50. **Ado's theorem**

50.1. **The nilradical.** Consider now a solvable Lie algebra $\mathfrak{a}$ over $\mathbb{C}$ and its adjoint representation. By Lie's theorem, in some basis $\mathfrak{a}$ acts in this representation by upper triangular matrices. Let $\mathfrak{n} \subset \mathfrak{a}$ be the subset of nilpotent elements (the **nilradical** of $\mathfrak{a}$). Thus $\mathfrak{n}$ is the set of $x \in \mathfrak{a}$ that act in this basis by strictly upper triangular matrices. In particular, $\mathfrak{n} \supset [\mathfrak{a}, \mathfrak{a}]$, so $\mathfrak{a}/\mathfrak{n}$ is abelian. Moreover, the diagonal entries of these upper triangular matrices gives rise to characters $\lambda_i \in (\mathfrak{a}/\mathfrak{n})^*$ which constitute a composition series of the adjoint representation of $\mathfrak{a}$. By the definition of $\mathfrak{n}$, the characters λ_i form a spanning set in $(\mathfrak{a}/\mathfrak{n})^*$.

Proposition 50.1. *If $d : \mathfrak{a} \to \mathfrak{a}$ is a derivation then $d(\mathfrak{a}) \subset \mathfrak{n}$. Thus if $\mathfrak{a} = \mathrm{rad}(\mathfrak{g})$ is the radical of $\mathfrak{g}$ then $\mathfrak{g}$ acts trivially on $\mathfrak{a}/\mathfrak{n}$.*

Proof. Since there are finitely many characters λ_i in the composition series of $\mathfrak{a}$, for each of them we have $e^{td}\lambda_i = \lambda_i$. It follows that d acts on $\mathfrak{a}/\mathfrak{n}$ trivially. □

50.2. **Algebraic Lie algebras.** Let us say that a finite dimensional complex Lie algebra $\mathfrak{g}$ is **algebraic** if $\mathfrak{g}$ is the Lie algebra of a group $G = K \ltimes N$, where K is a reductive group and N a unipotent group. It turns out that this is equivalent to being the Lie algebra of an **affine algebraic group** over $\mathbb{C}$ (i.e., a closed subgroup in $GL_n(\mathbb{C})$ defined by polynomial equations), which motivates the terminology.

A finite dimensional complex Lie algebra need not be algebraic:

Example 50.2. Let $\mathfrak{g}_1$ be a 3-dimensional Lie algebra with basis d, x, y and $[x, y] = 0$, $[d, x] = x$, $[d, y] = \sqrt{2}y$. Similarly, let $\mathfrak{g}_2$ have basis d, x, y with $[x, y] = 0$, $[d, x] = x$, $[d, y] = y + x$. Then $\mathfrak{g}_1, \mathfrak{g}_2$ are not algebraic (check it!).

Nevertheless, we have the following proposition.

Proposition 50.3. *Any finite dimensional complex Lie algebra is a Lie subalgebra of an algebraic one.*

Proof. Let us say that $\mathfrak{g}$ is n**-algebraic** if it is the Lie algebra of a group $G := K \ltimes A$, where K is reductive and $\mathfrak{a} = \mathrm{Lie}(A)$ is solvable with $\dim(\mathfrak{a}/\mathfrak{n}) \le n$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{a}$. Thus 0-algebraic is the same as algebraic. Note that for any $\mathfrak{g}$ we have the Levi decomposition $\mathfrak{g} = \mathfrak{g}_{ss} \ltimes \mathfrak{a}$, where $\mathfrak{a} = \mathrm{rad}(\mathfrak{g})$, which shows that any $\mathfrak{g}$ is n-algebraic for some n. So it suffices to show that any n-algebraic Lie algebra for $n > 0$ embeds into an $n-1$-algebraic one.

To this end, let $\mathfrak{g} = \mathrm{Lie}(G)$ be n-algebraic, with $G = K \ltimes A$ and A simply connected. Let $\mathfrak{a} = \mathrm{Lie}(A)$, so $\dim(\mathfrak{a}/\mathfrak{n}) = n$. Pick $d \in \mathfrak{a}$, $d \notin \mathfrak{n}$

such that d is K-invariant. This can be done since K acts trivially on $\mathfrak{a}/\mathfrak{n}$ and its representations are completely reducible. We have a decomposition $\mathfrak{a} = \oplus_{i=1}^r \mathfrak{a}[\beta_i]$ of $\mathfrak{a}$ into generalized eigenspaces of d. It is clear that K preserves each $\mathfrak{a}[\beta_i]$. Pick a character $\chi : \mathfrak{a} \to \mathbb{C}$ such that $\chi(d) = 1$.

Consider the subgroup Γ of $\mathbb{C}$ generated by β_i and let $\alpha_1, ..., \alpha_m$ be a basis of Γ, so that $\beta_i = \sum_j b_{ij}\alpha_j$ for $b_{ij} \in \mathbb{Z}$. Let $T = (\mathbb{C}^\times)^m$ and make T act on G so that it commutes with K and acts on $\mathfrak{a}[\beta_i]$ by $(z_1, ..., z_m) \mapsto \prod_j z_j^{b_{ij}}$. Now consider the group $\widetilde{G} := (K \times T) \ltimes A$. Let $\mathfrak{a}' \subset \mathrm{Lie}(T) \ltimes \mathfrak{a} \subset \mathrm{Lie}(\widetilde{G})$ be spanned by $\mathrm{Ker}\chi$ and $d - \alpha$ where $\alpha = (\alpha_1, ..., \alpha_m) \in \mathrm{Lie}(T)$. Then the nilradical $\mathfrak{n}'$ of $\mathfrak{a}'$ is spanned by $\mathfrak{n}$ and $d - \alpha$ (as the latter is nilpotent). Moreover, if A' is the simply connected group corresponding to $\mathfrak{a}'$, then $(K \times T) \ltimes A \cong (K \ltimes T) \ltimes A'$ Thus, the Lie algebra $\widetilde{\mathfrak{g}} := \mathrm{Lie}(\widetilde{G})$ is $n-1$-algebraic (as $\dim(\mathfrak{a}'/\mathfrak{n}') = n - 1$), and it contains $\mathfrak{g}$, as claimed. $\square$

Example 50.4. The Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ in the Example 50.2 are 1-algebraic.

To embed $\mathfrak{g}_1$ into an algebraic Lie algebra, add element δ with $[\delta, x] = 0$, $[\delta, y] = y$, $[\delta, d] = 0$. Then the Lie algebra $\mathfrak{g}_1'$ spanned by δ, d, x, y is $\mathfrak{b} \oplus \mathfrak{b}$, where $\mathfrak{b}$ is the non-abelian 2-dimensional Lie algebra (so it is algebraic). Namely, the first copy of $\mathfrak{b}$ is spanned by δ, y and the second by $d - \sqrt{2}\delta, x$.

To embed $\mathfrak{g}_2$ into an algebraic Lie algebra, add element δ with $[\delta, x] = 0$, $[\delta, y] = x$, $[\delta, d] = 0$. Then the Lie algebra $\mathfrak{g}_2'$ spanned by δ, d, x, y is $\mathbb{C} \ltimes \mathcal{H}$, where $\mathcal{H}$ is the 3-dimensional Heisenberg Lie algebra with basis δ, x, y, and $\mathbb{C}$ is spanned by $d - \delta$ (so it is algebraic, as $d - \delta$ acts diagonalizably with integer eigenvalues).

50.3. **Faithful representations of nilpotent Lie algebras.** Let $\mathfrak{n}$ be a finite dimensional nilpotent Lie algebra over $\mathbb{C}$. In this subsection we will show that $\mathfrak{n}$ has a finite dimensional faithful representation.

To this end, recall that by Theorem 49.1, $\mathfrak{n} = \mathrm{Lie}(N)$ where N is a simply connected Lie group, and the exponential map $\exp : \mathfrak{n} \to N$ is bijective. Moreover, the multiplication law of N, when rewritten on $\mathfrak{n}$ using the exponential map, is given by polynomials.

Proposition 50.5. *Let $\mathcal{O}(N)$ be the space of polynomial functions on $N \cong \mathfrak{n}$ (identified using the exponential map). Then $\mathcal{O}(N)$ is invariant under the action of $\mathfrak{n}$ by left-invariant vector fields. Moreover, we have a canonical filtration $\mathcal{O}(N) = \cup_{n\geq 1} V_n$, where $V_n \subset \mathcal{O}(N)$ are finite dimensional subspaces such that $V_1 \subset V_2 \subset ...$ and $\mathfrak{n}V_n \subset V_{n-1}$.*

Proof. Let $\mu : \mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$ be the polynomial multiplication law. Let $x \in \mathfrak{n}$ and L_x be the corresponding left-invariant vector field. Let $f \in \mathcal{O}(N) = S\mathfrak{n}^*$. Then for $y \in \mathfrak{n}$ we have

$$(L_x f)(y) = \frac{d}{dt}|_{t=0} f(\mu(y, tx)).$$

Since f and μ are polynomials, this is clearly a polynomial in y. Thus $L_x : \mathcal{O}(N) \to \mathcal{O}(N)$.

We have a lower central series filtration on $\mathfrak{n}$:

$$\mathfrak{n} = D_0(\mathfrak{n}) \supset [\mathfrak{n}, \mathfrak{n}] = D_1(\mathfrak{n}) \supset ... \supset D_m(\mathfrak{n}) = 0.$$

This gives an ascending filtration

$$0 = D_0(\mathfrak{n})^\perp \subset .... \subset D_m(\mathfrak{n})^\perp = \mathfrak{n}^*.$$

We assign to $D_j(\mathfrak{n})^\perp$ filtration degree d^j, where d is a sufficiently large positive integer. This gives rise to an ascending filtration $F^\bullet$ on $S\mathfrak{n}^* = \mathcal{O}(N)$. Note that

$$\mu(x, y) = x + y + \sum_{i \geq 1} Q_i(x, y),$$

where $Q_i : \mathfrak{n} \times \mathfrak{n} \to [\mathfrak{n}, \mathfrak{n}]$ has degree i in x. Thus

$$(L_x f)(y) = (\partial_x f)(y) + (\partial_{Q_1(x,y)} f)(y).$$

The first term clearly lowers the degree, and so does the second one if d is large enough. So we may take $V_n = F_n(S\mathfrak{n}^*)$ to be the space of polynomials of degree $\leq n$, then $L_x V_n \subset V_{n-1}$, as claimed. $\square$

Example 50.6. We illustrate this proof on the example of the Heisenberg algebra $\mathcal{H} = \langle x, y, c \rangle$ with $[x, y] = c$ and $[x, c] = [y, c] = 0$. In this case

$$e^{tx} e^{sy} = e^{tx + sy + \frac{1}{2}tsc},$$

so writing $u = px + qy + rc \in \mathcal{H}$, we get

$$\mu((p_1, q_1, r_1), (p_2, q_2, r_2)) = (p_1 + p_2, q_1 + q_2, r_1 + r_2 + \tfrac{1}{2}(p_1 q_2 - p_2 q_1)).$$

Thus

$$L_c = \partial_r,\ L_x = \partial_p - \tfrac{1}{2} q \partial_r,\ L_y = \partial_q + \tfrac{1}{2} p \partial_r.$$

We have $D_1(\mathcal{H}) = \mathbb{C}c$, so $D_1(\mathcal{H})^\perp$ is spanned by p, q. Thus we have $\deg(p) = \deg(q) = d$, $\deg(r) = d^2$. So for any $d > 1$, L_c, L_x, L_y lower the degree. So setting $V_n = F_n(S\mathcal{H}^*)$ to be the (finite dimensional) space of polynomials of degree $\leq n$, we see that L_c, L_x, L_y map V_n to V_{n-1}.

Corollary 50.7. *Every finite dimensional nilpotent Lie algebra $\mathfrak{n}$ over $\mathbb{C}$ has a faithful finite dimensional representation where all its elements act by nilpotent operators. Thus $\mathfrak{n}$ is isomorphic to a subalgebra of the Lie algebra of strictly upper triangular matrices of some size.*

Proof. By definition, $\mathcal{O}(N)$ is a faithful $\mathfrak{n}$-module. Hence so is V_n for some n. $\square$

50.4. **Faithful representations of general finite dimensional Lie algebras.**

Theorem 50.8. *(Ado's theorem) Every finite dimensional Lie algebra over $\mathbb{C}$ has a finite dimensional faithful representation.*

Proof. Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. By Proposition 50.3, $\mathfrak{g}$ can be embedded into an algebraic Lie algebra, so we may assume without loss of generality that $\mathfrak{g}$ is algebraic. Thus $\mathfrak{g} = \mathrm{Lie}(G)$ where $G = K \ltimes N$ for reductive K and unipotent N. Also we may assume that $\mathfrak{g} \neq \mathfrak{g}' \oplus \mathfrak{g}''$ for $\mathfrak{g}', \mathfrak{g}'' \neq 0$, otherwise the problem reduces to a smaller algebraic Lie algebra (indeed if V', V'' are faithful representations of $\mathfrak{g}', \mathfrak{g}''$ then $V' \oplus V''$ is a faithful representation of $\mathfrak{g}' \oplus \mathfrak{g}''$). Then $\mathfrak{k} = \mathrm{Lie}(K)$ acts faithfully on $\mathfrak{n} = \mathrm{Lie}(N)$. Now, $\mathfrak{g}$ acts on $\mathcal{O}(N)$ preserving the subspaces V_n ($\mathfrak{n} = \mathrm{Lie}(N)$ acts by left invariant vector fields and $\mathfrak{k}$ by the adjoint action).

As we have shown in the proof of Corollary 50.7, $\mathfrak{n}$ acts faithfully on V_n for some n. We claim that this V_n is, in fact, a faithful representation of the whole $\mathfrak{g}$, which implies the theorem. Indeed, let $\mathfrak{a} \subset \mathfrak{g}$ be the ideal of elements acting by zero on V_n, and let $\overline{\mathfrak{a}}$ be the projection of $\mathfrak{a}$ to $\mathfrak{k}$ (an ideal in $\mathfrak{k}$). Since $\mathfrak{n}$ acts faithfully on V_n, we have $\mathfrak{a} \cap \mathfrak{n} = 0$. Given $a \in \mathfrak{a}$, we have $a = \overline{a} + b$ where $\overline{a} \in \overline{\mathfrak{a}}$ is the projection of a and $b \in \mathfrak{n}$. For $x \in \mathfrak{n}$ we have $[a, x] \in \mathfrak{a} \cap \mathfrak{n} = 0$. Thus $[\overline{a}, x] = -[b, x]$. Hence the operator $x \mapsto [\overline{a}, x]$ on $\mathfrak{n}$ is nilpotent. So $\overline{\mathfrak{a}}$ acts on $\mathfrak{n}$ by nilpotent operators. Since K is reductive and $\overline{\mathfrak{a}} \subset \mathfrak{k}$ is an ideal, this means that $\overline{\mathfrak{a}}$ acts on $\mathfrak{n}$ by zero. Thus $\overline{\mathfrak{a}} = 0$ and $\mathfrak{a} \subset \mathfrak{n}$. Hence $\mathfrak{a} = 0$. $\square$

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