44. **Maximal tori in compact groups, Cartan decomposition**

44.1. **Maximal tori in connected compact Lie groups.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{g}^c$ its compact form, G a connected Lie group with Lie algebra $\mathfrak{g}$, $G^c \subset G$ its compact part (the connected Lie subgroup with Lie algebra $\mathfrak{g}^c$), as above.

A **Cartan subalgebra** $\mathfrak{h}^c \subset \mathfrak{g}^c$ is a maximal commutative Lie subalgebra (note that it automatically consists of semisimple elements since all elements of $\mathfrak{g}^c$ are semisimple). In other words, it is a subspace such that $\mathfrak{h}^c \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$.

Recall that all Cartan subalgebras of $\mathfrak{g}$ are conjugate, even if equipped with a system of simple roots (Theorem 20.10). Namely, given two such subalgebras $(\mathfrak{h}, \Pi)$ and $(\mathfrak{h}', \Pi')$, there is $g \in G$ such that $\mathrm{Ad}_g(\mathfrak{h}, \Pi) = (\mathfrak{h}', \Pi')$. It turns out that the same result holds for $\mathfrak{g}^c$.

Lemma 44.1. *Any two Cartan subalgebras in $\mathfrak{g}^c$ equipped with systems of simple roots are conjugate under G^c.*

Proof. Given $(\mathfrak{h}^c, \Pi)$ and $(\mathfrak{h}^{c'}, \Pi')$, there is $g \in G$ such that $\mathrm{Ad}_g(\mathfrak{h}^c, \Pi) = (\mathfrak{h}^{c'}, \Pi')$. Then we also have $\mathrm{Ad}_{\overline{g}}(\mathfrak{h}^c, \Pi) = (\mathfrak{h}^{c'}, \Pi')$, where $\overline{g} := \omega(g)$. So $\overline{g}^{-1}g$ commutes with $\mathfrak{h}^c$ and preserves Π, i.e., $\overline{g}h = g$, $h \in H := \exp(\mathfrak{h}^c_{\mathbb{C}})$. Writing $g = kp$, where $k \in G^c$, $p \in \mathbf{P}$, we have $kp^{-1}h = kp$, so $h = p^2$. Since p is positive, $p = h^{1/2}$, so it commutes with $\mathfrak{h}^c$ and preserves Π, thus $\mathrm{Ad}_k(\mathfrak{h}^c, \Pi) = (\mathfrak{h}^{c'}, \Pi')$, as claimed. $\square$

Note that for every Cartan subalgebra $\mathfrak{h}^c \subset \mathfrak{g}^c$, $H^c = \exp(\mathfrak{h}^c) \subset G^c$ is a torus, which is clearly a **maximal torus** (as the complexified Lie algebra of a larger torus would be a larger commutative subalgebra than $\mathfrak{h}^c$). Conversely, if $H^c \subset G^c$ is a maximal torus then $\mathrm{Lie}(H^c)$ can be included in a Cartan subalgebra, hence it is itself a Cartan subalgebra. So we have a bijection between Cartan subalgebras in $\mathfrak{g}^c$ and maximal tori in G^c. Similarly, there is a bijection between Cartan subalgebras in $\mathfrak{g}$ and maximal tori in G.

This implies

Corollary 44.2. *Any two maximal tori in G or G^c equipped with systems of simple roots are conjugate.*

We also have

Theorem 44.3. *Every element of a connected compact Lie group K is contained in a maximal torus, and all maximal tori in K are conjugate (even when equipped with systems of simple roots).*

Proof. We may assume without loss of generality that K is semisimple, i.e., $K = G^c$ for a connected semisimple complex Lie group G, which implies the second statement. To prove the first statement, let $K' \subset K$ be the set of elements contained in a maximal torus. Fix a maximal torus $T \subset K$ and consider the map $f : K \times T \to K$ given by $f(k,t) = ktk^{-1}$, whose image is K'. This implies that K' is compact, hence closed, so $K \setminus K'$ is open.

On the other hand, recall from Subsection 20.1 that a generic $x \in \mathfrak{g}^c$ is **regular**, meaning that its centralizer $\mathfrak{z}_x$ has dimension $\leq \mathrm{rank}(\mathfrak{g})$, in which case it must have dimension exactly $\mathrm{rank}(\mathfrak{g})$ and be a Cartan subalgebra. It is clear that every regular element x is contained in a unique maximal torus, namely $\exp(\mathfrak{z}_x)$, so the elements of $K \setminus K'$ are all non-regular. But the set of non-regular elements is defined by polynomial equations in Ad_x (the minors of Ad_x of codimension $\mathrm{rank}(\mathfrak{g})$ all vanish), so $K \setminus K'$ must be empty (as it is an open set contained in the set of solutions of nontrivial polynomial equations in Ad_x). $\square$

This immediately implies

Corollary 44.4. *The exponential map* $\exp : \mathfrak{g}^c \to G^c$ *is surjective.*[26]

Exercise 44.5. Is the exponential map surjective for the group $SL_2(\mathbb{C})$?

44.2. **Semisimple and unipotent elements.** Let G be a connected reductive complex Lie group. An element $g \in G$ is called **semisimple** if it acts in every finite dimensional representation of G by a semisimple (=diagonalizable) operator, and **unipotent** if it acts in every finite dimensional representation of G by a unipotent operator (all eigenvalues are 1).

Exercise 44.6. Let Y be a faithful finite dimensional representation of G (it exists by Corollary 36.5). Show that $g \in G$ is semisimple if and only if it acts semisimply on Y, and unipotent if and only if it acts unipotently on Y.

Hint: Use Proposition 36.12.

Exercise 44.7. Show that if G is semisimple then the exponential map defines a homeomorphism between the set of nilpotent elements in $\mathfrak{g} = \mathrm{Lie}G$ and the set of unipotent elements in G.

Exercise 44.8. Let Z be the center of a connected complex reductive group G.

[26]Here is another proof of this corollary. Let $K(x,y)$ be the Killing form of $\mathfrak{g}^c$. Since K is negative definite, the form $-K$ extends to a bi-invariant Riemannian metric on G_c. Since G^c is compact, the Hopf-Rinow theorem guarantees that for any $g \in G^c$ there is a geodesic on G^c in this metric connecting 1 and g. But it is easy to see that this geodesic is a segment of a one-parameter subgroup of G^c, which implies the statement.

(i) Show that the homomorphism $\pi : G \to G/Z$ defines a bijection between unipotent elements of G and G/Z.

(ii) Show that the set of semisimple elements of G is the preimage under π of the set of semisimple elements of G/Z.

Proposition 44.9. *(Jordan decomposition in G). Every element $g \in G$ has a unique factorization $g = g_s g_u$, where $g_s \in G$ is semisimple, $g_u \in G$ is unipotent and $g_s g_u = g_u g_s$.*

Exercise 44.10. Prove Proposition 44.9.

Hint. Use Exercise 44.8 to reduce to the case when $G = G_{\mathrm{ad}}$ is a semisimple adjoint group. In this case, write Ad_g as su, where s is a semisimple and u a unipotent operator with $su = us$ (Jordan decomposition for matrices). Show that $s = \mathrm{Ad}_{g_s}$ and $u = \mathrm{Ad}_{g_u}$ for some commuting $g_s, g_u \in G_{\mathrm{ad}}$. Then establish uniqueness using the uniqueness of Jordan decomposition of matrices.

44.3. **Maximal abelian subspaces of $\mathfrak{p}_\theta$.** Let G be a connected complex semisimple group, $G_\theta \subset G$ a real form, $\mathfrak{g}_\theta \subset \mathfrak{g}$ their Lie algebras. We have the polar decomposition $G_\theta = K^c P_\theta$ and the additive version $\mathfrak{g}_\theta = \mathfrak{k}^c \oplus \mathfrak{p}_\theta$, with $\mathfrak{p}_\theta = i\mathfrak{p}^c$. Also $\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c$.

Proposition 44.11. *(i) Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}_\theta$. Then the centralizer $\mathfrak{z}$ of $\mathfrak{a}$ in $\mathfrak{g}^c$ has the form $\mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m}$ is a reductive Lie algebra contained in $\mathfrak{k}^c$. Moreover, if $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{m}$ then $\mathfrak{t} \oplus i\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}^c$ and $\mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}_\theta$.*

(ii) If $a \in \mathfrak{a}$ is sufficiently generic then the centralizer of a in $\mathfrak{p}_\theta$ is $\mathfrak{a}$.

(iii) For any $p \in \mathfrak{p}_\theta$ there exists $k \in K^c$ such that $\mathrm{Ad}_k(p) \in \mathfrak{a}$.

(iv) All maximal abelian subspaces of $\mathfrak{p}_\theta$ are conjugate by K^c.

Proof. (i) Let $x \in \mathfrak{g}^c$, $[x, \mathfrak{a}] = 0$. Write $x = x_+ + x_-$, $x_+ \in \mathfrak{k}^c, x_- \in \mathfrak{p}^c$. Then $[x_\pm, \mathfrak{a}] = 0$, thus $x_- \in \mathfrak{a}$ by maximality of $\mathfrak{a}$. So $x \in \mathfrak{k}^c \oplus \mathfrak{a}$. Thus $\mathfrak{z} = \mathfrak{m} \oplus \mathfrak{a}$ where $\mathfrak{m} \subset \mathfrak{k}^c$ is a reductive Lie algebra. Moreover, if $\mathfrak{t} \subset \mathfrak{m}$ is a Cartan subalgebra then $\mathfrak{t} \oplus i\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{g}^c$, hence is a Cartan subalgebra. Similarly, $\mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}_\theta$.

(ii) Consider the group $T_\mathfrak{a} := \exp(i\mathfrak{a}) \subset G^c$. It is clear from (i) that this is a compact torus. Thus for a generic enough $a \in \mathfrak{a}$, the 1-parameter subgroup e^{ita} is dense in $T_\mathfrak{a}$. So if $p \in \mathfrak{p}_\theta$ and $[p, a] = 0$ then e^{ita} commutes with p, hence so do $T_\mathfrak{a}$ and $\mathfrak{a}$. So by maximality of $\mathfrak{a}$ we have $p \in \mathfrak{a}$.

(iii) Let $a \in \mathfrak{a}$ be generic enough as in (ii). Then by (ii), $\mathrm{Ad}_k(p) \in \mathfrak{a}$ if and only if $[\mathrm{Ad}_k(p), a] = 0$.

Consider the function $f : K^c \to \mathbb{R}$ given by $f(b) := (\mathrm{Ad}_b(p), a)$. This function is continuous, so attains a maximum on the compact group K^c. Suppose k is a maximum point of f. Let $p_0 := \mathrm{Ad}_k(p)$. Differentiating f at k, we get $([x, p_0], a) = 0$ for all $x \in \mathfrak{k}^c$. Thus $(x, [p_0, a]) = 0$ for all $x \in \mathfrak{k}^c$. But $[p_0, a] \in \mathfrak{k}^c$ and the inner product on $\mathfrak{k}^c$ is nondegenerate. Thus $[p_0, a] = 0$, as desired.

(iv) Let $\mathfrak{a}, \mathfrak{a}'$ be maximal abelian subspaces of $\mathfrak{p}_\theta$. Pick a generic element $p \in \mathfrak{a}'$ as in (ii). By (iii) we can find $k \in K^c$ such that $\mathrm{Ad}_k(p) = a \in \mathfrak{a}$. Moreover, a is generic in $\mathrm{Ad}_k(\mathfrak{a}')$. So for every $b \in \mathfrak{a}$ we have $[b, \mathrm{Ad}_k(\mathfrak{a}')] = 0$ (as $[b, a] = 0$). By maximality of $\mathfrak{a}'$ this implies that $b \in \mathrm{Ad}_k(\mathfrak{a}')$, i.e., $\mathfrak{a} \subset \mathrm{Ad}_k(\mathfrak{a}')$. Thus $\dim \mathfrak{a} \le \dim \mathfrak{a}'$. Switching $\mathfrak{a}, \mathfrak{a}'$, we also get $\dim \mathfrak{a}' \le \dim \mathfrak{a}$, hence $\dim \mathfrak{a} = \dim \mathfrak{a}'$ and $\mathfrak{a} = \mathrm{Ad}_k(\mathfrak{a}')$, as claimed. $\square$

44.4. **The Cartan decomposition of semisimple linear groups.** Let $\mathfrak{a} \subset \mathfrak{p}_\theta$ be a maximal abelian subspace and $A = \exp(\mathfrak{a}) \subset P_\theta \subset G_\theta$. This is a subgroup isomorphic to $\mathbb{R}^n$, where $n = \dim \mathfrak{a}$.

Theorem 44.12. *(The Cartan decomposition) We have $G_\theta = K^c A K^c$. In other words, every element $g \in G_\theta$ has a factorization $g = k_1 a k_2$, $k_1, k_2 \in K^c$, $a \in A$.*[27]

Proof. Recall that we have the polar decomposition $G_\theta = K^c P_\theta$. Thus it suffices to show that every K^c-orbit on P_θ intersects A. To do so, take $Y \in P_\theta$ and let $y = \log Y \in \mathfrak{p}_\theta$. By Proposition 44.11 there is $k \in K^c$ such that $\mathrm{Ad}_k(y) \in \mathfrak{a}$. Then $\mathrm{Ad}_k(Y) \in A$, as claimed. $\square$

Remark 44.13. Theorem 44.12 has a straightforward generalization to reductive groups.

Example 44.14. 1. For $G_\theta = GL_n(\mathbb{C})$, Theorem 44.12 reduces to a classical theorem in linear algebra: any invertible complex matrix can be written as $U_1 D U_2$, where U_1, U_2 are unitary and D is diagonal with positive entries.

2. Similarly, for $G_\theta = GL_n(\mathbb{R})$, Theorem 44.12 says that any invertible real matrix can be written as $O_1 D O_2$, where O_1, O_2 are orthogonal and D is diagonal with positive entries.

44.5. **Maximal compact subgroups.**

Theorem 44.15. *(E. Cartan) Let G_θ be a real form of a connected semisimple complex group G. Then any compact subgroup L of G_θ is conjugate to a subgroup of K^c by an element of P_θ. Also every compact subgroup of G_θ is contained in a maximal one. Thus all maximal compact subgroups of G_θ are conjugate (to K^c).*

[27] This factorization is not unique.

Proof. We give a simplified version of Cartan's proof, due to G. D. Mostow.

First note that K^c is a maximal compact subgroup of G^θ. Indeed, if $K \supset K_c$ is a compact subgroup then the polar decomposition implies that $K = K_c \cdot (P_\theta \cap K)$. But if $Y \in P_\theta \cap K$ and $Y \neq 1$ then the sequence $Y^n \in K$ has no convergent subsequence (which is clear by looking at the eigenvalues of Y^n on $\mathfrak{g}_\theta$. Thus $K = K_c$.

It remains to prove that every compact subgroup $L \subset G_\theta$ can be conjugated into K^c by an element of P_θ. The idea of proof is to define an L-invariant continuous real-valued function f on P_θ and show that it has a unique minimum Y using a convexity argument. Then the required conjugating element is obtained as $Y^{-\frac{1}{2}}$.

So let us proceed with this plan. Recall that we have a decomposition of the Lie algebra $\mathfrak{g}_\theta := \mathrm{Lie}(G_\theta)$ given by $\mathfrak{g}_\theta = \mathfrak{k}^c \oplus \mathfrak{p}_\theta$, which is the eigenspace decomposition of θ, and that the Killing form $B = B_{\mathfrak{g}}$ is positive on $\mathfrak{p}_\theta$, negative on $\mathfrak{k}^c$, and θ-invariant. Thus we have a positive definite inner product on the real vector space $\mathfrak{g}_\theta$ given by

$$B_\theta(x, y) := -B(x, \theta(y)).$$

Denote by $A^\dagger$ the adjoint operator to $A \in \mathrm{End}(\mathfrak{g}_\theta)$ under this inner product. Then $A := \mathrm{Ad}_g$ is orthogonal ($A^\dagger = A^{-1}$) for $g \in K^c$, while for $g \in P_\theta$ it is self-adjoint ($A^\dagger = A$), unimodular and positive definite as its eigenvalues are positive). So if $g = kp$ with $k \in K^c$, $p \in P_\theta$ then $\overline{g} = kp^{-1}$, hence

$$\mathrm{Ad}_g^\dagger = \mathrm{Ad}_{kp}^\dagger = \mathrm{Ad}_p^\dagger \mathrm{Ad}_k^\dagger = \mathrm{Ad}_p \mathrm{Ad}_k^{-1} = \mathrm{Ad}_{pk^{-1}} = \mathrm{Ad}\overline{g}^{-1}. \tag{44.1}$$

Let

$$S := \int_L \mathrm{Ad}_h^\dagger \mathrm{Ad}_h dh \in \mathrm{End}(\mathfrak{g}_\theta).$$

Then S is a self-adjoint positive definite operator. So it admits an orthonormal eigenbasis v_i with eigenvalues $\lambda_i > 0$. Let $\lambda_{\min}$ be the smallest of these eigenvalues.

Consider the function $f : P_\theta \to \mathbb{R}$ given by

$$f(X) := \mathrm{Tr}(\mathrm{Ad}_X \cdot S) = \sum_i \lambda_i B_\theta(\mathrm{Ad}_X v_i, v_i).$$

So, since Ad_X is positive definite, we have

$$f(X) \geq \lambda_{\min} \mathrm{Tr}(\mathrm{Ad}_X). \tag{44.2}$$

Note also that the group G_θ acts on P_θ by $g \circ X = gX\overline{g}^{-1}$, and by (44.1) the function f is L-invariant.

Recall that for any $R > 0$ the set of unimodular positive symmetric matrices A with $\mathrm{Tr}(A) \leq R$ is compact, since so is its subset of diagonal matrices, and any such matrix can be diagonalized by an orthogonal transformation. Since Ad_X is a positive self-adjoint operator on $\mathfrak{g}_\theta$ with respect to B_θ, it follows from (44.2) that the set of $X \in P_\theta$ with $f(X) \le R$ is compact. This implies that f, being continuous, attains a minimum on P_θ. Suppose it attains a minimum at the point $Y = \exp(y)$, $y \in \mathfrak{p}_\theta$.

Proposition 44.16. *This minimum point is unique.*

Proof. Suppose $Z = \exp(z)$, $z \in \mathfrak{p}_\theta$ is another minimum point. Consider the Cartan decomposition of the element $\exp(-\frac{z}{2})\exp(\frac{y}{2}) \in G_\theta$:

$$\exp(\tfrac{z}{2})\exp(-\tfrac{y}{2}) = k\exp(\tfrac{x}{2}),$$

$k \in K^c$, $x \in \mathfrak{p}_\theta$. It follows that

$$\exp(\tfrac{x}{2}) = \exp(-\tfrac{y}{2})\exp(\tfrac{z}{2})k = k^{-1}\exp(\tfrac{z}{2})\exp(-\tfrac{y}{2}),$$

so multiplying, we get

$$\exp(x) = \exp(-\tfrac{y}{2})\exp(z)\exp(-\tfrac{y}{2})$$

and thus

$$\exp(z) = \exp(\tfrac{y}{2})\exp(x)\exp(\tfrac{y}{2}). \tag{44.3}$$

Consider the function

$$F(t) = f(\exp(\tfrac{y}{2})\exp(tx)\exp(\tfrac{y}{2})),\ t \in \mathbb{R}.$$

This function has a global minimum at $t = 0$, and also at $t = 1$ in view of (44.3). Thus the function F is not strictly convex. On the other hand, we have the following lemma.

Lemma 44.17. *Let a, M be symmetric real matrices such that M is positive definite. Then the function*

$$\phi(t) := \mathrm{Tr}(\exp(ta)M),\ t \in \mathbb{R}$$

is convex, and is strictly convex if $a \neq 0$.

Proof. Conjugating a, M simultaneously by an orthogonal matrix, we may assume that a is diagonal, with diagonal entries a_i. Then we have

$$\phi(t) := \sum_i M_{ii}\exp(ta_i).$$

Since M is positive definite, $M_{ii} > 0$ and the statement follows. □

Using Lemma 44.17 for $a := \mathrm{ad}x$ and $M := \exp(\frac{\mathrm{ad}y}{2})S\exp(\frac{\mathrm{ad}y}{2})$ and the fact that $F(t)$ is not strictly convex, we get that $\mathrm{ad}x = 0$, hence $x = 0$ (as $\mathfrak{g}$ is semisimple) and $y = z$, as claimed. □

Now, since the function f has a unique minimum point and is L-invariant, this minimum point must also be L-invariant. Thus we have $h\exp(y) = \exp(y)\overline{h}$ for all $h \in L$. It follows that

$$\exp(-\tfrac{y}{2})h\exp(\tfrac{y}{2}) = \exp(\tfrac{y}{2})\overline{h}\exp(-\tfrac{y}{2}) = \overline{\exp(-\tfrac{y}{2})h\exp(\tfrac{y}{2})}.$$

Thus the element $p := \exp(-\frac{y}{2}) = Y^{-\frac{1}{2}}$ conjugates L into K^c. $\square$

44.6. **Cartan subalgebras in real semisimple Lie algebras.** We have seen that Cartan subalgebras in a complex semisimple Lie algebra are conjugate, but this is not so for real semisimple Lie algebras, as demonstrated by the following exercise.

Exercise 44.18. (i) Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{R})$. For $0 \le m \le \frac{n}{2}$, let $\mathfrak{h}_m$ be the space of matrices of the form

$$A = \bigoplus_{i=1}^{m} \begin{pmatrix} a_i & b_i \\ -b_i & a_i \end{pmatrix} \oplus \operatorname{diag}(c_1, ..., c_{n-2m})$$

such that $\operatorname{Tr}(A) = 0$. Show that $\mathfrak{h}_m$ is a Cartan subalgebra of $\mathfrak{g}$ and that $\mathfrak{h}_m$ is not conjugate to $\mathfrak{h}_n$ when $m \neq n$ (look at eigenvalues of elements of $\mathfrak{h}_m$ in the vector representation). Conclude that Lemma 44.1 does not necessarily hold for non-compact forms of $\mathfrak{g}$.

(ii) Show that every Cartan subalgebra in $\mathfrak{g}$ is conjugate to one of the form $\mathfrak{h}_m$ for some m.

(iii) Classify Cartan subalgebras in other classical real simple Lie algebras (up to conjugacy).

Let us say that a semisimple element of $\mathfrak{g}_\theta$ is **split** if it acts on $\mathfrak{g}_\theta$ with real eigenvalues, and say that a commutative Lie subalgebra of $\mathfrak{g}_\theta$ is a **split subalgebra** if it consists of split elements. An invariant of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_\theta$ under conjugation is the dimension $s(\mathfrak{h})$ of the largest split subalgebra of $\mathfrak{h}$ (consisting of all split elements of $\mathfrak{h}$). For example, a split real form $\mathfrak{g}_\theta$ has a split Cartan subalgebra with $s(\mathfrak{h}) = r = \operatorname{rank}(\mathfrak{g})$, and conversely, a real form that admits a split Cartan subalgebra is split. Also, in Exercise 44.18, $s(\mathfrak{h}_m) = n - 1 - m$.

Let us say that $\mathfrak{h}$ is **maximally split** if $s(\mathfrak{h})$ is the largest possible, and **maximally compact** if $s(\mathfrak{h})$ is the smallest possible. For example, in Exercise 44.18, $\mathfrak{h}_0$ is maximally split and $\mathfrak{h}_{[n/2]}$ is maximally compact (where $[n/2]$ is the floor of $n/2$). Also, a split Cartan subalgebra is maximally split and a compact one (i.e., one for which $\exp(\mathfrak{h})$ is a compact torus) is maximally compact, if they exist. Finally, the Cartan subalgebra $\mathfrak{h}_+^c \oplus i\mathfrak{h}_-^c$, where $\mathfrak{h}_+^c, \mathfrak{h}_-^c$ are as in the proof of Proposition 41.7, is maximally compact.

Note that $s(\mathfrak{h})$ may also be interpreted as the signature of the Killing form restricted to $\mathfrak{h}$, which equals $(s(\mathfrak{h}), r - s(\mathfrak{h}))$.

Theorem 44.19. *(i) A θ-stable Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_\theta$ is maximally split iff $\mathfrak{h}_- := \mathfrak{h} \cap \mathfrak{p}_\theta$ is a maximal abelian subspace in $\mathfrak{p}_\theta$.*

(ii) A θ-stable Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_\theta$ is maximally compact iff $\mathfrak{h}_+ := \mathfrak{h} \cap \mathfrak{k}^c$ is a Cartan subalgebra in $\mathfrak{k}^c$, and in this case $s(\mathfrak{h}) =$ $\mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{k})$.

(iii) Any two maximally split θ-stable Cartan subalgebras are conjugate by K^c.

(iv) Any two maximally compact θ-stable Cartan subalgebras are conjugate by K^c.

(v) Any Cartan subalgebra in $\mathfrak{g}_\theta$ is conjugate to a θ-stable one by an element of G_θ (or, equivalently, P_θ).

Proof. (i) It is clear that if $\mathfrak{h}_-$ is a maximal abelian subspace of $\mathfrak{p}_\theta$ then $\mathfrak{h}$ is maximally split, since by Proposition 44.11 any abelian subspace of $\mathfrak{p}_\theta$ can be conjugated into $\mathfrak{h}_-$. Conversely, if $\mathfrak{h}$ is maximally split, suppose that $a \in \mathfrak{p}_\theta, a \notin \mathfrak{h}_-$ with $[a, \mathfrak{h}_-] = 0$. Then $\mathfrak{h}'_- = \mathfrak{h}_- \oplus \mathbb{R}\mathfrak{a}$, and let $\mathfrak{h}'$ be a Cartan subalgebra of $\mathfrak{g}_\theta$ containing $\mathfrak{h}'_-$. Then $s(\mathfrak{h}') > s(\mathfrak{h})$, a contradiction.

(ii) It is clear that if $\mathfrak{h}_+$ is a Cartan subalgebra of $\mathfrak{k}^c$ then $\mathfrak{h}$ is maximally compact. Also given a Cartan subalgebra $\mathfrak{h}_+ \subset \mathfrak{k}^c$, take a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_\theta$ containing $\mathfrak{h}_+$. Then $s(\mathfrak{h}) \leq \mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{k})$. This implies that for any maximally compact $\mathfrak{h}$, we have that $\mathfrak{h} \cap \mathfrak{k}^c$ is a Cartan subalgebra in $\mathfrak{k}^c$, and $s(\mathfrak{h}) = \mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{k})$.

(iii) Let $\mathfrak{h}, \mathfrak{h}'$ be maximally split θ-stable Cartan subalgebras in $\mathfrak{g}_\theta$. Then $\mathfrak{h}_-, \mathfrak{h}'_-$ are maximal abelian subspaces of $\mathfrak{p}_\theta$. So they are conjugate by K^c by Proposition 44.11, thus we may assume that $\mathfrak{h}_- = \mathfrak{h}'_-$. Let Z^c_- be the centralizer of $\mathfrak{h}_-$ in K^c. It is a compact group, and it is clear that $\mathfrak{h}_+, \mathfrak{h}'_+ \subset \mathrm{Lie}(Z^c_-)$ are Cartan subalgebras. Hence they are conjugate by an element of Z^c_-, as desired.

(iv) Let $\mathfrak{h}, \mathfrak{h}'$ be maximally compact θ-stable Cartan subalgebras in $\mathfrak{g}_\theta$. Then $\mathfrak{h}_+, \mathfrak{h}'_+$ are Cartan subalgebras of $\mathfrak{k}^c$, so they are conjugate by K^c and we may assume that $\mathfrak{h}_+ = \mathfrak{h}'_+$. Let Z_+ be the centralizer of $\mathfrak{h}_+$ in G_θ and $\mathfrak{z}_+ = \mathrm{Lie}(Z_+)$. This is a θ-stable reductive subalgebra of $\mathfrak{g}_\theta$ containing $\mathfrak{h}, \mathfrak{h}'$ whose center contains $\mathfrak{h}_+$. Thus $\mathfrak{h}_-, \mathfrak{h}'_- \subset \mathrm{Lie}(Z_+)/\mathfrak{h}_+$ are θ-stable split Cartan subalgebras, so they are conjugate by $Z^c_+ :=$ $Z_+ \cap K^c$ owing to (iii). This implies the statement.

(v) The proof is by induction in the rank r of $\mathfrak{g}_\theta$, with obvious base $r = 0$. Suppose the statement is known for rank $< r$ and let us prove it for rank r. Let $\mathfrak{h} \subset \mathfrak{g}_\theta$ be a Cartan subalgebra. We have $\mathfrak{h} = \mathfrak{h}_+ \oplus \mathfrak{h}_-$ where $\mathfrak{h}_+, \mathfrak{h}_-$ are the subspaces of elements with imaginary and real eigenvalues on the adjoint representation, respectively. The Lie group $H_+ = \exp(\mathfrak{h}_+)$ is a compact torus, so it is contained in a maximal compact subgroup. Hence by Theorem 44.15 H_+ is conjugate to a subgroup of K^c. We may thus assume that $\mathfrak{h}_+ \subset \mathfrak{k}^c$.

As in (iv), let $Z_+ \subset G_\theta$ be the centralizer of $\mathfrak{h}_+$ and $\mathfrak{z}_+ = \mathrm{Lie}(Z_+)$. It suffices to show that $\mathfrak{h}$ is conjugate to a θ-stable Cartan subalgebra under Z_+. This is equivalent to saying that $\mathfrak{h}_-$ is conjugate to a θ-stable Cartan subalgebra of $\mathfrak{z}_+/\mathfrak{h}_+$ under Z_+/H_+. So if $\mathfrak{h}_+ \neq 0$ then the statement follows by the induction assumption, since the rank of $\mathfrak{z}_+/\mathfrak{h}_+$ is smaller than r. On the other hand, if $\mathfrak{h}_+ = 0$ then $\mathfrak{h}$ is split, so $\mathfrak{g}_\theta$ is split. In this case, let $\mathfrak{h}_0$ be the standard Cartan subalgebra of $\mathfrak{g}_\theta$. Fixing systems of simple roots Π for $\mathfrak{h}$ and Π_0 for $\mathfrak{h}_0$, there exists an isomorphism $\phi : (\mathfrak{g}_\theta, \mathfrak{h}, \Pi) \to (\mathfrak{g}_\theta, \mathfrak{h}_0, \Pi_0)$ which is given by an inner automorphism of $\mathfrak{g}_\theta$, i.e., an element $g \in G_{\mathrm{ad},\theta}$, which completes the induction step and the proof. □

44.7. **Integral form of the Weyl character formula.**

Proposition 44.20. *Let f be a conjugation-invariant continuous function on a compact connected Lie group K with a maximal torus $T \subset K$ and Haar probability measure dk. Then*

$$\int_K f(k)dk = \frac{1}{|W|}\int_T f(t)|\Delta(t)|^2 dt,$$

where $\Delta(t)$ is the Weyl denominator,[28]

$$\Delta(t) = \rho(t)^{-1} \prod_{\alpha \in R^+} (\alpha(t) - 1).$$

Proof. Since characters of irreducible representations span a dense subspace in the space of conjugation-invariant continuous functions on K, it suffices to check this for $f = \chi_\lambda$, the character of the irreducible representation L_λ. Then the left hand side is $\delta_{0\lambda}$ by orthogonality of characters. On the other hand, the Weyl character formula implies that the right hand side also equals $\delta_{0\lambda}$. □

Example 44.21. Let f be a conjugation-invariant continuous function on $U(n)$. Then

$$\int_{U(n)} f(k)dk =$$

[28]Note that the function $\rho(t)$ may be multivalued, but its branches differ from each other by a root of unity, so the function $|\Delta(t)|$ is well defined. Namely, $|\Delta(t)| = |\Delta_0(t)|$ where $\Delta_0(t) = \prod_{\alpha \in R^+}(\alpha(t) - 1)$.

$$\frac{1}{(2\pi)^n n!}\int_{|z_1|=...=|z_n|=1} f(\mathrm{diag}(z_1,...,z_n))\prod_{m<j}|z_m-z_j|^2 d\theta_1....d\theta_n$$

where $z_j = e^{i\theta_j}$.

Thus we see that the orthogonality of characters can be written as

$$\frac{1}{|W|}\int_T \chi_\lambda(t)\overline{\chi_\mu(t)}|\Delta(t)|^2 dt = \delta_{\lambda,\mu}.$$

Exercise 44.22. (i) Let $\mathfrak{k} = \mathrm{Lie}K$ with Cartan subalgebra $\mathfrak{t}$ and f be a compactly supported K-invariant continuous function on $\mathfrak{k}$. Show that

$$\int_{\mathfrak{k}} f(a)da = \frac{1}{|W|}\int_{\mathfrak{t}} f(u)|\Delta_{\mathrm{rat}}(u)|^2 du,$$

where $\Delta_{\mathrm{rat}}(u) = \prod_{\alpha\in R_+}\alpha(u)$ is the rational version of the Weyl denominator.

(ii) Write explicitly the identity you get if you set $f(a) := e^{B_{\mathfrak{k}}(a,a)}$ (compute the Gaussian integral on the left hand side).

Hint. In Proposition 44.20, make a change of variable $k = \exp(\varepsilon a)$, $t = e^{\varepsilon u}$ for small $\varepsilon > 0$ and then send ε to zero.