45. **Topology of Lie groups and homogeneous spaces, I**

45.1. **The Chevalley-Eilenberg complex of a compact connected Lie group.** We would now like to study topology of connected Lie groups. The Cartan decomposition implies that any real semisimple Lie group G_θ is diffeomorphic to the product of its maximal compact subgroup K^c and a Euclidean space. This combined with weak Levi decomposition (Theorem 16.6) implies that topology of connected Lie groups essentially reduces to topology of compact ones, as any simply-connected solvable Lie group has a filtration by normal subgroups with successive quotients being the 1-dimensional group $\mathbb{R}$, hence is diffeomorphic to $\mathbb{R}^n$ (cf. Theorem 49.1, Corollary 49.6 below).

So let us study cohomology of compact connected Lie groups.

We first recall some generalities on cohomology of manifolds. As we mentioned before, the cohomology of an n-dimensional manifold M can be computed by the **de Rham complex**

$$0 \to \Omega^0(M) \to \Omega^1(M) \to ... \to \Omega^n(M) \to 0,$$

where $\Omega^i(M)$ is the space of smooth (complex-valued) differential i-forms on M. The maps in this complex are given by the differential $d : \Omega^i(M) \to \Omega^{i+1}(M)$, which satisfies the equation $d^2 = 0$. Namely, we define the i-th **de Rham cohomology** of M as the quotient

$$H^i(M, \mathbb{C}) := \Omega^i_{\text{closed}}(M)/\Omega^i_{\text{exact}}(M)$$

where $\Omega^i_{\text{closed}}(M) \subset \Omega^i(M)$ is the space of **closed forms** (such that $d\omega = 0$) and $\Omega^i_{\text{exact}}(M) \subset \Omega^i(M)$ is the space of **exact forms** (such that $\omega = d\eta$ for some $\eta \in \Omega^{i-1}(M)$).

If M is compact then the spaces $H^i(M, \mathbb{C})$ are known to be finite dimensional, so we can define the **Betti numbers** of M, $b_i(M) := \dim H^i(M, \mathbb{C})$. Note that $b_0(M)$ is the number of connected components of M, so if M is connected then $b_0(M) = 1$.

The wedge product of differential forms descends to the cohomology, which makes $H^\bullet(M, \mathbb{C}) := \oplus_{i=0}^n H^i(M, \mathbb{C})$ into a graded algebra. This algebra is associative and **graded-commutative**: $ab = (-1)^{\deg(a)\deg(b)} ba$ (since the wedge product of differential forms has these properties). Moreover, if $f : M \to N$ is a differentiable map of manifolds then we have the pullback map $f^* : \Omega^i(N) \to \Omega^i(M)$ which commutes with d and hence descends to the cohomology. Also f^* preserves the wedge product, hence defines a graded algebra homomorphism $f^* : H^\bullet(N, \mathbb{C}) \to H^\bullet(M, \mathbb{C})$.

Exercise 45.1. Let $f : [0,1] \times M \to N$ be a differentiable map and $f_t : M \to N$ be given by $f_t(x) = f(t,x)$. Then $f_0^* = f_1^*$ on $H^\bullet(N, \mathbb{C})$. In other words, f^* is invariant under (smooth) homotopies of f.

Recall that for a vector field v on M, the **Lie derivative**

$$L_v : \Omega^\bullet(M) \to \Omega^\bullet(M)$$

is the unique derivation of the algebra of differential forms which commutes with the de Rham differential and equals the usual derivative of a function along v on $\Omega^0(M)$.

Lemma 45.2. *(Cartan's magic formula) Let v be a vector field on M, $L_v : \Omega^i(M) \to \Omega^i(M)$ the Lie derivative and $\iota_v : \Omega^i(M) \to \Omega^{i-1}(M)$ the contraction operator. Then*

$$L_v = \iota_v d + d\iota_v.$$

Proof. It suffices to check this identity on local charts. It is easy to see that both sides are derivations, so it suffices to check the equation on functions (0-forms) and on 1-forms of the form df where f is a function. For functions we have $L_v f = \iota_v df$, which is essentially the definition of L_v, while for $\omega = df$ we have

$$L_v(df) = d(L_v f) = d\iota_v(df) = (\iota_v d + d\iota_v)(df),$$

since $d^2 = 0$. □

Corollary 45.3. *L_v maps closed forms to exact forms, hence acts trivially in cohomology.*

Corollary 45.4. *If a connected Lie group G acts on a manifold M then G acts trivially on $H^\bullet(M, \mathbb{C})$.*

Suppose now that a compact connected Lie group G acts on a manifold M. Then we have the averaging operator $P : \Omega^\bullet(M) \to \Omega^\bullet(M)$ over G which commutes with d and satisfies the equation $P^2 = P$, so we have a decomposition of complexes

$$\Omega^\bullet(M) = \Omega^\bullet(M)^G \oplus \Omega^\bullet(M)_0$$

where the first summand is the image of P and the second one is the kernel of P.

Theorem 45.5. *The complex $\Omega^\bullet(M)_0$ is exact. Thus the cohomology $H^\bullet(M, \mathbb{C})$ is computed by the complex of invariant differential forms $\Omega^\bullet(M)^G$.*

Proof. If $\omega \in \Omega^i(M)_0$ is closed then by Corollary 45.4 the cohomology class $[\omega]$ of ω coincides with the cohomology class of $[g\omega]$ for all $g \in G$. Thus

$$[\omega] = \int_G [g\omega]dg = \big[\int_G g\omega dg \big] = 0.$$

It follows that $\omega = d\eta$ for some $\eta \in \Omega^i(M)$. Then $\omega = (1-P)\omega = d(1-P)\eta$, and $(1-P)\eta \in \Omega^i(M)_0$. So the complex $\Omega^\bullet(M)_0$ is exact, which implies the statement. □

Corollary 45.6. *If G is a compact Lie group then $H^\bullet(G,\mathbb{C})$ is computed by the complex $\Omega^\bullet(G)^G$ of left-invariant differential forms on G.*

The complex $\Omega^\bullet(G)^G$ is called the **Chevalley-Eilenberg complex** of G.

45.2. **Cohomology of Lie algebras.** It turns out that the Chevalley-Eilenberg complex of G can be described purely algebraically in terms of the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$. To this end, we will need another lemma from basic differential geometry.

Lemma 45.7. *(Cartan differentiation formula) Let $\omega \in \Omega^m(M)$ and $v_0, ..., v_m$ be vector fields on M. Then*

$$d\omega(v_0, ..., v_m) = \sum_i (-1)^i L_{v_i}(\omega(v_0, ..., \widehat{v}_i, ..., v_m)) +$$

$$\sum_{i<j} (-1)^{i+j} \omega([v_i, v_j], v_0, ..., \widehat{v}_i, ..., \widehat{v}_j, ..., v_m)$$

(where the hats indicate the omitted terms).

Proof. It is easy to show that the right hand side is linear over functions on M with respect to each v_i (the first derivatives of the function cancel out). Therefore, it suffices to assume that $v_i = \frac{\partial}{\partial x_{k_i}}$ (in local coordinates), and $\omega = f dx_{j_1} \wedge ... \wedge dx_{j_m}$. Then the second summand on the RHS vanishes and the verification is straightforward. □

Corollary 45.8. *Let G be a Lie group and $\omega \in \Omega^m(G)^G$ be a left-invariant differential form. Then for any left-invariant vector fields $v_0, ..., v_m$ we have*

(45.1) $$d\omega(v_0, ..., v_m) = \sum_{i<j} (-1)^{i+j} \omega([v_i, v_j], v_0, ..., \widehat{v}_i, ..., \widehat{v}_j, ..., v_m).$$

Proof. This follows since the functions $\omega(v_0, ..., \widehat{v}_i, ..., v_m)$ are constant. □

Now observe that $\Omega^m(G)^G = \wedge^m \mathfrak{g}^*$. Thus we get

Corollary 45.9. *For any Lie group G the complex $\Omega^\bullet(G)^G$ coincides with the complex*

$$0 \to \mathbb{C} \to \mathfrak{g}^* \to (\wedge^2\mathfrak{g})^* \to ...(\wedge^m\mathfrak{g})^* \to ...$$

with differential defined by (45.1)*, where* $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$*.*

This purely algebraic complex can be defined for any Lie algebra $\mathfrak{g}$ over any field (the equality $d^2 = 0$ follows from the Jacobi identity).[29] It is called the **standard complex** or the **Chevalley-Eilenberg complex** of $\mathfrak{g}$, denoted $CE^\bullet(\mathfrak{g})$, and its cohomology is called the **Lie algebra cohomology** of $\mathfrak{g}$, denoted $H^\bullet(\mathfrak{g})$.[30]

Also note that the complex $CE^\bullet(\mathfrak{g})$ has wedge product multiplication, which descends to the cohomology. Thus $H^\bullet(\mathfrak{g})$ is a graded-commutative associative algebra. Furthermore, if $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ for a compact connected Lie group G then $H^\bullet(\mathfrak{g}) \cong H^\bullet(G, \mathbb{C})$ as a graded algebra. However, this may fail even at the level of vector spaces (i.e., Betti numbers) if G is not compact.

Example 45.10. Let $\mathfrak{g}$ be abelian, $\dim \mathfrak{g} < \infty$. Then $CE^\bullet(\mathfrak{g}) = \wedge^\bullet\mathfrak{g}^*$, with zero differential, so $H^\bullet(\mathfrak{g}) = \wedge^\bullet\mathfrak{g}^*$. So if $G = (S^1)^n$ is a torus then we get $H^\bullet(G, \mathbb{C}) = \wedge^\bullet\mathfrak{g}^* = \wedge^\bullet(\xi_1, ..., \xi_n)$ where ξ_i have degree 1. In particular, $H^\bullet(S^1) = \wedge^\bullet(\xi)$. However, for the universal cover $\mathbb{R}$ of S^1 this is clearly false.

Remark 45.11. Corollary 45.9 implies that for compact Lie groups K_1, K_2 the map $\Omega^\bullet(K_1)\otimes\Omega^\bullet(K_2) \to \Omega^\bullet(K_1\times K_2)$ (i.e., in components, $\Omega^i(K_1) \otimes \Omega^j(K_2) \to \Omega^{i+j}(K_1 \times K_2)$) defines an isomorphism of cohomology rings $H^\bullet(K_1, \mathbb{C}) \otimes H^\bullet(K_2, \mathbb{C}) \to H^\bullet(K_1 \times K_2, \mathbb{C})$. This is a special case of the **Künneth theorem**, which actually holds for any manifolds (and more generally for sufficiently nice topological spaces), which need not have any group structure. We warn the reader, however, that the **tensor product of algebras here is in the graded sense**, i.e.

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(b)\deg(a')}(aa' \otimes bb').$$

Theorem 45.12. *If G is a connected compact Lie group with* $\mathrm{Lie}(G)_{\mathbb{C}} = \mathfrak{g}$ *then $H^\bullet(G, \mathbb{C}) \cong (\wedge^\bullet\mathfrak{g}^*)^{\mathfrak{g}}$ as a ring.*

Proof. We have an action of $G \times G$ on G, so the cohomology of G is computed by the complex of invariants $\Omega^\bullet(G)^{G\times G} = (\wedge^\bullet\mathfrak{g}^*)^G$. So our job is to show that the differential in this complex is actually zero.

[29]Note that if $\mathfrak{g}$ is finite dimensional then $\wedge^i\mathfrak{g}^* = (\wedge^i\mathfrak{g})^*$.

[30]Note that $H^1(\mathfrak{g})$ already appeared earlier in Section 18.

But this follows immediately from the definition of the differential in $\wedge^\bullet \mathfrak{g}^*$. $\square$

We also have

Proposition 45.13. *If G is a connected Lie group, $\Gamma \subset G$ a finite subgroup, and $\pi : G \to G/\Gamma$ is the canonical map then π^* defines an isomorphism $H^\bullet(G/\Gamma, \mathbb{C}) \to H^\bullet(G, \mathbb{C})$.*

Proof. The map π^* is an isomorphism $H^\bullet(G/\Gamma, \mathbb{C}) \to H^\bullet(G, \mathbb{C})^\Gamma$, but Γ, being a subgroup of G, acts trivially on $H^\bullet(G, \mathbb{C})$. $\square$

Thus it suffices to determine the cohomology of simple, simply connected compact Lie groups.