48. **Levi decomposition**

48.1. **Cohomology of Lie algebras with coefficients.** The definition of cohomology of Lie algebras may be generalized to define the cohomology with coefficients in a module, so that the cohomology considered above is the one for the trivial module.

Let $\mathfrak{g}$ be a Lie algebra and V a $\mathfrak{g}$-module. The **Chevalley-Eilenberg (or standard) complex of $\mathfrak{g}$ with coefficients in V** is defined by

$$CE^\bullet(\mathfrak{g}, V) := \mathrm{Hom}(\wedge^\bullet \mathfrak{g}, V)$$

with differential defined by the full Cartan formula (without dropping the first term):

$$d\omega(a_0, ..., a_m) = \sum_i (-1)^i a_i \omega(a_0, ..., \widehat{a}_i, ..., a_m) +$$

$$\sum_{i<j} (-1)^{i+j} \omega([a_i, a_j], a_0, ..., \widehat{a}_i, ..., \widehat{a}_j, ..., a_m).$$

The cohomology of this complex is called the **cohomology of $\mathfrak{g}$ with coefficients in V** and denoted $H^\bullet(\mathfrak{g}, V)$. Note that the previously defined cohomology $H^\bullet(\mathfrak{g})$ is $H^\bullet(\mathfrak{g}, \mathbb{C})$.

If $\mathfrak{g}$ is the Lie algebra of a Lie group G (or its complexification) and V is finite dimensional, then we simply have $CE^\bullet(\mathfrak{g}, V) := (\Omega^\bullet(G) \otimes V)^{\mathfrak{g}}$ (and the differential is just the de Rham differential). So in particular by Theorem 45.5 we have (using that the smallest $i > 0$ such that $H^i(\mathfrak{g}, \mathbb{C}) \neq 0$ is 3):

Proposition 48.1. *(i) If G is compact and V is a nontrivial irreducible representation then*

$$H^i(\mathfrak{g}, V) = 0, \ i > 0.$$

In particular, this is so for any non-trivial irreducible finite dimensional reporesentation V of a semisimple Lie algebra $\mathfrak{g}$.

(ii) (Whitehead's theorem) For semisimple $\mathfrak{g}$ and any finite dimensional V we have $H^1(\mathfrak{g}, V) = H^2(\mathfrak{g}, V) = 0$.[29]

However, this cohomology is non-trivial in general if $\mathfrak{g}$ is not semisimple or V is infinite dimensional.

Let us explore the meaning of $H^i(\mathfrak{g}, V)$ for small i.

1. We have $H^0(\mathfrak{g}, V) = V^{\mathfrak{g}}$, the $\mathfrak{g}$-invariants in V.

2. $H^1(\mathfrak{g}, V)$ is the quotient of the space $Z^1(\mathfrak{g}, V)$ of 1-cocycles $\omega : \mathfrak{g} \to V$, i.e., linear maps satisfying

$$\omega([x, y]) = x\omega(y) - y\omega(x)$$

[29]Note that Whitehead's theorem in the case of H^1 was proved previously in Subsection 18.2.

by the space of 1-coboundaries $B^1(\mathfrak{g}, V)$, of the form $\omega(x) = xv$ for some $v \in V$.

Proposition 48.2. *(i) If V, W are representations of $\mathfrak{g}$ then* $\mathrm{Ext}^1(V, W) = H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(V, W))$.

(ii) Consider the action of the additive group of V on the Lie algebra $\mathfrak{g} \ltimes V$ (with trivial commutator on V) by

$$v \circ (x, w) = (x, w + xv).$$

Then $H^1(\mathfrak{g}, V)$ classifies Lie algebra homomorphisms $\mathfrak{g} \to \mathfrak{g} \ltimes V$ of the form $x \mapsto (x, \omega(x))$ modulo this action.

Proof. (i) Suppose the space $W \oplus V$ is equipped with the action of $\mathfrak{g}$ so that W is a submodule and V the quotient. Thus the action of $\mathfrak{g}$ on $W \oplus V$ is given by

$$\rho(x) = \begin{pmatrix} \rho_W(x) & \omega(x) \\ 0 & \rho_V(x) \end{pmatrix},$$

where $\omega : \mathfrak{g} \to \mathrm{Hom}_{\mathbf{k}}(V, W)$. So the identity $\rho([x, y]) = [\rho(x), \rho(y)]$ translates into

$$\omega([x, y]) = \rho_W(x)\omega(y) - \omega(y)\rho_V(x) - \rho_W(y)\omega(x) + \omega(x)\rho_V(y).$$

i.e., $\rho \in Z^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(V, W))$. Also it is easy to check that for two such representations ρ_1, ρ_2 there is an isomorphism $\rho_1 \to \rho_2$ acting trivially on W and V/W if and only if the corresponding maps ω_1, ω_2 differ by a coboundary: $\omega_1 - \omega_2 \in B^1(\mathfrak{g}, \mathrm{Hom}_{\mathbf{k}}(V, W))$. This implies the statement.

(ii) We leave this to the reader as an exercise. □

3. $Z^1(\mathfrak{g}, \mathfrak{g})$ is the Lie algebra of derivations of $\mathfrak{g}$, and $B^1(\mathfrak{g}, \mathfrak{g})$ is the ideal of inner derivations. So $H^1(\mathfrak{g}, \mathfrak{g})$ is the Lie algebra of **outer derivations**, the quotient of all derivations by inner derivations. In particular, we rederive the fact proved earlier that all derivations of a semisimple complex Lie algebra $\mathfrak{g}$ are inner ($H^1(\mathfrak{g}, \mathfrak{g}) = 0$).

4. Suppose we want to define an **abelian extension** $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ by V, i.e., a Lie algebra which can be included in the short exact sequence

$$0 \to V \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0$$

where V is an abelian ideal. To classify such extensions, pick a vector space splitting $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus V$, then the commutator looks like

$$[(x, v), (y, w)] = ([x, y], xw - yv + \omega(x, y)),$$

where $\omega : \wedge^2\mathfrak{g} \to V$ is a linear map. The Jacobi identity is then equivalent to ω being in the space $Z^2(\mathfrak{g}, V)$ of 2-cocycles. Moreover, it is easy to check that for two such extensions $\widetilde{\mathfrak{g}}_1, \widetilde{\mathfrak{g}}_2$ there is an isomorphism

$\phi : \widetilde{\mathfrak{g}}_1 \to \widetilde{\mathfrak{g}}_2$ which acts trivially on V and $\mathfrak{g}$ if and only if the corresponding cocycles ω_1, ω_2 differ by a coboundary: $\omega_1 - \omega_2 \in B^2(\mathfrak{g}, V)$. Thus, we get

Proposition 48.3. *Abelian extensions of $\mathfrak{g}$ by V modulo isomorphisms which act trivially on V and $\mathfrak{g}$ are classified by $H^2(\mathfrak{g}, V)$. For example, the space $H^2(\mathfrak{g}, \mathbb{C})$ classifies* **1-dimensional central extensions** *of $\mathfrak{g}$:*

$$0 \to \mathbb{C} \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

Example 48.4. Let $\mathfrak{g} = \mathbb{C}^2$ be the 2-dimensional abelian Lie algebra. Then we have seen that the Poincaré polynomial of the cohomology of $\mathfrak{g}$ is $1 + 2q + q^2$ (cohomology of the 2-torus). So $H^2(\mathfrak{g}, \mathbb{C}) = \mathbb{C}$. The only cocycle up to scaling is given by $\omega(x, y) = 1$, where x, y is a basis of $\mathfrak{g}$, and all coboundaries are zero. So we have a central extension of $\mathfrak{g}$ defined by this cocycle with basis x, y, c and $[x, y] = c$, $[x, c] = [y, c] = 0$. This is the **Heisenberg Lie algebra**, which is isomorphic to the Lie algebra of strictly upper-triangular 3 by 3 matrices.

5. Let us now study deformations of Lie algebras. Suppose $\mathfrak{g}$ is a Lie algebra over a field $\mathbf{k}$ and we want to deform the bracket, with deformation parameter t. So the new bracket will be

$$[x, y]_t = [x, y] + tc_1(x, y) + t^2 c_2(x, y) + ...,$$

where $c_i : \wedge^2\mathfrak{g} \to \mathfrak{g}$ are linear maps. This bracket should satisfy the Jacobi identity, i.e., define a new Lie algebra structure on $\mathfrak{g}[[t]]$ (over $\mathbf{k}[[t]]$). Such deformations are distinguished up to linear isomorphisms

$$a = 1 + ta_1 + t^2 a_2 + ...$$

where $a_i \in \mathrm{End}_{\mathbf{k}}(\mathfrak{g})$.

In particular, in first order, i.e., modulo t^2, we get a new Lie algebra structure on $\mathfrak{g}[t]/t^2\mathfrak{g}[t] = \mathfrak{g} \oplus t\mathfrak{g}$ such that this Lie algebra can be included in the short exact sequence

$$0 \to t\mathfrak{g} \to \mathfrak{g} \oplus t\mathfrak{g} \to \mathfrak{g} \to 0$$

where $t\mathfrak{g} \cong \mathfrak{g}$ is an abelian ideal with adjoint action of $\mathfrak{g}$ (note that this Lie algebra structure is automatically $\mathbf{k}[t]/t^2$-linear). So this is an abelian extension of $\mathfrak{g}$ by $t\mathfrak{g}$, and we know that such extensions are classified by $H^2(\mathfrak{g}, \mathfrak{g})$. So we obtain

Proposition 48.5. *First-order deformations of $\mathfrak{g}$ as a Lie algebra are classified by $H^2(\mathfrak{g}, \mathfrak{g})$.*

Thus if $H^2(\mathfrak{g}, \mathfrak{g}) = 0$, every deformation is isomorphic to the trivial one, with $c_1 = c_2 = ... = 0$. Indeed, applying automorphisms

$a = 1 + ta_1 + t^2 a_2 + ...$, we can kill successively c_1, then c_2, then c_3, and so on. Thus from Whitehead's theorem we obtain

Corollary 48.6. *If $\mathfrak{g}$ is semisimple then it is rigid, i.e., has no nontrivial Lie algebra deformations.*

Example 48.7. Let $\mathfrak{g}$ be the 2-dimensional abelian Lie algebra over $\mathbb{C}$. Then $H^2(\mathfrak{g}, \mathfrak{g}) = \mathbb{C}^2$, and we get a 2-parameter family of deformations with bracket $[x, y] = tx + sy$. These, however, turn out to be all equivalent (for $(t, s) \neq (0, 0)$) under the action of $GL_2(\mathbb{C})$: they are all isomorphic to the Lie algebra with basis x, y and commutator $[x, y] = y$.

However, not all first order deformations of a Lie algebra lift to second order, i.e., modulo t^3. Namely, the Jacobi identity in the second order tells us that $dc_2 = [c_1, c_1]$, where $[c_1, c_1]$ is the **Schouten bracket** of c_1 with itself:

$$[c_1, c_1](x, y, z) = c_1(c_1(x, y), z) + c_1(c_1(y, z), x) + c_1(c_1(z, x), y).$$

This expression is automatically a cocycle (check it!), but we need it to be a coboundary. So the cohomology class of $[c_1, c_1]$ in $H^3(\mathfrak{g}, \mathfrak{g})$ is an obstruction to lifting the deformation modulo t^3. Thus the space $H^3(\mathfrak{g}, \mathfrak{g})$ is the home for **obstructions to deformations**. For example, if $\mathfrak{g}$ is abelian then $H^2(\mathfrak{g}, \mathfrak{g}) = \mathrm{Hom}_{\mathbf{k}}(\wedge^2 \mathfrak{g}, \mathfrak{g})$, and the obstruction to extending $c = tc_1$ modulo t^3 is

$$\mathrm{Jacobi}(c_1) := [c_1, c_1] \in H^3(\mathfrak{g}, \mathfrak{g}) = \mathrm{Hom}_{\mathbf{k}}(\wedge^3 \mathfrak{g}, \mathfrak{g}).$$

6. In a similar way we can study deformations $V[[t]]$ of a module V over $\mathfrak{g}$:

$$\rho_t(x) = \rho(x) + t\rho_1(x) + t^2 \rho_2(x) + ...$$

Modulo t^2 we get a $\mathfrak{g}$-module structure on $V[t]/t^2 V[t] = V \oplus tV$ such that we have a short exact sequence

$$0 \to tV \to V \oplus tV \to V \to 0.$$

Thus first order deformations of V are classified by $\mathrm{Ext}^1_{\mathfrak{g}}(V, V) = H^1(\mathfrak{g}, \mathrm{End}_{\mathbf{k}} V)$. Again, lifting of this deformation modulo t^3 is not automatic, and we get an obstruction in $\mathrm{Ext}^2_{\mathfrak{g}}(V, V) = H^2(\mathfrak{g}, \mathrm{End}_{\mathbf{k}}(V))$.

Exercise 48.8. (i) Let $\mathfrak{a}, \mathfrak{g}$ be Lie algebras and $\phi : \mathfrak{a} \to \mathfrak{g}$ a homomorphism. Show that first order deformations of ϕ are classified by $H^1(\mathfrak{a}, \mathfrak{g})$, where $a \in \mathfrak{a}$ acts on $\mathfrak{g}$ by $\mathrm{ad}\phi(a)$.

(ii) Show that if $\mathfrak{a}$ is semisimple and $\mathfrak{g}$ finite dimensional over $\mathbb{C}$ then $H^1(\mathfrak{a}, \mathfrak{g}) = 0$.

(iii) Show that if $\mathfrak{a}, \mathfrak{g}$ are semisimple complex Lie algebras then there are only finitely many homomorphisms $\mathfrak{a} \to \mathfrak{g}$ up to conjugation by

G_{ad}. (**Hint**: Consider the affine algebraic variety $X \subset \text{Hom}_{\mathbb{C}}(\mathfrak{a}, \mathfrak{g})$ of all homomorphisms and show that the tangent space $T_\phi X$ is $Z^1(\mathfrak{a}, \mathfrak{g})$, the space of 1-cocycles. Then use (ii) to deduce that X is the union of finitely many orbits of G_{ad}.)

(iv) How many conjugacy classes do we have in (iii) if $\mathfrak{a} = \mathfrak{sl}_2$ and $\mathfrak{g} = \mathfrak{sl}_n, \mathfrak{so}_n, \mathfrak{sp}_{2n}$?

48.2. **Levi decomposition.**

Theorem 48.9. *(Levi decomposition, Theorem 16.7) Over real or complex numbers we have* $\mathfrak{g} \cong \text{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{\text{ss}}$*, where* $\mathfrak{g}_{\text{ss}} \subset \mathfrak{g}$ *is a semisimple subalgebra (but not necessarily an ideal); i.e.,* $\mathfrak{g}$ *is isomorphic to the semidirect product* $\mathfrak{g}_{\text{ss}} \ltimes \text{rad}(\mathfrak{g})$*. In other words, the projection* $p : \mathfrak{g} \to \mathfrak{g}_{\text{ss}}$ *admits an (in general, non-unique) splitting* $q : \mathfrak{g}_{\text{ss}} \to \mathfrak{g}$*, i.e., a Lie algebra map such that* $p \circ q = \text{Id}$.

Proof. We can write $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \text{rad}(\mathfrak{g})$ as a vector space. Then the commutator looks like

$$[(a, x), (b, y)] = ([x, b]-[y, a]+[a, b]+\omega(x, y), [x, y]),\ x, y \in \mathfrak{g}_{ss}, a, b \in \text{rad}(\mathfrak{g}).$$

Let $\text{rad}(\mathfrak{g}) = D^0 \supset D^1 \supset ...$ be the upper central series of $\text{rad}(\mathfrak{g})$, i.e., $D^{i+1} = [D^i, D^i]$. Suppose $D^n \neq 0$ but $D^{n+1} = 0$ (so D^n is an abelian ideal). Using induction in dimension of $\mathfrak{g}$ and replacing $\mathfrak{g}$ by $\mathfrak{g}/D^n$, we may assume that $\omega(x, y) \in D^n$. But then $\omega \in Z^2(\mathfrak{g}_{ss}, D^n)$, which equals $B^2(\mathfrak{g}_{ss}, D^n)$ by Whitehead's theorem, i.e., $\omega = d\eta$. Using η, we can modify the splitting $\mathfrak{g} = \mathfrak{g}_{ss} \oplus \text{rad}(\mathfrak{g})$ to make sure that $\omega = 0$. This implies the statement. $\square$

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