51. Borel subgroups and the flag manifold of a complex reductive Lie group

51.1. Borel subgroups and subalgebras. Let $G$ be a connected complex reductive Lie group, $\mathfrak{g} = \text{Lie}(G)$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with a system of simple positive roots $\Pi$, and consider the corresponding triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_+$ is spanned by positive root elements and $\mathfrak{n}_-$ by negative root elements. Let $H$ be the maximal torus in $G$ corresponding to $\mathfrak{h}$, $N_+$ the unipotent subgroup of $G$ corresponding to $\mathfrak{n}_+$, and $B_+ = HN_+$ the solvable subgroup with $\text{Lie}(B_+) = \mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$; these are all closed Lie subgroups.

Definition 51.1. A Borel subalgebra of $\mathfrak{g}$ is a Lie subalgebra conjugate to $\mathfrak{b}_+$. A Borel subgroup of $G$ is a Lie subgroup conjugate to $B_+$.

Since all pairs $(\mathfrak{h}, \Pi)$ are conjugate, this definition does not depend on the choice of $(\mathfrak{h}, \Pi)$.

Lemma 51.2. $B_+$ is its own normalizer in $G$.

Proof. Let $\gamma \in G$ be such that $\text{Ad} \gamma(B_+) = B_+$. Let $H' = \text{Ad} \gamma(H) \subset B_+$. It is easy to show that we can conjugate $H'$ back into $H$ inside $B_+$, so we may assume without loss of generality that $H' = H$. Then $\gamma \in N(H)$, and it preserves positive roots. Hence the image of $\gamma$ in $W$ is 1, so $\gamma \in H \subset B_+$, as claimed. $\square$

51.2. The flag manifold of a connected complex reductive group. Thus the set of all Borel subalgebras (or subgroups) in $G$ is the homogeneous space $G/B_+$, a complex manifold. It is called the flag manifold of $G$. Note that it only depends on the semisimple part $\mathfrak{g}_{ss} \subset \mathfrak{g}$ and does not depend on the choice of the Cartan subalgebra and triangular decomposition.

Let $G^c \subset G$ be the compact form of $G$, with Lie algebra $\mathfrak{g}^c \subset \mathfrak{g}$. It is easy to see that $\mathfrak{g}^c + \mathfrak{b}_+ = \mathfrak{g}$. Thus the $G^c$-orbit $G^c \cdot 1$ of $1 \in G/B_+$ contains a neighborhood of $1$ in $G/B_+$. Hence the same holds for any point of this orbit, i.e., $G^c \cdot 1 \subset G/B_+$ is an open subset. But it is also compact, since $G^c$ is compact, hence closed. As $G/B_+$ is connected, we get that $G^c \cdot 1 = G/B_+$, i.e., $G^c$ acts transitively on $G/B_+$.

Also the Cartan involution $\omega$ maps positive root elements to negative ones, so $G^c \cap B_+ \subset w_0(B_+) \cap B_+ = H$. Thus $G^c \cap B_+ = H$, a maximal torus in $G^c$. So we get

Proposition 51.3. We have $G/B_+ = G^c/H^c$. In particular, $G/B_+$ is a compact complex manifold of dimension $|R_+| = \frac{1}{2}(\dim \mathfrak{g} - \text{rank} \mathfrak{g})$. 274
Example 51.4. 1. For $G = SL_2$ we have $G/B_+ = SU(2)/U(1) = S^2$, the Riemann sphere.

2. For $G = GL_n$ we have $G/B_+ = U(n)/U(1)^n = \mathcal{F}_n$, the set of flags in $\mathbb{C}^n$ that we considered in Subsection 47.3.

Another realization of the flag manifold is one as the $G$-orbit of the line spanned by the highest weight vector in an irreducible representation with a regular highest weight. Namely, let $\lambda \in P_+$ be a dominant integral weight with $\lambda(h_i) \geq 1$ for all $i$ (i.e., $\lambda = \mu + \rho$ for $\mu \in P_+$). Let $L_\lambda$ be the corresponding irreducible representation with highest weight vector $v_\lambda$. We have $b_+ \cdot C_{v_\lambda} = C_{v_\lambda}$, but $e^{-\alpha}v_\lambda \neq 0$ for any $\alpha \in R_+$ (as $e_\alpha e^{-\alpha}v_\lambda = h_\alpha v_\lambda = (\lambda, \alpha^\vee)v_\lambda$, and $(\lambda, \alpha^\vee) > 0$). Moreover, these vectors have different weights, so are linearly independent. Thus $b_+$ is the stabilizer of $C_{v_\lambda}$ in $g$. Hence any $g \in G$ which preserves $C_{v_\lambda}$ belongs to the normalizer of $b_+$ (or, equivalently, $B_+$), i.e., $g \in B_+$. Thus $\mathcal{O} := G \cdot C_{v_\lambda} \subset \mathbb{P}L_\lambda$ is identified with $G/B_+$. This shows that $\mathcal{O}$ is compact, hence closed, i.e., $\mathcal{O} = G/B_+$ is a smooth complex projective variety.

Let $A = \exp(i\mathfrak{h})c \subset H$, $K = G^c$, $N = N_+$. Proposition 51.3 immediately implies

**Corollary 51.5. (The Iwasawa decomposition of $G$)** The multiplication map $K \times A \times N \to G$ is a diffeomorphism. In particular, we have $G = KAN$.

A similar theorem holds for real reductive groups (Theorem 51.14).

51.3. **The Borel fixed point theorem.** Let $V$ be a finite dimensional representation of a finite dimensional $\mathbb{C}$-Lie algebra $\mathfrak{a}$, and $X \subset \mathbb{P}V$ be a subset. We will say that $X$ is $\mathfrak{a}$-invariant (or fixed by $\mathfrak{a}$) if it is $\exp(\mathfrak{a})$-invariant.

**Theorem 51.6.** Let $\mathfrak{a}$ be a solvable Lie algebra over $\mathbb{C}$, $V$ a finite dimensional $\mathfrak{a}$-module. Let $X \subset \mathbb{P}V$ be a closed $\mathfrak{a}$-invariant subset. Then there exists $x \in X$ fixed by $\mathfrak{a}$.

**Proof.** The proof is by induction in $n = \dim \mathfrak{a}$. The base $n = 0$ is trivial, so we only need to justify the induction step. Since $\mathfrak{a}$ is solvable, it has an ideal $\mathfrak{a}'$ of codimension 1. By the induction assumption, $Y := X^{\mathfrak{a}'}$ (the set of $\exp(\mathfrak{a}')$-fixed points in $X$) is a nonempty closed subset of $X$, so it suffices to show that the 1-dimensional Lie algebra $\mathfrak{a}/\mathfrak{a}'$ has a fixed point on $Y$. Thus it suffices to prove the theorem for $n = 1$.

So let $\mathfrak{a}$ be 1-dimensional, spanned by $a \in \mathfrak{a}$. We can choose the normalization of $a$ so that all eigenvalues of $a$ on $V$ have different real parts. Fix $x_0 \in X$ and consider the curve $e^{ta}x_0$ for $t \in \mathbb{R}$. It is easy
to see that there exists \( x := \lim_{t \to \infty} e^{t a} x_0 \in P V \). Then \( x \in X \) as \( X \) is closed and, and \( x \) is fixed by \( a \), as desired. \( \square \)

51.4. **Parabolic and Levi subalgebras.** A Lie subalgebra \( p \supset b \) of a reductive Lie algebra \( g \) containing some Borel subalgebra \( b \subset g \) is called a **parabolic subalgebra** of \( g \). The corresponding connected Lie subgroup \( P \subset G \) is called a **parabolic subgroup**. It is easy to see that \( P \subset G \) is necessarily closed (check it!).

**Exercise 51.7.** Show that parabolic subalgebras \( p \) containing \( b_+ \) are in bijection with subsets \( S \subset \Pi \) of the set of simple roots of \( b_+ \), namely, \( p \) is sent to the set \( S_p \) of \( \alpha \in \Pi \) such that \( e_{-\alpha} \in p \), and \( S \) is sent to the Lie subalgebra \( p_S \) of \( g \) generated by \( b_+ \) and \( f_i \) where \( \alpha \) runs through positive roots for which \( e_{-\alpha} \in \). Such a subalgebra \( l \) is called a **Levi subalgebra** of \( p \), and we have \( p = l \ltimes u \), which is \( l \oplus u \) as a vector space.

Let \( P \subset G \) be a parabolic subgroup with Lie algebra \( p \). Let \( u \subset p \) be the nilpotent radical of \( p \); for instance, if \( p \supset b_+ \) then \( u \) is the Lie subalgebra spanned by \( e_{\alpha} \) such that \( e_{-\alpha} \notin p \). It is easy to see that there exists a (non-unique) Lie subalgebra \( l \subset p \) complementary to \( u \), which therefore projects isomorphically to \( p/u \); indeed, if \( p \supset b_\) then we can take \( l \) to be the Lie subalgebra spanned by \( h \) and \( e_{\alpha}, e_{-\alpha} \) where \( \alpha \) runs through positive roots for which \( e_{-\alpha} \in p \). Such a subalgebra \( l \) is called a **Levi subalgebra** of \( p \), and we have \( p = l \ltimes u \), which is \( l \oplus u \) as a vector space.

Let \( U = \exp(u) \). The quotient \( P/U \) is a reductive group with Lie algebra \( p/u \). A **Levi subgroup** of \( P \) is a subgroup \( L \) in \( P \) such that \( l := \text{Lie}(L) \) is a Levi subalgebra of \( p \); equivalently, \( L \) projects isomorphically to \( P/U \), so we have \( P = L \ltimes U \), written shortly as \( P = LU \). It is not difficult to show that all Levi subgroups of \( P \) (or, equivalently, all Levi subalgebras of \( p \)) are conjugate by the action of \( U \) (check it!).

For example, \( L \) is a maximal torus if and only if \( P \) is a Borel subgroup, and \( L = G \) if and only if \( P = G \).

**Example 51.8.** Let \( n = n_1 + \ldots + n_k \) where \( n_i \) are positive integers. Then the subgroup \( P \) of block upper triangular matrices with diagonal blocks of size \( n_1, \ldots, n_k \) is a parabolic subgroup of \( GL_n(C) \), and the subgroup \( L \) of block diagonal matrices in \( P \) is a Levi subgroup. The unipotent radical \( U \) of \( P \) is the subgroup of block upper triangular matrices with identity matrices on the diagonal.

51.5. **Maximal solvable and maximal nilpotent subalgebras.** Note that \( b_+ \) is a maximal solvable subalgebra of \( g \); indeed, any bigger parabolic subalgebra contains a negative root vector, hence the corresponding root \( \mathfrak{sl}_2 \)-subalgebra, so it is not solvable. Moreover, \( B_+ \) is a maximal solvable subgroup of \( G \); if \( P \supset B_+ \) then some element \( g \in P \)
does not normalize $\mathfrak{b}_+$, so $\operatorname{Lie}(P)$ has to be larger than $\mathfrak{b}_+$, hence not solvable. Thus any Borel subalgebra (subgroup) is a maximal solvable one. It turns out that the converse also holds.

**Proposition 51.9.** Any solvable Lie subalgebra of $\mathfrak{g}$ (respectively, connected solvable subgroup of $G$) is contained in a Borel subalgebra (subgroup).

*Proof.* Let $\mathfrak{a} \subset \mathfrak{g}$ be a solvable Lie subalgebra. By the Borel fixed point theorem, $\mathfrak{a}$ has a fixed point $b \in G/B_+$. Thus $\mathfrak{a}$ normalizes $\mathfrak{b}$. Hence $\mathfrak{a} \subset \mathfrak{b}$, as claimed. $\square$

**Corollary 51.10.** Any element of $\mathfrak{g}$ is contained in a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$.

Let us say that a Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is a **nilpotent subalgebra** if it consists of nilpotent elements. Note that this is a stronger condition than just being nilpotent as a Lie algebra; for example, a Cartan subalgebra is a nilpotent Lie algebra (since it is abelian) but it is not a nilpotent subalgebra of $\mathfrak{g}$.

**Corollary 51.11.** Any nilpotent subalgebra of $\mathfrak{g}$ is conjugate to a Lie subalgebra of $\mathfrak{n}_+$. Thus $\mathfrak{n}_+$ is a maximal nilpotent subalgebra of $\mathfrak{g}$, and any maximal nilpotent subalgebra of $\mathfrak{g}$ is conjugate to $\mathfrak{n}_+$.

*Proof.* By Proposition 51.9 there is $g \in G$ such that $\operatorname{Ad}_g \mathfrak{a} \subset \mathfrak{b}_+$, but since $\mathfrak{a}$ is nilpotent we actually have $\operatorname{Ad}_g \mathfrak{a} \subset \mathfrak{n}_+$. $\square$

A similar result holds for groups, with the same proof:

**Corollary 51.12.** Any unipotent subgroup of $G$ is conjugate to a (closed) Lie subgroup of $\mathcal{N}_+$. Thus $\mathcal{N}_+$ is a maximal unipotent subgroup of $\mathfrak{g}$, and any maximal unipotent subgroup of $\mathfrak{g}$ is conjugate to $\mathcal{N}_+$.

We also have

**Proposition 51.13.** The normalizer of $\mathfrak{n}_+$ and $\mathcal{N}_+$ in $G$ is $B_+$. Thus every maximal nilpotent subalgebra (unipotent subgroup) is contained in a unique Borel subgroup. Hence such subalgebras (subgroups) are parametrized by the flag manifold $G/B_+$.

*Proof.* Clearly $B_+$ is contained in the normalizer of $\mathcal{N}_+$, so this normalizer is a parabolic subgroup. We have seen that such a subgroup, if larger than $B_+$, must have a Lie algebra larger than $\mathfrak{b}_+$, so it must be $\mathfrak{p}_S$ for some $S \neq \emptyset$, hence contains some root $\mathfrak{sl}_2$-subalgebra. But the group corresponding to such a subalgebra does not normalize $\mathfrak{n}_+$, a contradiction. $\square$
51.6. Iwasawa decomposition of a real semisimple linear group.

Let \( G_{\theta} = K^c P_{\theta} \) be the polar decomposition of a real form of a complex semisimple group \( G \), \( g_{\theta} = \mathfrak{k}^c \oplus \mathfrak{p}_{\theta} \) the additive version, \( a \subset \mathfrak{p}_{\theta} \) a maximal abelian subspace. Let \( A = \exp(a) \subset P_{\theta} \) be the corresponding abelian subgroup of \( G_{\theta} \). Pick a generic element \( a \in a \). Let \( \mathfrak{z} = g_{\theta}^a \) be the centralizer of \( a \) in \( g_{\theta} \) and let \( n_{a,\pm} \) be the (nilpotent) Lie subalgebras of \( g_{\theta} \) spanned by eigenvectors of \( \text{ad}a \) with positive, respectively negative eigenvalues, so that \( g_{\theta} = n_{a, -} \oplus \mathfrak{z} \oplus n_{a, +} \). Let \( N_{a, \pm} = \exp(n_{a, \pm}) \).

The following theorem is a generalization of Proposition 51.5.

**Theorem 51.14.** (Iwasawa decomposition) The multiplication map \( K^c \times A \times N_{a, +} \to G_{\theta} \) is a diffeomorphism.

Theorem 51.14 is proved in the following exercise.

**Exercise 51.15.** (i) Let \( m = \mathfrak{z} \cap \mathfrak{k}^c \). Show that \( \mathfrak{z} = m \oplus a \) (use Proposition 44.11(ii)).

(ii) Given \( x \in \mathfrak{p} \), write \( x = x_- + x_0 + x_+ \), \( x_0 \in \mathfrak{z} \). Show that \( (x_\pm) = -x_\pm, (x_0) = -x_0 \). Deduce the additive Iwasawa decomposition \( g_{\theta} = \mathfrak{k}^c \oplus a \oplus n_{a, +} \) (write \( x \) as \( (x_- - x_+) + x_0 + 2x_+ \)).

(iii) Show that \( \mathfrak{z} \oplus n_{a, +} = m \oplus a \oplus n_{a, +} \) is a parabolic subalgebra in \( g_{\theta} \) with Levi subalgebra \( \mathfrak{z} \) (i.e., their complexifications are a parabolic subalgebra in \( g \) and its Levi subalgebra) and its unipotent radical is \( n_{a, +} \).

(iv) Let \( M \) be the centralizer of \( a \) in \( K^c \). Show that \( \mathbb{P} := MAN_{a, +} \) is a subgroup of \( G_{\theta} \) and \( X := G_{\theta}/\mathbb{P} \) is a compact homogeneous space.

(v) Show that \( K^c \) acts transitively on \( X \), and \( X \cong K^c/M \) as a homogeneous space for \( K^c \) (generalize the argument in Subsection 51.2). Deduce Theorem 51.14.

51.7. The Bruhat decomposition. Let \( G \) be a connected complex reductive group, \( H \subset G \) a maximal torus, \( B = B_+ \supset H \) a Borel subgroup. The Bruhat decomposition is the decomposition of \( G \) into double cosets of \( B \).

Let \( N(H) \) be the normalizer of \( H \) in \( G \) and \( W = N(H)/H \) be the Weyl group. Given \( w \in W \), let \( \tilde{w} \) be a lift of \( w \) to \( N(H) \) and consider the double coset \( B\tilde{w}B \subset G \). Since any two lifts of \( w \) differ by an element of \( H \) which is contained in \( B \), the set \( B\tilde{w}B \) does not depend on the choice of \( \tilde{w} \), so we will denote it by \( BwB \).

**Proposition 51.16.** The double cosets \( BwB, w \in W \) are disjoint.

**Proof.** Let \( w_1, w_2 \in N(H) \) be such that \( Bw_1B = Bw_2B \). Then there exist elements \( b_1, b_2 \in B \) such that \( b_1w_1 = w_2b_2 \). Let us apply this identity to a highest weight vector \( v_\lambda \) of an irreducible representation
$L_\lambda$ of $G$, where $\lambda \in P_+$ is regular. We have $w_2b_2v_\lambda = Cv_{w_2\lambda}$ for some $C \in \mathbb{C}^\times$, where $v_{w_2\lambda}$ is an extremal vector of weight $w_2\lambda$. On the other hand, $b_1w_1v_\lambda = C'b_1v_{w_1\lambda}$ for some $C' \in \mathbb{C}^\times$. Thus $Cv_{w_2\lambda} = C'b_1v_{w_1\lambda}$. But $b_1v_{w_1\lambda}$ equals $C''v_{w_1\lambda}$ plus terms of weight $> w_1\lambda$, where $C'' \in \mathbb{C}^\times$. It follows that $w_1\lambda = w_2\lambda$, hence $w_1 = w_2h$, $h \in H$. 

**Theorem 51.17.** (Bruhat decomposition) The union of the double cosets $BwB$, $w \in W$ is the entire group $G$. Thus they define a partition of $G$ into double cosets of $B$.

Theorem 51.17 can be reformulated as a classification of $B$-orbits on the flag manifold $G/B$. Namely, given $w \in W$, the set $BwB/B$ is an orbit of $B$ on $G/B$, which we will denote by $C_w$. By Theorem 51.16, $C_w$ are disjoint, and Theorem 51.17 is equivalent to

**Theorem 51.18.** (Schubert decomposition) $C_w, w \in W$ give the partition of $G/B$ into $B$-orbits.

The sets $BwB$ are called Bruhat cells and the sets $C_w$ are called Schubert cells.\(^{35}\)

Note that for type $A_{n-1}$ ($G = SL_n(\mathbb{C})$ or its quotient), we have already proved Theorem 51.18 in Subsection 47.3, where we decomposed the flag manifold $\mathcal{F}$ into Schubert cells labeled by permutations.

A proof of Theorem 51.18 can be found, for example, in the textbook [CG]. It is also sketched in the following exercise.

**Exercise 51.19.** (i) Let $B = B_+$ and $w \in W$. Consider the multiplication map $\mu_{i,w} : B s_i B \times_B C_w \to G/B$. Show that if $\ell(s_iw) = \ell(w) + 1$ then $\mu_{i,w}$ is an isomorphism onto $C_{s_iw}$, while if $\ell(s_iw) = \ell(w) - 1$ then the image of $\mu_{i,w}$ consists of $C_w$ and $C_{s_iw}$.

**Hint:** Reduce to the $SL_2$-case.

(ii) For $i \in I$ let $P_i$ be the minimal parabolic subgroup of $G$ generated by $B$ and the 1-parameter subgroup $exp(tf_i)$. Show that $P_i/B = C_{s_i} \cap C_1 \cong \mathbb{CP}^1 \subset G/B$ (where $C_1$ is a point and $C_{s_i} \cong \mathbb{C}$).

(iii) Let $w = s_{i_1}...s_{i_l}$ be a reduced decomposition of $w \in W$ (so $l = \ell(w)$); denote this decomposition by $\overline{w}$. The product $\prod_{k=1}^l P_{i_k}$ carries a free action of $B^l$ via

$$(b_1, ..., b_l) \circ (p_1, ..., p_l) := (p_1b_1^{-1}, b_1p_2b_2^{-1}, ..., b_{l-1}p_lb_l^{-1}).$$

Define the Bott-Samelson variety $X_{\overline{w}} := (\prod_{k=1}^l P_{i_k})/B^l$. Use (ii) to show that if $\overline{w} = s_i\overline{w}$ then $X_{\overline{w}}$ fibers over $\mathbb{CP}^1$ with fiber $X_{\overline{w}}$. Deduce that $X_{\overline{w}}$ is a smooth projective variety of dimension $\ell(w)$.

\(^{35}\)We note that Bruhat cells, unlike Schubert cells, are not literally cells in the topological sense – they are not homeomorphic to an affine space, but are homeomorphic to the product of an affine space and a torus.
(iv) Define the Bott-Samelson map $$\mu_w : X_w \to G/B$$ given by multiplication. Use (i) to show that the image of $$\mu_w$$ is the Schubert variety $$C_w$$, the closure of $$C_w$$ in $$G/B$$. Moreover, show that $$C_w \setminus C_w$$ is the union of $$C_u$$ over some $$u \in W$$ with $$\ell(u) < \ell(w)$$.

(v) Apply (iv) to the maximal element $$w = w_0 \in W$$. In this case, show that $$\mu_w$$ is surjective, and deduce Theorem 51.18.

Let us derive some corollaries of Theorem 51.18.

**Corollary 51.20.**
(i) Any pair of Borel subgroups of $$G$$ is conjugate to the pair $$(B, w(B))$$ for a unique $$w \in W$$. In particular, any two Borel subgroups of $$G$$ share a maximal torus.

(ii) The cell $$C_w$$ is isomorphic to $$C^{\ell(w)}$$.

**Proof.** (i) Let $$(B_1, B_2)$$ be a pair of Borel subgroups in $$G$$. Then we can conjugate $$B_1$$ to $$B$$, and $$B_2$$ will be conjugated to some Borel subgroup $$B_3$$. This subgroup is conjugate to $$B$$, i.e., is of the form $$gBg^{-1}$$ for some $$g \in G$$. By Bruhat decomposition, we can write $$g$$ as $$g = b_1 \tilde{w} b_2$$, $$b_1, b_2 \in B$$, $$\tilde{w} \in N(H)$$. So conjugating by $$b_1^{-1}$$, we will bring our pair to the required form $$(B, w(B))$$, where $$w$$ is the image of $$\tilde{w}$$ in $$W$$. Uniqueness follows from Proposition 51.16.

(ii) By (i) we have $$C_w \cong B/(B \cap w(B))$$. Since $$B = NH$$, where $$N = [B, B]$$ and $$B \cap w(B) \supset H$$, we get $$C_w = N/(N \cap w(B)) = N/(N \cap w(N))$$. This is a complex affine space of dimension equal to the number of positive roots mapped to negative roots by $$w$$, i.e., $$\ell(w)$$. \(\square\)

**Corollary 51.21.** The Poincaré polynomial of the flag manifold $$G/B$$ is

$$\sum_{i \geq 0} b_{2i}(G/B)q^i = \sum_{w \in W} q^{\ell(w)}.$$

**Remark 51.22.** Similarly to the type $$A$$ case, one can show that this polynomial can also be written as $$\prod_{i=1}^r [m_i + 1]_q$$, where $$m_i$$ are the exponents of $$G$$, but we will not give a proof of this identity.

**References**


