46. **Topology of Lie groups and homogeneous spaces, II**

46.1. **The coproduct on the cohomology ring.** To understand the algebra $R := H^\bullet(G) = H^\bullet(G,\mathbb{C})$ better, note that the multiplication map $G \times G \to G$ induces the graded algebra homomorphism $\Delta : H^\bullet(G) \to H^\bullet(G \times G) = H^\bullet(G) \otimes H^\bullet(G)$, which is coassociative:

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

(Note that the warning in Remark 45.11 about tensor product in the graded sense still applies here!) Such a map Δ is called a **coproduct** since it defines an algebra structure on the dual space R^* (see Subsection 12.3). We also have the augmentation map $\varepsilon : R \to \mathbb{C}$ such that

$$(\varepsilon \otimes 1)(\Delta(x)) = (1 \otimes \varepsilon)(\Delta(x)) = x$$

for all $x \in R$. Such a structure is called a **graded bialgebra**.[26]

Exercise 46.1. (Hopf theorem) Let R be a finite dimensional graded-commutatitive bialgebra over a field $\mathbf{k}$ of characteristic zero, and $R[0] = \mathbf{k}$ (where the grading is by nonnegative integers). Show that R is a **free** graded commutative algebra on some homogeneous generators of odd degrees, i.e., $R = \wedge^\bullet_{\mathbf{k}}(\xi_1, ..., \xi_r)$ with $\deg \xi_i = 2m_i + 1$ for some nonnegative integers m_i. Thus $\dim R = 2^r$.

Hint. Recall from Subsection 14.1 that an element $x \in R$ is **primitive** if $\Delta(x) = x \otimes 1 + 1 \otimes x$. Show that any homogeneous primitive x has odd degree (use that $\dim R < \infty$), thus $x^2 = 0$, and that R is generated by homogeneous primitive elements. Then show that linearly independent primitive elements in R cannot satisfy any nontrivial relation (take a relation of lowest degree, compute its coproduct and find a relation of even lower degree, getting a contradiction).

For more hints see [C], Subsection 2.4.

Let us now determine the number r. We have $2^r = \dim(\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}}$. But this dimension can be computed using the Weyl character formula. Namely, the character of $\wedge^\bullet \mathfrak{g}^*$ is

$$\chi_{\wedge^\bullet \mathfrak{g}^*}(t) = 2^{\mathrm{rank}(\mathfrak{g})} \prod_{\alpha>0} (1 + \alpha(t))(1 + \alpha(t)^{-1}),$$

where $T \subset G$ is a maximal torus and $t \in T$. So

$$\dim(\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}} = \frac{2^{\mathrm{rank}(\mathfrak{g})}}{|W|} \int_T \prod_{\alpha>0} (\alpha(t^2) - 1)(1 - \alpha(t^{-2}))dt = 2^{\mathrm{rank}(\mathfrak{g})}.$$

[26] Moreover, we have an algebra homomorphism $S : R \to R$ induced by the inversion map $G \to G$ called the **antipode**. This makes R into what is called a **graded Hopf algebra**.

So $r = \text{rank}(\mathfrak{g})$.

Thus we have

$$H^\bullet(G) = H^\bullet(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}^*)^{\mathfrak{g}} = \wedge^\bullet(\xi^{(1)}, ..., \xi^{(r)}),$$

where $r = \text{rank}(\mathfrak{g})$. and $\deg(\xi^{(i)}) = 2m_i + 1$. Moreover, it suffices to consider the case when $\mathfrak{g}$ is simple. What are the numbers m_i in this case?

Let us order m_i as follows: $m_1 \le m_2 \le ... \le m_r$. We know that $r + 2\sum m_i = \dim \mathfrak{g}$, so $\sum_i m_i = |R_+|$. Also it is not hard to see that $m_1 = 1$, $m_2 > 1$:

Exercise 46.2. Show that for a simple Lie algebra $\mathfrak{g}$ we have $(\wedge^3\mathfrak{g}^*)^{\mathfrak{g}} = \mathbb{C}$, spanned by the triple product $([xy], z)$.

Hint. Let $\omega \in (\wedge^3\mathfrak{g}^*)^{\mathfrak{g}}$.

1. Show that

$$\omega(e_i, [f_i, h_i], h) + \omega(e_i, h_i, [f_i, h]) = 0$$

for $h \in \mathfrak{h}$ and deduce that

$$\omega(e_i, f_i, h) = \tfrac{1}{2}\alpha_i(h)\omega(e_i, f_i, h_i).$$

2. Take $y, z \in \mathfrak{h}$ and show that

$$\omega(h_i, y, z) + \omega(f_i, [e_i, y], z) + \omega(f_i, y, [e_i, z]) = 0.$$

Deduce that $\omega(x, y, z) = 0$ for $x, y, z \in \mathfrak{h}$. Conclude that ω is completely determined by $\omega(e_\alpha, e_{-\alpha}, h)$ for all roots α and $h \in \mathfrak{h}$. Use the Weyl group to reduce to $\omega(e_i, f_i, h)$ and then to $\omega(e_i, f_i, h_i)$.

3. Finally, use that

$$\omega([e_i, e_j], f_i, f_j) = \omega(e_j, f_j, h_i) = \omega(e_i, f_i, h_j)$$

to show that all possible ω are proportional.

In particular, we see that for a simple compact connected Lie group G, one has $H^3(G, \mathbb{C}) \cong \mathbb{C}$. Thus, the sphere S^n admits a Lie group structure if and only if $n = 0, 1, 3$.

Example 46.3. We get $m_2 = 2$ for A_2, $m_2 = 3$ for $B_2 = C_2$, $m_2 = 5$ for G_2. Thus the Poincaré polynomials $P_{\mathfrak{g}}(q) := \sum_{n \ge 0} \dim H^n(G, \mathbb{C})q^n$ for compact simple Lie groups of rank ≤ 2 are:

$$P_{A_1}(q) = 1 + q^3, \ P_{A_2}(q) = (1 + q^3)(1 + q^5),$$

$$P_{B_2}(q) = (1 + q^3)(1 + q^7), \ P_{G_2}(q) = (1 + q^3)(1 + q^{11}).$$

46.2. **The cohomology ring of a simple compact connected Lie group.** In fact, we have the following classical theorem, which we will not prove in general, but will prove below for type A and also in exercises for classical groups and G_2.

Theorem 46.4. *Let G be a simple compact Lie group with complexified Lie algebra $\mathfrak{g}$. Then the numbers m_i are the exponents of $\mathfrak{g}$ defined in Subsection 32.3. In other words, the degrees $2m_i+1$ of generators of the cohomology ring are the dimensions of simple modules occurring in the decomposition of $\mathfrak{g}$ over its principal $\mathfrak{sl}_2$-subalgebra. Thus the cohomology ring $H^\bullet(G,\mathbb{C})$ is the exterior algebra $\wedge^\bullet(\xi_{2m_1+1},...,\xi_{2m_r+1})$, where ξ_j has degree j.*

A modern general proof of this theorem can be found in [R].

Remark 46.5. The Poincaré polynomial $P_{\mathfrak{g}}(q)$ of $(\wedge^\bullet\mathfrak{g}^*)^{\mathfrak{g}}$ is given by the formula

$$P_{\mathfrak{g}}(q) = \frac{(1+q)^r}{|W|}\int_T \prod_{\alpha\in R}(1+q\alpha(t))\prod_{\alpha>0}(\alpha(t)^{\frac{1}{2}}-\alpha(t)^{-\frac{1}{2}})^2.$$

So Theorem 46.4 is equivalent to the statement that this integral equals $\prod_i(1+q^{2m_i+1})$.

We will prove Theorem 46.4 in the case of type A.

Corollary 46.6. *For $\mathfrak{g}=\mathfrak{sl}_n$ we have $m_i=i$. Equivalently, the same is true for $\mathfrak{g}=\mathfrak{gl}_n$ if we add $m_0=0$.*

Proof. Let $\mathfrak{g}=\mathfrak{gl}_n$, $V=\mathbb{C}^n$. We need to compute the Poincaré polynomial of $\wedge^\bullet(V\otimes V^*)^{\mathfrak{g}}$. The skew Howe duality (Proposition 30.11) implies that this Poincaré polynomial is

$$P(q)=\sum_{\lambda=\lambda^t} q^{|\lambda|},$$

where the summation is over λ with $\le n$ parts. But there are exactly 2^n such symmetric partitions λ: they consist of a sequence of hooks $(k,1^{k-1})$ with decreasing values of k, with each of them either present or not. The degree of such a hook is $2k-1$, which implies that

$$P_{\mathfrak{gl}_n}(q)=(1+q)(1+q^3)(1+q^5)...(1+q^{2n-1}). \tag{46.1}$$

$\square$

Thus we get that the cohomology $H^\bullet(U(n),\mathbb{C})=H^\bullet(GL_n(\mathbb{C}),\mathbb{C})$ is $\wedge^\bullet(\xi_1,\xi_3,...,\xi_{2n-1})$ (where subscripts are degrees) with Poincaré polynomial (46.1), and $H^\bullet(SU(n),\mathbb{C})=H^\bullet(SL_n(\mathbb{C}),\mathbb{C})=\wedge^\bullet(\xi_3,...,\xi_{2n-1})$ with Poincaré polynomial $(1+q^3)(1+q^5)...(1+q^{2n-1})$.

In the next exercise and the following subsections we will use the notions of a **cell complex** and its **cellular homology and cohomology** with coefficients in any commutative ring, and the fact that if a manifold is equipped with a cell decomposition (i.e., represented as a disjoint union of cells) then its cellular cohomology with $\mathbb{C}$-coefficients (=dual to the cellular homology) is canonically isomorphic to the de Rham cohomology via the integration pairing (the **de Rham theorem**). More details can be found, for instance, in [H].

Exercise 46.7. (i) Give another proof of Theorem 46.4 for type A_{n-1} as follows. Use that $SU(n)/SU(n-1) = S^{2n-1}$ to construct a cellular decomposition of $SU(n)$ into 2^{n-1} cells (use the decomposition of S^{2n-1} into a point and its complement). Then show that the differential in the corresponding cochain complex with $\mathbb{C}$-coefficients is zero (compare its dimension to the dimension of the cohomology). Derive Theorem 46.4 for $SU(n)$ by induction in n.

(ii) Use the same idea and the fact that $U(n,\mathbb{H})/U(n-1,\mathbb{H}) = S^{4n-1}$ to establish Theorem 46.4 in type C_n. Conclude that the cohomology ring of $U(n,\mathbb{H})$ (and $\mathrm{Sp}_{2n}(\mathbb{C})$) is $\wedge(\xi_3,\xi_7,...,\xi_{4n-1})$ with Poincaré polynomial is $(1+q^3)(1+q^7)...(1+q^{4n-1})$.

(iii) Show that these Poincaré polynomials are valid for cohomology of the same Lie groups with any coefficients.[27]

46.3. **Cohomology of homogeneous spaces.** Let G be a connected compact Lie group, $\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$, $K \subset G$ a closed subgroup, $\mathfrak{k} = \mathrm{Lie}(K)_{\mathbb{C}}$, and consider the homogeneous space G/K. How to compute the cohomology $H^\bullet(G/K,\mathbb{C})$?

Since the group G acts on G/K, this cohomology is computed by the complex $\Omega^\bullet(G/K)^G = (\wedge^\bullet(\mathfrak{g}/\mathfrak{k})^*)^K$. Let us denote this complex by $CE^\bullet(\mathfrak{g},K)$. It is called the **relative Chevalley-Eilenberg complex**.

For example, if $K=\Gamma$ is finite, this is just the Γ-invariant part of the usual Chevalley-Eilenberg complex. But Γ acts trivially on the cohomology, so we get $H^\bullet(G/\Gamma) = H^\bullet(G)$ (as already noted above).

But what happens if $\dim K > 0$? Can we describe the differential in this complex algebraically as we did for $K=1$?

This question is answered by the following proposition. Let $\mathfrak{k} \subset \mathfrak{g}$ be a pair of Lie algebras (not necessarily finite dimensional, over any field). Denote by $CE^i(\mathfrak{g},\mathfrak{k})$ the spaces $(\wedge^\bullet(\mathfrak{g}/\mathfrak{k})^*)^{\mathfrak{k}}$.

Proposition 46.8. *$CE^\bullet(\mathfrak{g},\mathfrak{k})$ is a subcomplex of $CE^\bullet(\mathfrak{g})$.*

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[27] A similar idea can be used to find the cohomology of $\mathrm{Spin}(n)$ (see Exercise 46.13 below) but it is a bit more complicated since there is no cell decomposition with zero boundary map, and thus any cell decomposition has strictly more than 2^r cells for sufficiently large n (as there is 2-torsion in the integral cohomology).

Exercise 46.9. Prove Proposition 46.8.

Definition 46.10. The complex $CE^\bullet(\mathfrak{g},\mathfrak{k})$ is called the **relative Chevalley-Eilenberg complex**, and its cohomology is called the **relative Lie algebra cohomology**, denoted by $H^\bullet(\mathfrak{g},\mathfrak{k})$.

Now note that, going back to the setting of compact Lie groups, we have $CE^\bullet(\mathfrak{g},K) = CE^\bullet(\mathfrak{g},\mathfrak{k})^{K/K^\circ}$, so we obtain

Corollary 46.11. *$H^\bullet(G/K,\mathbb{C}) \cong H^\bullet(\mathfrak{g},\mathfrak{k})^{K/K^\circ}$ as algebras.*

Thus, the computation of the cohomology of G/K reduces to the computation of the relative Lie algebra cohomology, which is again a purely algebraic problem.

Corollary 46.12. *Suppose $z \in K$ is an element that acts by -1 on $\mathfrak{g}/\mathfrak{k}$. Then $(\wedge^i(\mathfrak{g}/\mathfrak{k})^*)^K = 0$ for odd i. Hence the differential in $CE^\bullet(\mathfrak{g},K)$ vanishes and thus $H^\bullet(G/K,\mathbb{C}) \cong (\wedge^\bullet(\mathfrak{g}/\mathfrak{k})^*)^K$, with cohomology present only in even degrees.*

Exercise 46.13. The real **Stiefel manifold** $\mathrm{St}_{n,k}(\mathbb{R})$, $k < n$, is the manifold of all orthonormal k-tuples of vectors in $\mathbb{R}^n$. For example, $\mathrm{St}_{n,1}(\mathbb{R}) = S^{n-1}$ and $\mathrm{St}_{n,n-1}(\mathbb{R}) = SO(n)$.

(i) Show that $\mathrm{St}_{n,k}(\mathbb{R}) = SO(n)/SO(n-k)$ and hence $\dim \mathrm{St}_{n,k}(\mathbb{R}) = k(n-k) + \frac{k(k-1)}{2}$.

(ii) Show that for $n \geq 3$, the manifold $\mathrm{St}_{n,2}(\mathbb{R})$ is a fiber bundle over S^{n-1} with fiber S^{n-2}. Conclude that $\mathrm{St}_{n,2}(\mathbb{R})$ has a cell decomposition with four cells of dimensions $0, n-2, n-1, 2n-3$. Show that the boundary of the $n-1$-dimensional cell is zero if n is even and twice the $n-2$-dimensional cell if n is odd. Compute the cohomology groups of $\mathrm{St}_{n,2}(\mathbb{R})$ with any coefficient ring. In particular, show that if n is odd then the cohomology groups with coefficients in any field of characteristic $\neq 2$ are the same as for the sphere S^{2n-3}.

(iii) Use the relative Chevalley-Eilenberg complex to compute the cohomology $H^*(\mathrm{St}_{n,2}(\mathbb{R}),\mathbb{C})$ in another way. Compare to (ii).

Exercise 46.14. (i) Prove Theorem 46.4 for type B_n using the method of Exercise 46.7. Namely, use that $SO(2n+1)/SO(2n-1) = \mathrm{St}_{2n+1,2}(\mathbb{R})$ and Exercise 46.13(ii) or (iii). Conclude that the cohomology ring of $SO(2n+1)$ (and $SO_{2n+1}(\mathbb{C})$) over $\mathbb{C}$ is $\wedge^\bullet(\xi_3,\xi_7,...,\xi_{4n-1})$ with Poincaré polynomial is $(1+q^3)(1+q^7)...(1+q^{4n-1})$.

(ii) Use the conclusion of (i) for B_{n-1} and that $SO(2n)/SO(2n-1) = S^{2n-1}$ to prove Theorem 46.4 for type D_n (again using the method of Exercise 46.7). Conclude that the cohomology ring of $SO(2n)$ (and $SO_{2n}(\mathbb{C})$) over $\mathbb{C}$ is $\wedge^\bullet(\xi_3,\xi_7,...,\xi_{4n-5},\eta_{2n-1})$ with Poincaré polynomial having the form $(1+q^3)(1+q^7)...(1+q^{4n-5})\cdot(1+q^{2n-1})$.

(iii) Show that these Poincaré polynomials are valid for cohomology of the same Lie groups with coefficients in any ring containing $\frac{1}{2}$.

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