in \mathfrak{a}$ .

(iv) All maximal abelian subspaces of  $\mathfrak{p}_\theta$  are conjugate by  $K^c$ .

*Proof.* (i) Let  $x \in \mathfrak{g}^c$ ,  $[x, \mathfrak{a}] = 0$ . Write  $x = x_+ + x_-$ ,  $x_+ \in \mathfrak{k}^c$ ,  $x_- \in \mathfrak{p}^c$ . Then  $[x_\pm, \mathfrak{a}] = 0$ , thus  $x_- \in \mathfrak{a}$  by maximality of  $\mathfrak{a}$ . So  $x \in \mathfrak{k}^c \oplus \mathfrak{a}$ . Thus  $\mathfrak{z} = \mathfrak{m} \oplus \mathfrak{a}$  where  $\mathfrak{m} \subset \mathfrak{k}^c$  is a reductive Lie algebra. Moreover, if  $\mathfrak{t} \subset \mathfrak{m}$  is a Cartan subalgebra then  $\mathfrak{t} \oplus i\mathfrak{a}$  is a maximal abelian subalgebra of  $\mathfrak{g}^c$ , hence is a Cartan subalgebra. Similarly,  $\mathfrak{t} \oplus \mathfrak{a}$  is a Cartan subalgebra of  $\mathfrak{g}_\theta$ .

(ii) Consider the group  $T_{\mathfrak{a}} := \exp(i\mathfrak{a}) \subset G^c$ . It is clear from (i) that this is a compact torus. Thus for a generic enough  $a \in \mathfrak{a}$ , the 1-parameter subgroup  $e^{ita}$  is dense in  $T_{\mathfrak{a}}$ . So if  $p \in \mathfrak{p}_\theta$  and  $[p, a] = 0$  then  $e^{ita}$  commutes with  $p$ , hence so do  $T_{\mathfrak{a}}$  and  $\mathfrak{a}$ . So by maximality of  $\mathfrak{a}$  we have  $p \in \mathfrak{a}$ .

(iii) Let  $a \in \mathfrak{a}$  be generic enough as in (ii). Then by (ii),  $\text{Ad}_k(p) \in \mathfrak{a}$  if and only if  $[\text{Ad}_k(p), a] = 0$ .

Consider the function  $f : K^c \rightarrow \mathbb{R}$  given by  $f(b) := (\text{Ad}_b(p), a)$ . This function is continuous, so attains a maximum on the compact group  $K^c$ . Suppose  $k$  is a maximum point of  $f$ . Let  $p_0 := \text{Ad}_k(p)$ . Differentiating  $f$  at  $k$ , we get  $([x, p_0], a) = 0$  for all  $x \in \mathfrak{k}^c$ . Thus  $(x, [p_0, a]) = 0$  for all  $x \in \mathfrak{k}^c$ . But  $[p_0, a] \in \mathfrak{k}^c$  and the inner product on  $\mathfrak{k}^c$  is nondegenerate. Thus  $[p_0, a] = 0$ , as desired.

(iv) Let  $\mathfrak{a}, \mathfrak{a}'$  be maximal abelian subspaces of  $\mathfrak{p}_\theta$ . Pick a generic element  $p \in \mathfrak{a}'$  as in (ii). By (iii) we can find  $k \in K^c$  such that

$\text{Ad}_k(p) = a \in \mathfrak{a}$ . Moreover,  $a$  is generic in  $\text{Ad}_k(\mathfrak{a}')$ . So for every  $b \in \mathfrak{a}$  we have  $[b, \text{Ad}_k(\mathfrak{a}')] = 0$  (as  $[b, a] = 0$ ). By maximality of  $\mathfrak{a}'$  this implies that  $b \in \text{Ad}_k(\mathfrak{a}')$ , i.e.,  $\mathfrak{a} \subset \text{Ad}_k(\mathfrak{a}')$ . Thus  $\dim \mathfrak{a} \leq \dim \mathfrak{a}'$ . Switching  $\mathfrak{a}, \mathfrak{a}'$ , we also get  $\dim \mathfrak{a}' \leq \dim \mathfrak{a}$ , hence  $\dim \mathfrak{a} = \dim \mathfrak{a}'$  and  $\mathfrak{a} = \text{Ad}_k(\mathfrak{a}')$ , as claimed.  $\square$

#### 44.4. The Cartan decomposition of semisimple linear groups.

Let  $\mathfrak{a} \subset \mathfrak{p}_\theta$  be a maximal abelian subspace and  $A = \exp(\mathfrak{a}) \subset P_\theta \subset G_\theta$ . This is a subgroup isomorphic to  $\mathbb{R}^n$ , where  $n = \dim \mathfrak{a}$ .

**Theorem 44.12.** (*The Cartan decomposition*) We have  $G_\theta = K^c A K^c$ . In other words, every element  $g \in G_\theta$  has a factorization  $g = k_1 a k_2$ ,  $k_1, k_2 \in K^c$ ,  $a \in A$ .<sup>23</sup>

*Proof.* Recall that we have the polar decomposition  $G_\theta = K^c P_\theta$ . Thus it suffices to show that every  $K^c$ -orbit on  $P_\theta$  intersects  $A$ . To do so, take  $Y \in P_\theta$  and let  $y = \log Y \in \mathfrak{p}_\theta$ . By Proposition 44.11 there is  $k \in K^c$  such that  $\text{Ad}_k(y) \in \mathfrak{a}$ . Then  $\text{Ad}_k(Y) \in A$ , as claimed.  $\square$

**Remark 44.13.** Theorem 44.12 has a straightforward generalization to reductive groups.

**Example 44.14.** 1. For  $G_\theta = GL_n(\mathbb{C})$ , Theorem 44.12 reduces to a classical theorem in linear algebra: any invertible complex matrix can be written as  $U_1 D U_2$ , where  $U_1, U_2$  are unitary and  $D$  is diagonal with positive entries.

2. Similarly, for  $G_\theta = GL_n(\mathbb{R})$ , Theorem 44.12 says that any invertible real matrix can be written as  $O_1 D O_2$ , where  $O_1, O_2$  are orthogonal and  $D$  is diagonal with positive entries.

#### 44.5. Maximal compact subgroups.

**Theorem 44.15.** (*E. Cartan*) Let  $G_\theta$  be a real form of a connected semisimple complex group  $G$ . Then any compact subgroup  $L$  of  $G_\theta$  is conjugate to a subgroup of  $K^c$  by an element of  $P_\theta$ . Also every compact subgroup of  $G_\theta$  is contained in a maximal one. Thus all maximal compact subgroups of  $G_\theta$  are conjugate (to  $K^c$ ).

*Proof.* We give a simplified version of Cartan's proof, due to G. D. Mostow.

First note that  $K^c$  is a maximal compact subgroup of  $G^\theta$ . Indeed, if  $K \supset K_c$  is a compact subgroup then the polar decomposition implies that  $K = K_c \cdot (P_\theta \cap K)$ . But if  $Y \in P_\theta \cap K$  and  $Y \neq 1$  then the sequence  $Y^n \in K$  has no convergent subsequence (which is clear by looking at the eigenvalues of  $Y^n$  on  $\mathfrak{g}_\theta$ ). Thus  $K = K_c$ .

<sup>23</sup>This factorization is not unique.

It remains to prove that every compact subgroup  $L \subset G_\theta$  can be conjugated into  $K^c$  by an element of  $P_\theta$ . The idea of proof is to define an  $L$ -invariant continuous real-valued function  $f$  on  $P_\theta$  and show that it has a unique minimum  $Y$  using a convexity argument. Then the required conjugating element is obtained as  $Y^{-\frac{1}{2}}$ .

So let us proceed with this plan. Recall that we have a decomposition of the Lie algebra  $\mathfrak{g}_\theta := \text{Lie}(G_\theta)$  given by  $\mathfrak{g}_\theta = \mathfrak{k}^c \oplus \mathfrak{p}_\theta$ , which is the eigenspace decomposition of  $\theta$ , and that the Killing form  $B = B_{\mathfrak{g}}$  is positive on  $\mathfrak{p}_\theta$ , negative on  $\mathfrak{k}^c$ , and  $\theta$ -invariant. Thus we have a positive definite inner product on the real vector space  $\mathfrak{g}_\theta$  given by

$$B_\theta(x, y) := -B(x, \theta(y)).$$

Denote by  $A^\dagger$  the adjoint operator to  $A \in \text{End}(\mathfrak{g}_\theta)$  under this inner product. Then  $A := \text{Ad}_g$  is orthogonal ( $A^\dagger = A^{-1}$ ) for  $g \in K^c$ , while for  $g \in P_\theta$  it is self-adjoint ( $A^\dagger = A$ ), unimodular and positive definite as its eigenvalues are positive). So if  $g = kp$  with  $k \in K^c$ ,  $p \in P_\theta$  then  $\bar{g} = kp^{-1}$ , hence

$$(44.1) \quad \text{Ad}_g^\dagger = \text{Ad}_{kp}^\dagger = \text{Ad}_p^\dagger \text{Ad}_k^\dagger = \text{Ad}_p \text{Ad}_k^{-1} = \text{Ad}_{pk^{-1}} = \text{Ad}_{\bar{g}}^{-1}.$$

Let

$$S := \int_L \text{Ad}_h^\dagger \text{Ad}_h dh \in \text{End}(\mathfrak{g}_\theta).$$

Then  $S$  is a self-adjoint positive definite operator. So it admits an orthonormal eigenbasis  $v_i$  with eigenvalues  $\lambda_i > 0$ . Let  $\lambda_{\min}$  be the smallest of these eigenvalues.

Consider the function  $f : P_\theta \rightarrow \mathbb{R}$  given by

$$f(X) := \text{Tr}(\text{Ad}_X \cdot S) = \sum_i \lambda_i B_\theta(\text{Ad}_X v_i, v_i).$$

So, since  $\text{Ad}_X$  is positive definite, we have

$$(44.2) \quad f(X) \geq \lambda_{\min} \text{Tr}(\text{Ad}_X).$$

Note also that the group  $G_\theta$  acts on  $P_\theta$  by  $g \circ X = gX\bar{g}^{-1}$ , and by (44.1) the function  $f$  is  $L$ -invariant.

Recall that for any  $R > 0$  the set of unimodular positive symmetric matrices  $A$  with  $\text{Tr}(A) \leq R$  is compact, since so is its subset of diagonal matrices, and any such matrix can be diagonalized by an orthogonal transformation. Since  $\text{Ad}_X$  is a positive self-adjoint operator on  $\mathfrak{g}_\theta$  with respect to  $B_\theta$ , it follows from (44.2) that the set of  $X \in P_\theta$  with  $f(X) \leq R$  is compact. This implies that  $f$ , being continuous, attains a minimum on  $P_\theta$ . Suppose it attains a minimum at the point  $Y = \exp(y)$ ,  $y \in \mathfrak{p}_\theta$ .

**Proposition 44.16.** *This minimum point is unique.*

*Proof.* Suppose  $Z = \exp(z)$ ,  $z \in \mathfrak{p}_\theta$  is another minimum point. Consider the Cartan decomposition of the element  $\exp(-\frac{z}{2})\exp(\frac{y}{2}) \in G_\theta$ :

$$\exp(\frac{z}{2})\exp(-\frac{y}{2}) = k \exp(\frac{x}{2}),$$

$k \in K^c$ ,  $x \in \mathfrak{p}_\theta$ . It follows that

$$\exp(\frac{x}{2}) = \exp(-\frac{y}{2})\exp(\frac{z}{2})k = k^{-1}\exp(\frac{z}{2})\exp(-\frac{y}{2}),$$

so multiplying, we get

$$\exp(x) = \exp(-\frac{y}{2})\exp(z)\exp(-\frac{y}{2})$$

and thus

$$(44.3) \quad \exp(z) = \exp(\frac{y}{2})\exp(x)\exp(\frac{y}{2}).$$

Consider the function

$$F(t) = f(\exp(\frac{y}{2})\exp(tx)\exp(\frac{y}{2})), \quad t \in \mathbb{R}.$$

This function has a global minimum at  $t = 0$ , and also at  $t = 1$  in view of (44.3). Thus the function  $F$  is not strictly convex. On the other hand, we have the following lemma.

**Lemma 44.17.** *Let  $a, M$  be symmetric real matrices such that  $M$  is positive definite. Then the function*

$$\phi(t) := \text{Tr}(\exp(ta)M), \quad t \in \mathbb{R}$$

*is convex, and is strictly convex if  $a \neq 0$ .*

*Proof.* Conjugating  $a, M$  simultaneously by an orthogonal matrix, we may assume that  $a$  is diagonal, with diagonal entries  $a_i$ . Then we have

$$\phi(t) := \sum_i M_{ii} \exp(ta_i).$$

Since  $M$  is positive definite,  $M_{ii} > 0$  and the statement follows.  $\square$

Using Lemma 44.17 for  $a := \text{adx}$  and  $M := \exp(\frac{\text{ady}}{2})S\exp(\frac{\text{ady}}{2})$  and the fact that  $F(t)$  is not strictly convex, we get that  $\text{adx} = 0$ , hence  $x = 0$  (as  $\mathfrak{g}$  is semisimple) and  $y = z$ , as claimed.  $\square$

Now, since the function  $f$  has a unique minimum point and is  $L$ -invariant, this minimum point must also be  $L$ -invariant. Thus we have  $h\exp(y) = \exp(y)\bar{h}$  for all  $h \in L$ . It follows that

$$\exp(-\frac{y}{2})h\exp(\frac{y}{2}) = \exp(\frac{y}{2})\bar{h}\exp(-\frac{y}{2}) = \overline{\exp(-\frac{y}{2})h\exp(\frac{y}{2})}.$$

Thus the element  $p := \exp(-\frac{y}{2}) = Y^{-\frac{1}{2}}$  conjugates  $L$  into  $K^c$ .  $\square$

**44.6. Cartan subalgebras in real semisimple Lie algebras.** We have seen that Cartan subalgebras in a complex semisimple Lie algebra are conjugate, but this is not so for real semisimple Lie algebras, as demonstrated by the following exercise.

**Exercise 44.18.** (i) Let  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$ . For  $0 \leq m \leq \frac{n}{2}$ , let  $\mathfrak{h}_m$  be the space of matrices of the form

$$A = \bigoplus_{i=1}^m \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix} \oplus \text{diag}(c_1, \dots, c_{n-2m})$$

such that  $\text{Tr}(A) = 0$ . Show that  $\mathfrak{h}_m$  is a Cartan subalgebra of  $\mathfrak{g}$  and that  $\mathfrak{h}_m$  is not conjugate to  $\mathfrak{h}_n$  when  $m \neq n$  (look at eigenvalues of elements of  $\mathfrak{h}_m$  in the vector representation). Conclude that Lemma 44.1 does not necessarily hold for non-compact forms of  $\mathfrak{g}$ .

(ii) Show that every Cartan subalgebra in  $\mathfrak{g}$  is conjugate to one of the form  $\mathfrak{h}_m$  for some  $m$ .

(iii) Classify Cartan subalgebras in other classical real simple Lie algebras (up to conjugacy).

Let us say that a semisimple element of  $\mathfrak{g}_\theta$  is **split** if it acts on  $\mathfrak{g}_\theta$  with real eigenvalues, and say that a commutative Lie subalgebra of  $\mathfrak{g}_\theta$  is a **split subalgebra** if it consists of split elements. An invariant of a Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}_\theta$  under conjugation is the dimension  $s(\mathfrak{h})$  of the largest split subalgebra of  $\mathfrak{h}$  (consisting of all split elements of  $\mathfrak{h}$ ). For example, a split real form  $\mathfrak{g}_\theta$  has a split Cartan subalgebra with  $s(\mathfrak{h}) = r = \text{rank}(\mathfrak{g})$ , and conversely, a real form that admits a split Cartan subalgebra is split. Also, in Exercise 44.18,  $s(\mathfrak{h}_m) = n - 1 - m$ .

Let us say that  $\mathfrak{h}$  is **maximally split** if  $s(\mathfrak{h})$  is the largest possible, and **maximally compact** if  $s(\mathfrak{h})$  is the smallest possible. For example, in Exercise 44.18,  $\mathfrak{h}_0$  is maximally split and  $\mathfrak{h}_{[n/2]}$  is maximally compact (where  $[n/2]$  is the floor of  $n/2$ ). Also, a split Cartan subalgebra is maximally split and a compact one (i.e., one for which  $\exp(\mathfrak{h})$  is a compact torus) is maximally compact, if they exist. Finally, the Cartan subalgebra  $\mathfrak{h}_+^c \oplus i\mathfrak{h}_-^c$ , where  $\mathfrak{h}_+^c, \mathfrak{h}_-^c$  are as in the proof of Proposition 41.7, is maximally compact.

Note that  $s(\mathfrak{h})$  may also be interpreted as the signature of the Killing form restricted to  $\mathfrak{h}$ , which equals  $(s(\mathfrak{h}), r - s(\mathfrak{h}))$ .

**Theorem 44.19.** (i) A  $\theta$ -stable Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}_\theta$  is maximally split iff  $\mathfrak{h}_- := \mathfrak{h} \cap \mathfrak{p}_\theta$  is a maximal abelian subspace in  $\mathfrak{p}_\theta$ .

(ii) A  $\theta$ -stable Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}_\theta$  is maximally compact iff  $\mathfrak{h}_+ := \mathfrak{h} \cap \mathfrak{k}^c$  is a Cartan subalgebra in  $\mathfrak{k}^c$ , and in this case  $s(\mathfrak{h}) = \text{rank}(\mathfrak{g}) - \text{rank}(\mathfrak{k})$ .

(iii) Any two maximally split  $\theta$ -stable Cartan subalgebras are conjugate by  $K^c$ .

(iv) Any two maximally compact  $\theta$ -stable Cartan subalgebras are conjugate by  $K^c$ .

(v) Any Cartan subalgebra in  $\mathfrak{g}_\theta$  is conjugate to a  $\theta$ -stable one by an element of  $G_\theta$  (or, equivalently,  $P_\theta$ ).

*Proof.* (i) It is clear that if  $\mathfrak{h}_-$  is a maximal abelian subspace of  $\mathfrak{p}_\theta$  then  $\mathfrak{h}$  is maximally split, since by Proposition 44.11 any abelian subspace of  $\mathfrak{p}_\theta$  can be conjugated into  $\mathfrak{h}_-$ . Conversely, if  $\mathfrak{h}$  is maximally split, suppose that  $a \in \mathfrak{p}_\theta, a \notin \mathfrak{h}_-$  with  $[a, \mathfrak{h}_-] = 0$ . Then  $\mathfrak{h}'_- = \mathfrak{h}_- \oplus \mathbb{R}a$ , and let  $\mathfrak{h}'$  be a Cartan subalgebra of  $\mathfrak{g}_\theta$  containing  $\mathfrak{h}'_-$ . Then  $s(\mathfrak{h}') > s(\mathfrak{h})$ , a contradiction.

(ii) It is clear that if  $\mathfrak{h}_+$  is a Cartan subalgebra of  $\mathfrak{k}^c$  then  $\mathfrak{h}$  is maximally compact. Also given a Cartan subalgebra  $\mathfrak{h}_+ \subset \mathfrak{k}^c$ , take a Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}_\theta$  containing  $\mathfrak{h}_+$ . Then  $s(\mathfrak{h}) \leq \text{rank}(\mathfrak{g}) - \text{rank}(\mathfrak{k})$ . This implies that for any maximally compact  $\mathfrak{h}$ , we have that  $\mathfrak{h} \cap \mathfrak{k}^c$  is a Cartan subalgebra in  $\mathfrak{k}^c$ , and  $s(\mathfrak{h}) = \text{rank}(\mathfrak{g}) - \text{rank}(\mathfrak{k})$ .

(iii) Let  $\mathfrak{h}, \mathfrak{h}'$  be maximally split  $\theta$ -stable Cartan subalgebras in  $\mathfrak{g}_\theta$ . Then  $\mathfrak{h}_-, \mathfrak{h}'_-$  are maximal abelian subspaces of  $\mathfrak{p}_\theta$ . So they are conjugate by  $K^c$  by Proposition 44.11, thus we may assume that  $\mathfrak{h}_- = \mathfrak{h}'_-$ . Let  $Z_-^c$  be the centralizer of  $\mathfrak{h}_-$  in  $K^c$ . It is a compact group, and it is clear that  $\mathfrak{h}_+, \mathfrak{h}'_+ \subset \text{Lie}(Z_-^c)$  are Cartan subalgebras. Hence they are conjugate by an element of  $Z_-^c$ , as desired.

(iv) Let  $\mathfrak{h}, \mathfrak{h}'$  be maximally compact  $\theta$ -stable Cartan subalgebras in  $\mathfrak{g}_\theta$ . Then  $\mathfrak{h}_+, \mathfrak{h}'_+$  are Cartan subalgebras of  $\mathfrak{k}^c$ , so they are conjugate by  $K^c$  and we may assume that  $\mathfrak{h}_+ = \mathfrak{h}'_+$ . Let  $Z_+$  be the centralizer of  $\mathfrak{h}_+$  in  $G_\theta$  and  $\mathfrak{z}_+ = \text{Lie}(Z_+)$ . This is a  $\theta$ -stable reductive subalgebra of  $\mathfrak{g}_\theta$  containing  $\mathfrak{h}, \mathfrak{h}'$  whose center contains  $\mathfrak{h}_+$ . Thus  $\mathfrak{h}_-, \mathfrak{h}'_- \subset \text{Lie}(Z_+)/\mathfrak{h}_+$  are  $\theta$ -stable split Cartan subalgebras, so they are conjugate by  $Z_+^c := Z_+ \cap K^c$  owing to (iii). This implies the statement.

(v) The proof is by induction in the rank  $r$  of  $\mathfrak{g}_\theta$ , with obvious base  $r = 0$ . Suppose the statement is known for rank  $< r$  and let us prove it for rank  $r$ . Let  $\mathfrak{h} \subset \mathfrak{g}_\theta$  be a Cartan subalgebra. We have  $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$  where  $\mathfrak{h}_+, \mathfrak{h}_-$  are the subspaces of elements with imaginary and real eigenvalues on the adjoint representation, respectively. The Lie group  $H_+ = \exp(\mathfrak{h}_+)$  is a compact torus, so it is contained in a maximal compact subgroup. Hence by Theorem 44.15  $H_+$  is conjugate to a subgroup of  $K^c$ . We may thus assume that  $\mathfrak{h}_+ \subset \mathfrak{k}^c$ .

As in (iv), let  $Z_+ \subset G_\theta$  be the centralizer of  $\mathfrak{h}_+$  and  $\mathfrak{z}_+ = \text{Lie}(Z_+)$ . It suffices to show that  $\mathfrak{h}$  is conjugate to a  $\theta$ -stable Cartan subalgebra



under  $Z_+$ . This is equivalent to saying that  $\mathfrak{h}_-$  is conjugate to a  $\theta$ -stable Cartan subalgebra of  $\mathfrak{z}_+/\mathfrak{h}_+$  under  $Z_+/H_+$ . So if  $\mathfrak{h}_+ \neq 0$  then the statement follows by the induction assumption, since the rank of  $\mathfrak{z}_+/\mathfrak{h}_+$  is smaller than  $r$ . On the other hand, if  $\mathfrak{h}_+ = 0$  then  $\mathfrak{h}$  is split, so  $\mathfrak{g}_\theta$  is split. In this case, let  $\mathfrak{h}_0$  be the standard Cartan subalgebra of  $\mathfrak{g}_\theta$ . Fixing systems of simple roots  $\Pi$  for  $\mathfrak{h}$  and  $\Pi_0$  for  $\mathfrak{h}_0$ , there exists an isomorphism  $\phi : (\mathfrak{g}_\theta, \mathfrak{h}, \Pi) \rightarrow (\mathfrak{g}_\theta, \mathfrak{h}_0, \Pi_0)$  which is given by an inner automorphism of  $\mathfrak{g}_\theta$ , i.e., an element  $g \in G_{\text{ad},\theta}$ , which completes the induction step and the proof.  $\square$

#### 44.7. Integral form of the Weyl character formula.

**Proposition 44.20.** *Let  $f$  be a conjugation-invariant continuous function on a compact connected Lie group  $K$  with a maximal torus  $T \subset K$  and Haar probability measure  $dk$ . Then*

$$\int_K f(k)dk = \frac{1}{|W|} \int_T f(t)|\Delta(t)|^2 dt,$$

where  $\Delta(t)$  is the Weyl denominator,<sup>24</sup>

$$\Delta(t) = \rho(t)^{-1} \prod_{\alpha \in R^+} (\alpha(t) - 1).$$

*Proof.* Since characters of irreducible representations span a dense subspace in the space of conjugation-invariant continuous functions on  $K$ , it suffices to check this for  $f = \chi_\lambda$ , the character of the irreducible representation  $L_\lambda$ . Then the left hand side is  $\delta_{0\lambda}$  by orthogonality of characters. On the other hand, the Weyl character formula implies that the right hand side also equals  $\delta_{0\lambda}$ .  $\square$

**Example 44.21.** Let  $f$  be a conjugation-invariant continuous function on  $U(n)$ . Then

$$\int_{U(n)} f(k)dk = \frac{1}{(2\pi)^n n!} \int_{|z_1|=\dots=|z_n|=1} f(\text{diag}(z_1, \dots, z_n)) \prod_{m<j} |z_m - z_j|^2 d\theta_1 \dots d\theta_n$$

where  $z_j = e^{i\theta_j}$ .

Thus we see that the orthogonality of characters can be written as

$$\frac{1}{|W|} \int_T \chi_\lambda(t) \overline{\chi_\mu(t)} |\Delta(t)|^2 dt = \delta_{\lambda,\mu}.$$

<sup>24</sup>Note that the function  $\rho(t)$  may be multivalued, but its branches differ from each other by a root of unity, so the function  $|\Delta(t)|$  is well defined. Namely,  $|\Delta(t)| = |\Delta_0(t)|$  where  $\Delta_0(t) = \prod_{\alpha \in R^+} (\alpha(t) - 1)$ .

**Exercise 44.22.** (i) Let  $\mathfrak{k} = \text{Lie}K$  with Cartan subalgebra  $\mathfrak{t}$  and  $f$  be a compactly supported  $K$ -invariant continuous function on  $\mathfrak{k}$ . Show that

$$\int_{\mathfrak{k}} f(a) da = \frac{1}{|W|} \int_{\mathfrak{t}} f(u) |\Delta_{\text{rat}}(u)|^2 du,$$

where  $\Delta_{\text{rat}}(u) = \prod_{\alpha \in R_+} \alpha(u)$  is the rational version of the Weyl denominator.

(ii) Write explicitly the identity you get if you set  $f(a) := e^{B_{\mathfrak{t}}(a,a)}$  (compute the Gaussian integral on the left hand side).

**Hint.** In Proposition 44.20, make a change of variable  $k = \exp(\varepsilon a)$ ,  $t = e^{\varepsilon u}$  for small  $\varepsilon > 0$  and then send  $\varepsilon$  to zero.

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