49. **The third fundamental theorem of Lie theory**

49.1. **Exponentiating nilpotent and solvable Lie algebras and the third fundamental theorem of Lie theory.** The following theorem implies the third fundamental theorem of Lie theory for solvable Lie algebras. Let $\mathfrak{g}$ be a finite dimensional solvable Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ of dimension n.

Theorem 49.1. *There is a simply connected Lie group G over $\mathbb{K}$ with* $\mathrm{Lie}(G) = \mathfrak{g}$*, diffeomorphic to $\mathbb{K}^n$. Moreover, if $\mathfrak{g}$ is nilpotent then the exponential map* $\exp : \mathfrak{g} \to G$ *is a diffeomorphism, and if we use it to identify G with $\mathfrak{g}$ then the multiplication map $\mu : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is polynomial.*

Proof. The proof is by induction in n, with trivial base $n = 0$. Namely, fix a nonzero homomorphism $\chi : \mathfrak{g} \to \mathbb{K}$ (which exists since $\mathfrak{g}$ is solvable), and let $\mathfrak{g}_0 = \mathrm{Ker}\chi$. Then we have $\mathfrak{g} = \mathbb{K}\mathbf{d} \ltimes \mathfrak{g}_0$, the semidirect product, where $\mathbf{d} \in \mathfrak{g}$ acts as a derivation d on $\mathfrak{g}_0$. Let G_0 be the simply connected Lie group corresponding to $\mathfrak{g}_0$, which is defined by the induction assumption. So we have a 1-parameter group of automorphisms $e^{td} : \mathfrak{g}_0 \to \mathfrak{g}_0$ which by the second fundamental theorem of Lie theory gives rise to a 1-parameter group of automorphisms $e^{td} : G_0 \to G_0$. Thus we can define a group structure on $G := G_0 \times \mathbb{K}$ by the formula

$$(x,t) \cdot (y,s) = (x \cdot e^{td}(y), t+s),\ \ x,y \in G_0,\ \ t,s \in \mathbb{K}.$$

Otherwise formulated, $G = \mathbb{K} \ltimes G_0$. This gives a desired group G with Lie algebra $\mathfrak{g}$.

Moreover, if $\mathfrak{g}$ is nilpotent then by the induction assumption the exponential map $\mathfrak{g}_0 \to G_0$ is a diffeomorphism, and if we use it to identify $\mathfrak{g}_0$ with G_0 then the multiplication $\mu_0 : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$ is polynomial. So we may realize G as $\mathfrak{g} = \mathfrak{g}_0 \times \mathbb{K}$ with multiplication law

$$(X,t)*(Y,s) = \mu((X,t),(Y,s)) = (\mu_0(X, e^{td}(Y)), t+s),\ X,Y \in \mathfrak{g}_0,\ t,s \in \mathbb{K}.$$

By nilpotency $d^N = 0$ for some N, so

$$e^{td}(Y) = \sum_{n=0}^{N-1} \frac{t^n d^n(Y)}{n!},$$

so we see that μ is polynomial. Also

$$\exp(X,t) = (\exp(X_t), t),$$

where

$$X_t = \frac{e^{td}-1}{td}(X) = \sum_{n\ge 1}^{N} \frac{t^{n-1}d^{n-1}(X)}{n!}.$$

Thus

$$X = \left(\sum_{n=1}^{N} \frac{t^{n-1}d^{n-1}}{n!}\right)^{-1} (X_t),$$

which makes sense since $d^N = 0$. This implies that the exponential map for $\mathfrak{g}$ is a diffeomorphism. $\square$

Example 49.2. Let $\mathfrak{g}$ be the Heisenberg Lie algebra, i.e. the Lie algebra of strictly upper triangular 3-by-3 matrices. Then under such identification the multiplication map in the corresponding Heisenberg group G has the form

$$(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + \tfrac{1}{2}(xy' - x'y)).$$

Exercise 49.3. Show that if $\mathfrak{g}$ is the 2-dimensional non-abelian complex Lie algebra and G the corresponding simply connected Lie group then $\exp : \mathfrak{g} \to G$ is not injective.

Definition 49.4. The simply connected Lie group whose Lie algebra is nilpotent is called **unipotent**.[30]

Corollary 49.5. *(Third fundamental theorem of Lie theory, Theorem 9.13) For any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$ there is a simply connected Lie group G with* $\mathrm{Lie}(G) = \mathfrak{g}$.

Proof. By Theorem 49.1, we have such a group A for $\mathfrak{a} = \mathrm{rad}(\mathfrak{g})$. Moreover, by the Levi decomposition theorem, the simply connected semisimple group G_{ss} corresponding to $\mathfrak{g}_{ss}$ acts on $\mathrm{rad}(\mathfrak{g})$. Hence by the second fundamental theorem of Lie theory, G_{ss} acts on A, and the simply connected Lie group $G_{ss} \ltimes A$ has the Lie algebra $\mathfrak{g}_{ss} \ltimes \mathrm{rad}(\mathfrak{g}) = \mathfrak{g}$. $\square$

Corollary 49.6. *A simply connected complex Lie group G is of the form $G_{ss} \ltimes A$, where A is solvable simply connected, hence diffeomorphic to $\mathbb{C}^n$, and G_{ss} is a simply connected semisimple complex Lie group. Thus G has the homotopy type of G_{ss}^c.*

49.2. **Formal groups.** The third fundamental theorem of Lie theory assigns a simply connected Lie group G to any finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$, such that $\mathrm{Lie}G = \mathfrak{g}$. But what about infinite dimensional Lie algebras? There are some examples when this is possible, for instance for $\mathfrak{g} = \mathrm{Vect}(M)$, the Lie algebra of vector fields for a smooth manifold M, we can take G to be the universal cover of

[30] The reason for this terminology is that these groups act by unipotent operators on the adjoint representation.

$\mathrm{Diff}_0(M)$, the group of diffeomorphisms of M homotopic to the identity, and for $\mathfrak{g} = C^\infty(S^1, \mathfrak{k})$ for a finite dimensional Lie algebra $\mathfrak{k}$ we can take $G = C^\infty(S^1, K)$, where K is the simply connected Lie group corresponding to $\mathfrak{k}$ (although we would need to explain in what sense G is a Lie group and $\mathrm{Lie}G = \mathfrak{g}$). However, for a general infinite dimensional $\mathfrak{g}$, such an assignment is typically impossible and a suitable group G does not exist.

However, this assignment becomes possible (and in fact not just over $\mathbb{R}$ and $\mathbb{C}$ but over any field of characteristic zero) if we replace the notion of a Lie group with a purely algebraic notion of a **formal group**. Roughly speaking, the notion of a formal group is the analog of the notion of a real or complex analytic Lie group where analytic functions are replaced by formal power series, and we don't worry about their convergence. This allows us to work with infinite dimensional Lie algebras and over arbitrary fields of characteristic zero.

Let us give a precise definition. Given a vector space V over a field $\mathbf{k}$ of characteristic zero, define the algebra $\mathbf{k}[[V]]$ of **formal regular functions** on V to be $(SV)^*$, the dual of the symmetric algebra of V. Since SV has a bialgebra structure $\Delta_0 : SV \to SV \otimes SV$ defined by $\Delta_0(v) = v \otimes 1 + 1 \otimes v$ for $v \in V$, the dual map Δ_0^* gives a commutative associative product on $\mathbf{k}[[V]]$, which is continuous in the weak topology of the dual space. If $x_i, i \in I$ is a linear coordinate system on V corresponding to a basis $v_i, i \in I$, then we have a natural identification $\mathbf{k}[[V]] \cong \mathbf{k}[[x_i, i \in I]]$ of $\mathbf{k}[[V]]$ with the algebra of formal power series in x_i. Note that here I can be a set of any cardinality, not necessarily finite or countable. Moreover, if $\dim V < \infty$ then $\mathbf{k}[[V]] = \prod_{n \geq 0} S^n V^*$.

Finally, note that we have the augmentation homomorphism (counit) $\varepsilon : \mathbf{k}[[V]] \to \mathbf{k}$ given by $\varepsilon(f) = f(0)$, i.e., obtained by taking the quotient by the maximal ideal $\mathfrak{m} \subset \mathbf{k}[[V]]$.

Definition 49.7. A **formal group structure** on V is a (topological) coproduct $\Delta : \mathbf{k}[[V]] \to \mathbf{k}[[V \oplus V]]$, i.e., a continuous[31] homomorphism which is coassociative and compatible with the counit:

$$(\Delta \otimes \mathrm{Id}) \circ \Delta(f) = (\mathrm{Id} \otimes \Delta) \circ \Delta(f),\ (\varepsilon \otimes \mathrm{Id}) \circ \Delta(f) = (\mathrm{Id} \otimes \varepsilon) \circ \Delta(f) = f.$$

A **formal group** over $\mathbf{k}$ is a pair $G = (V, \Delta)$. We will denote $\mathbf{k}[[V]]$ by $\mathcal{O}(G)$ and call it the **algebra of regular functions** on G. We define the **dimension** of G by $\dim G := \dim V$.

[31]Note that if $\dim V < \infty$, any such homomorphism is automatically continuous.

A **(homo)morphism of formal groups** $\phi : G_1 \to G_2$ is a (continuous) algebra homomorphism $\mathcal{O}(G_2) \to \mathcal{O}(G_1)$ preserving the coproduct.[32]

For example, a 1-dimensional formal group is defined by a power series $F(x, y) \in \mathbf{k}[[x, y]]$, $F(x, y) = x + y + ...$, where ... denotes quadratic and higher terms, which is associative:

$$F(F(x, y), z) = F(x, F(y, z)).$$

Such a series F is called a **formal group law**. Namely, the map $\Delta : \mathbf{k}[[x]] \to \mathbf{k}[[x_1, x_2]]$ is defined by the formula

$$\Delta(f)(x_1, x_2) = f(F(x_1, x_2)).$$

Higher-dimensional formal groups G can also be presented in this way, with F, x, y being vectors with $\dim G$ entries rather than scalars.

Example 49.8. 1. The **additive formal group**: $\Delta(f) = f \otimes 1 + 1 \otimes f$, $f \in V^*$. In other words, $F(x, y) = x + y$ and $\Delta(f)(x, y) = f(x + y)$.

2. Let G be a real or complex Lie group. Then the multiplication map $G \times G \to G$ is smooth. So we can take its Taylor expansion at the unit element, which defines a formal group G_{formal} called the **formal completion of G at the identity.** Its coproduct is defined by the formula $\Delta(f)(x, y) = f(x \circ y)$ where $(x, y) \mapsto x \circ y$ denotes the group law of G. The same construction is valid for an algebraic group over any field.

So what does it have to do with groups? In fact, a lot: if G is a formal group then it defines a functor from the category of local commutative finite dimensional $\mathbf{k}$-algebras to the category of groups,

$$R \mapsto G(R) = \mathrm{Hom}_{\text{continuous}}(\mathcal{O}(G), R)$$

(where the topology on R is discrete).[33] Namely, the group law on such homomorphisms is defined by

$$(a \circ b)(f) = (a \otimes b)(\Delta(f)).$$

This makes sense even though $\Delta(f)$ does not belong to $\mathbf{k}[[V]] \otimes \mathbf{k}[[V]]$ but only to its completion $\mathbf{k}[[V \oplus V]]$ since R is finite dimensional.

Exercise 49.9. Show that $(G(R), \circ)$ is a group.

[32] Thus we forget the linear structure on V (it does not have to be preserved by homomorphisms). In other words, to specify a formal group, we don't need to specify a vector space V but only need to specify a (topological) ring isomorphic to $\mathbf{k}[[V]]$ for some V and equipped with a coproduct.

[33] Again, continuity is automatic if $\dim G < \infty$.

Moreover, any (homo)morphism of formal groups $G_1 \to G_2$ defines a morphism of functors $G_1(?) \to G_2(?)$, and this assignment is compatible with composition. Furthermore, it is not hard to show that this assignment can be inverted, which allows us to define formal groups as representable functors from local finite dimensional commutative algebras to groups.

Any formal group G defines a Lie algebra Lie$G$, which as a vector space is the continuous dual $\mathfrak{g} := (\mathfrak{m}/\mathfrak{m}^2)^*$. In other words, it is the underlying vector space V of $G = (V, \Delta)$. Note that by compatibility of Δ with ε, for $f \in \mathfrak{m}$ the element $\Delta(f) - f \otimes 1 - 1 \otimes f$ belongs to the completed tensor product $\mathfrak{m}\widehat{\otimes}\mathfrak{m}$, thus projects to a well defined element of $(\mathfrak{g}\otimes\mathfrak{g})^*$. Thus the same is true for the element $\Delta(f)-\Delta^{\rm op}(f)$ (where $\Delta^{\rm op}$ is obtained from Δ by swapping components); in fact, it defines an element of $(\wedge^2\mathfrak{g})^*$. Moreover, this element only depends on the residue $\overline{f}$ of f in $\mathfrak{g}^* = \mathfrak{m}/\mathfrak{m}^2$ (check it!). Denote the projection of $\Delta(f) - \Delta^{\rm op}(f)$ to $(\wedge^2\mathfrak{g})^*$ by $\delta(\overline{f})$. Then $\delta : \mathfrak{g}^* \to (\wedge^2\mathfrak{g})^*$ is continuous, so it is dual to the map $[,] = \delta^* : \wedge^2\mathfrak{g} \to \mathfrak{g}$, and it is easy to show that $[,]$ is a Lie bracket on $\mathfrak{g}$; namely, the Jacobi identity follows from the coassociativity of Δ (check it!).

Conversely, given a Lie algebra $\mathfrak{g}$ over $\mathbf{k}$ (not necessarily finite dimensional), we can use the Baker-Campbell-Hausdorff formula (Subsection 14.3) to assign a formal group to $\mathfrak{g}$. Namely, take $V = \mathfrak{g}$ and define $\Delta : \mathbf{k}[[\mathfrak{g}]] \to \mathbf{k}[[\mathfrak{g} \oplus \mathfrak{g}]]$ by

$$\Delta(f)(x, y) = f(\mu(x, y)),$$

where $\mu(x, y) = x + y + \frac{1}{2}[x, y] + ...$ is the Baker-Campbell-Hausdorff series. Then the coassociativity of Δ follows from the associativity of μ. In other words, we define G by setting its formal group law F to be equal to μ.

Example 49.10. Let $\mathfrak{g}$ be a Lie algebra and G be the corresponding formal group. Let R be a finite dimensional local commutative algebra with maximal ideal $\mathfrak{m}_R$. Then $G(R) = \mathfrak{m}_R \otimes \mathfrak{g}$ with group law

$$(x, y) \mapsto \mu(x, y)$$

(which makes sense since the series terminates).

Theorem 49.11. *(The fundamental theorems of Lie theory for formal groups) These assignments are mutually inverse equivalences between the category of formal groups over* $\mathbf{k}$ *and the category of Lie algebras over* $\mathbf{k}$.

Proof. The proof is analogous to the proof of the first two fundamental theorems for usual Lie groups (but without the analytic details), and we leave it as an exercise. Note that the third theorem, which was the hardest for usual Lie groups, assigning a group to a Lie algebra, has already been proved above by using the series $\mu(x,y)$. $\square$

Corollary 49.12. *Every 1-dimensional formal group G over a field of characteristic zero is isomorphic to the additive formal group, with $\Delta(f)(x,y) = f(x+y)$.*

Over a field of positive characteristic (or over a commutative ring, such as $\mathbb{Z}$), much, but not all, of this story extends; let us for simplicity consider the finite dimensional case over a field. Namely, the definition of a formal group structure (say, on a finite dimensional space) is the same: it's a coproduct on $\mathbf{k}[[x_1, ..., x_n]]$ with the same properties as above.[34] The definition of the Lie algebra of a formal group also goes along for the ride. However, the reverse assignment fails, since the series $\mu(x,y)$ is only defined over $\mathbb{Q}$ and has all primes occurring in denominators of its coefficients. As a result, not any Lie algebra gives rise to a formal group, and the fundamental theorems of Lie theory for formal groups don't hold.

In particular, there are many non-isomorphic 1-dimensional formal groups. For example, we have the additive group law $F(x,y) = x+y$ as above, but also the **multiplicative group law** $F(x,y) = x+y+xy$, which is called so because this means that $1+F(x,y) = (1+x)(1+y)$. In characteristic zero these are isomorphic by the map

$$x \mapsto e^x - 1 = \sum_{n\geq 1} \frac{x^n}{n!},$$

(not surprisingly in view of Corollary 49.12), but in positive characteristic this series does not make sense and in fact the additive and multiplicative formal groups are not isomorphic (check it!). There are also many other 1-dimensional formal group laws, commutative and not. Such (commutative) formal group laws are very important in algebraic topology, since they parametrize cohomology theories. For example, the additive group law corresponds to ordinary cohomology and the multiplicative one to K-theory. In characteristic zero the isomorphism between the additive and multiplicative formal groups leads to the **Chern character map** which identifies cohomology and K-theory of a topological space with $\mathbb{Q}$-coefficients.

[34]More precisely, instead of SV we should take the **symmetric algebra with divided powers** ΓV, defined by $\Gamma^m V := (S^m V^*)^*$. Note that in characteristic p, $\Gamma^m V$ is not naturally isomorphic to $S^m V$ for $m \geq p$.

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