47. Topology of Lie groups and homogeneous spaces, III

47.1. **Grassmannians.** Let $G = U(m+n), K = U(n) \times U(m)$, so that G/K is the **Grassmannian** $\mathrm{G}_{m+n,n}(\mathbb{C}) \cong \mathrm{G}_{m+n,m}(\mathbb{C})$ (the manifold of m-dimensional or n-dimensional subspaces of $\mathbb{C}^{m+n}$). The element $z = I_n \oplus (-I_m)$ acts by -1 on $\mathfrak{g}/\mathfrak{k} = V \otimes W^* \oplus W \otimes V^*$, where V, W are the tautological representations of $U(n)$ and $U(m)$. So we get that the Grassmannian has cohomology only in even degrees, and

$$H^{2i}(\mathrm{G}_{m+n,m}(\mathbb{C})) = \wedge^{2i}(V \otimes W^* \oplus W \otimes V^*)^{U(n)\times U(m)}.$$

We can therefore use the skew Howe duality (Proposition 30.11) to see that

$$\dim H^{2i}(\mathrm{G}_{m+n,m}(\mathbb{C})) = N_i(n, m),$$

where $N_i(n, m)$ is the number of partitions $\lambda = (\lambda_1, ..., \lambda_k)$ whose Young diagrams has i boxes and fit into the rectangle $m \times n$ (i.e., such that $k \leq m, \lambda_1 \leq n$).

To compute $N_i(m, n)$, consider the generating function

$$f_{n,m}(q) = \sum_i N_i(n, m)q^i.$$

Then, denoting by p_i the jumps $\lambda_i - \lambda_{i+1}$ of λ (with $p_0 = n - \lambda_1$), we have

$$\sum_{n\geq 0} f_{n,m}(q)z^n =$$

$$\sum_{p_0,p_1,\dots,p_m\geq 0} z^{p_0+p_1+\dots+p_m} q^{p_1+2p_2+\dots+mp_m} = \prod_{j=0}^{m} \frac{1}{1-q^j z}.$$

So the Betti numbers of Grassmannians are the coefficients of this series. For example, if $m = 1$ we get

$$\sum_{n\geq 0} f_{n,m}(q)z^n = \frac{1}{(1-z)(1-qz)} = \sum_n (1 + q + ... + q^n)z^n.$$

So we recover the Poincaré polynomial $1 + q + ... + q^n$ of the complex projective space $\mathbb{CP}^n$. More precisely, this is the Poincaré polynomial evaluated at $q^{\frac{1}{2}}$, which is actually a polynomial in q since we have nontrivial cohomology only in even degrees.

The polynomials $f_{n,m}(q)$ are called the **Gaussian binomial coefficients** and they can be computed explicitly. Namely, we have

Proposition 47.1.

$$f_{m,n}(q) = \binom{m+n}{n}_q = \binom{m+n}{m}_q = \frac{[m+n]_q!}{[m]_q![n]_q!},$$

where $[m]_q := \frac{q^m-1}{q-1}$ *and* $[m]_q! := [1]_q...[m]_q$.

Proof. This follows immediately from the q-**binomial theorem**[28]

$$\sum_{n\geq 0} \binom{m+n}{n}_q z^n = \prod_{j=0}^{m} \frac{1}{1-q^j z}. \tag{47.1}$$

□

Exercise 47.2. Prove (47.1).

Hint. Let $F(z)$ be the RHS of this identity. Write a q-difference equation expressing $F(qz)$ in terms of $F(z)$. Show that this equation has a unique solution such that $F(0) = 1$. Then prove that the LHS satisfies the same equation.

Exercise 47.3. Compute the Betti numbers of $\mathrm{G}_{N,2}(\mathbb{C})$.

47.2. **Schubert cells.** There is actually a more geometric way to obtain the same result. This way is based on decomposing the Grassmannians into **Schubert cells**. Namely, let $F_i \subset \mathbb{C}^{m+n}$ be spanned by the first i basis vectors $e_1, ..., e_i$; thus

$$0 = F_0 \subset F_1 \subset ... \subset F_{m+n} = \mathbb{C}^{m+n}.$$

Given an m-dimensional subspace $V \subset \mathbb{C}^{m+n}$, let ℓ_j be the smallest integer for which $\dim(F_{\ell_j} \cap V) = j$. Then

$$1 \leq \ell_1 < \ell_2 < ... < \ell_m \leq m+n,$$

which defines a partition with parts

$$\lambda_1 = \ell_m - m, \lambda_2 = \ell_{m-1} - m + 1, ..., \lambda_m = \ell_1 - 1$$

fitting in the $m \times n$ box. Let $S_\lambda \subset \mathrm{G}_{m+n,m}(\mathbb{C})$ be the set of V giving such numbers λ_i.

Exercise 47.4. Show that S_λ is a locally closed embedded complex submanifold of the Grassmannian isomorphic to the affine space $\mathbb{C}^{|\lambda|}$ of dimension $|\lambda| = \sum_i \lambda_i$ (i.e., a closed embedded submanifold in an open subset of the Grassmannian).

Hint. Show that for $V \in S_\lambda$, the elements $f_k := e^*_{\ell_k}|_V$ form a basis of V^*. For $\ell_j + 1 \leq i \leq \ell_{j+1}$ (with $\ell_{m+1} := m+n$), show that $e^*_i|_V$ is a linear combination of f_k, $j+1 \leq k \leq m$, and denote the corresponding

[28]Note that setting $q = 1$ in the q-binomial theorem, we get the familiar formula from calculus, often called the binomial theorem:

$$\sum_{n\geq 0} \binom{m+n}{m} z^n = \frac{1}{(1-z)^{m+1}}.$$

coefficients by $a_{ik}(V)$. Show that the assignment $V \mapsto (a_{ik}(V))$ is an isomorphism $S_\lambda \cong \mathbb{C}^{|\lambda|}$.

Definition 47.5. The subset S_λ of the Grassmannian is called the **Schubert cell** corresponding to λ.

So we see that $\mathrm{G}_{m+n,m}(\mathbb{C})$ has a **cell decomposition** into a disjoint union of Schubert cells.

Now we can rederive the same formula for the Poincaré polynomial of the Grassmannian from the following well-known fact from algebraic topology:

Proposition 47.6. *If X is a connected cell complex which only has even-dimensional cells, then the cohomology of X vanishes in odd degrees, and the groups $H^{2i}(X,\mathbb{Z})$ are free abelian groups of ranks $b_{2i}(X)$, where the Betti number $b_{2i}(X)$ is just the number of cells in X of dimension i. Moreover, X is simply connected.*

Indeed, the boundary map in this cell complex has to be zero, and its fundamental group must be trivial, as it is a quotient of the fundamental group of the 1-skeleton of X, which is a single point (why?).

So we obtain an even stronger statement than before:

Corollary 47.7. *$H^{2i}(\mathrm{G}_{m+n,n}(\mathbb{C}),\mathbb{Z})$ are free abelian groups of ranks given by coefficients of $\binom{m+n}{m}_q$, and the odd cohomology groups are zero. Moreover, Grassmannians are simply connected.*

In particular, this gives Betti numbers over any field (including positive characteristric), not just $\mathbb{C}$.

47.3. **Flag manifolds.** The **flag manifold** $\mathcal{F}_n(\mathbb{C})$ is the space of all **complete flags** $0 = V_0 \subset V_1 \subset ... \subset V_n = \mathbb{C}^n$, where $\dim V_i = i$. Note that the flag manifold is a homogeneous space: $\mathcal{F}_n = G/T$, where $G = U(n)$ and $T = U(1)^n$ is a maximal torus in G. It can also be written as $G_{\mathbb{C}}/B$, where $G_{\mathbb{C}} = GL_n(\mathbb{C})$ and $B = B_n$ is the subgroup of upper triangular matrices.

We have fibrations $\pi : \mathcal{F}_n(\mathbb{C}) \to \mathbb{CP}^{n-1}$ sending $(V_1, ..., V_{n-1})$ to V_{n-1}, whose fiber is the space of flags in V_{n-1}, i.e., $\mathcal{F}_{n-1}(\mathbb{C})$. This shows, by induction, that flag manifolds can be decomposed into even-dimensional cells isomorphic to $\mathbb{C}^k$.

More precisely, to define actual cells, we need to trivialize the fibration π over each cell in $\mathbb{CP}^{n-1}$. These cells are C_{in}, $i = 1, ..., n$, where C_{in} is the set of hyperplanes $E \subset \mathbb{C}^n$ defined by an equation $a_1x_1 + ... + a_nx_n = 0$ where the first nonzero coefficient is a_i (so $C_{in} \cong \mathbb{C}^{n-i}$). This means that for $(x_1, ..., x_n) \in E$, the coordinates

$x_j, j \neq i$ can be chosen arbitrarily, and then x_i is uniquely determined. So we may identify E with $\mathbb{C}^{n-1}$ by sending $(x_1, ..., x_n)$ to $(x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$, which defines the required trivialization.

Thus we obtain a stratification of $\mathcal{F}_n$ into cells C_w labeled by permutations $w \in S_n$, which we'll represent as orderings of $1, 2, ..., n$. Namely, this stratification and labeling are defined by induction in n: for $w \in S_{n-1}$, $C_w \times C_{in} = C_{w_i'}$, where $w_i' \in S_n$ is obtained from w by inserting n in the i-th place (namely, $w_i' = w \circ (i, i+1, ..., n)$). By analogy with the Grassmannian, the cells C_w are called **Schubert cells**.

It follows that the Betti numbers of $\mathcal{F}_n$ vanish in odd degrees, and in even degrees are given by the generating function

$$\sum b_{2i}(\mathcal{F}_n) q^n = [n]_q! = (1+q)(1+q+q^2)...(1+q+...+q^{n-1}).$$

Moreover, it is easy to see that $\dim_{\mathbb{C}} C_w = \ell(w)$, so we get the identity

$$\sum_{w \in S_n} q^{\ell(w)} = [n]_q!$$

Finally, note that the group B_n of upper triangular matrices preserves each C_w. In fact, it is easy to check by induction in n that C_w are simply B_n-orbits on $\mathcal{F}_n$.

Remark 47.8. We have a map $\pi_m : \mathcal{F}_{m+n}(\mathbb{C}) \to \mathrm{G}_{m+n,m}(\mathbb{C})$ sending $(V_1, ..., V_{m+n-1})$ to V_m. This is a fibration with fiber $\mathcal{F}_m(\mathbb{C}) \times \mathcal{F}_n(\mathbb{C})$. This gives another proof of the formula for Betti numbers of the Grassmannian (Proposition 47.1).

We can also define the **partial flag manifold** $\mathcal{F}_S(\mathbb{C})$, where $S \subset [1, n-1]$ is a subset, namely the space of **partial flags** $(V_s, s \in S)$, $V_s \subset \mathbb{C}^n$, $\dim V_s = s$, $V_s \subset V_t$ if $s < t$.

Exercise 47.9. Let $S = \{n_1, n_1 + n_2, ..., n_1 + ... + n_{k-1}\}$, and $n_k = n - n_1 - ... - n_{k-1}$. Show that the even Betti numbers of the partial flag manifold are the coefficients of the polynomial

$$P_S(q) := \frac{[n]_q!}{[n_1]_q! ... [n_k]_q!}$$

called the **Gaussian multinomial coefficient** (and the odd Betti numbers vanish). Show that the partial flag manifold is simply connected.

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