51. **Borel subgroups and the flag manifold of a complex reductive Lie group**

51.1. **Borel subgroups and subalgebras.** Let G be a connected complex reductive Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with a system of simple positive roots Π, and consider the corresponding triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_+$ is spanned by positive root elements and $\mathfrak{n}_-$ by negative root elements. Let H be the maximal torus in G corresponding to $\mathfrak{h}$, N_+ the unipotent subgroup of G corresponding to $\mathfrak{n}_+$, and $B_+ = HN_+$ the solvable subgroup with $\mathrm{Lie}(B_+) = \mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$; these are all closed Lie subgroups.

Definition 51.1. A **Borel subalgebra** of $\mathfrak{g}$ is a Lie subalgebra conjugate to $\mathfrak{b}_+$. A **Borel subgroup** of G is a Lie subgroup conjugate to B_+.

Since all pairs $(\mathfrak{h}, \Pi)$ are conjugate, this definition does not depend on the choice of $(\mathfrak{h}, \Pi)$.

Lemma 51.2. *B_+ is its own normalizer in G.*

Proof. Let $\gamma \in G$ be such that $\mathrm{Ad}\gamma(B_+) = B_+$. Let $H' = \mathrm{Ad}\gamma(H) \subset B_+$. It is easy to show that we can conjugate H' back into H inside B_+, so we may assume without loss of generality that $H' = H$. Then $\gamma \in N(H)$, and it preserves positive roots. Hence the image of γ in W is 1, so $\gamma \in H \subset B_+$, as claimed. □

51.2. **The flag manifold of a connected complex reductive group.** Thus the set of all Borel subalgebras (or subgroups) in G is the homogeneous space G/B_+, a complex manifold. It is called the **flag manifold** of G. Note that it only depends on the semisimple part $\mathfrak{g}_{ss} \subset \mathfrak{g}$ and does not depend on the choice of the Cartan subalgebra and triangular decomposition.

Let $G^c \subset G$ be the compact form of G, with Lie algebra $\mathfrak{g}^c \subset \mathfrak{g}$. It is easy to see that $\mathfrak{g}^c + \mathfrak{b}_+ = \mathfrak{g}$. Thus the G^c-orbit $G^c \cdot 1$ of $1 \in G/B_+$ contains a neighborhood of 1 in G/B_+. Hence the same holds for any point of this orbit, i.e., $G^c \cdot 1 \subset G/B_+$ is an open subset. But it is also compact, since G^c is compact, hence closed. As G/B_+ is connected, we get that $G^c \cdot 1 = G/B_+$, i.e., G^c acts transitively on G/B_+.

Also the Cartan involution ω maps positive root elements to negative ones, so $G^c \cap B_+ \subset w_0(B_+) \cap B_+ = H$. Thus $G^c \cap B_+ = H^c$, a maximal torus in G^c. So we get

Proposition 51.3. *We have $G/B_+ = G^c/H^c$. In particular, G/B_+ is a compact complex manifold of dimension $|R_+| = \frac{1}{2}(\dim \mathfrak{g} - \mathrm{rank}\mathfrak{g})$.*

Example 51.4. 1. For $G = SL_2$ we have $G/B_+ = SU(2)/U(1) = S^2$, the Riemann sphere.

2. For $G = GL_n$ we have $G/B_+ = U(n)/U(1)^n = \mathcal{F}_n$, the set of flags in $\mathbb{C}^n$ that we considered in Subsection 47.3.

Another realization of the flag manifold is one as the G-orbit of the line spanned by the highest weight vector in an irreducible representation with a regular highest weight. Namely, let $\lambda \in P_+$ be a dominant integral weight with $\lambda(h_i) \geq 1$ for all i (i.e., $\lambda = \mu + \rho$ for $\mu \in P_+$). Let L_λ be the corresponding irreducible representation with highest weight vector v_λ. We have $\mathfrak{b}_+ \cdot \mathbb{C}v_\lambda = \mathbb{C}v_\lambda$, but $e_{-\alpha}v_\lambda \neq 0$ for any $\alpha \in R_+$ (as $e_\alpha e_{-\alpha} v_\lambda = h_\alpha v_\lambda = (\lambda, \alpha^\vee)v_\lambda$, and $(\lambda, \alpha^\vee) > 0$). Moreover, these vectors have different weights, so are linearly independent. Thus $\mathfrak{b}_+$ is the stabilizer of $\mathbb{C}v_\lambda$ in $\mathfrak{g}$. Hence any $g \in G$ which preserves $\mathbb{C}v_\lambda$ belongs to the normalizer of $\mathfrak{b}_+$ (or, equivalently, B_+), i.e., $g \in B_+$. Thus $\mathcal{O} := G \cdot \mathbb{C}v_\lambda \subset \mathbb{P}L_\lambda$ is identified with G/B_+. This shows that $\mathcal{O}$ is compact, hence closed, i.e., $\mathcal{O} = G/B_+$ is a smooth complex projective variety.

Let $A = \exp(i\mathfrak{h}^c) \subset H$, $K = G^c$, $N = N_+$. Proposition 51.3 immediately implies

Corollary 51.5. *(The* **Iwasawa decomposition** *of G) The multiplication map $K \times A \times N \to G$ is a diffeomorphism. In particular, we have $G = KAN$.*

A similar theorem holds for *real* reductive groups (Theorem 51.14).

51.3. **The Borel fixed point theorem.** Let V be a finite dimensional representation of a finite dimensional $\mathbb{C}$-Lie algebra $\mathfrak{a}$, and $X \subset \mathbb{P}V$ be a subset. We will say that X is $\mathfrak{a}$-invariant (or fixed by $\mathfrak{a}$) if it is $\exp(\mathfrak{a})$-invariant.

Theorem 51.6. *Let $\mathfrak{a}$ be a solvable Lie algebra over $\mathbb{C}$, V a finite dimensional $\mathfrak{a}$-module. Let $X \subset \mathbb{P}V$ be a closed $\mathfrak{a}$-invariant subset. Then there exists $x \in X$ fixed by $\mathfrak{a}$.*

Proof. The proof is by induction in $n = \dim \mathfrak{a}$. The base $n = 0$ is trivial, so we only need to justify the induction step. Since $\mathfrak{a}$ is solvable, it has an ideal $\mathfrak{a}'$ of codimension 1. By the induction assumption, $Y := X^{\mathfrak{a}'}$ (the set of $\exp(\mathfrak{a}')$-fixed points in X) is a nonempty closed subset of X, so it suffices to show that the 1-dimensional Lie algebra $\mathfrak{a}/\mathfrak{a}'$ has a fixed point on Y. Thus it suffices to prove the theorem for $n = 1$.

So let $\mathfrak{a}$ be 1-dimensional, spanned by $a \in \mathfrak{a}$. We can choose the normalization of a so that all eigenvalues of a on V have different real parts. Fix $x_0 \in X$ and consider the curve $e^{ta}x_0$ for $t \in \mathbb{R}$. It is easy to see that there exists $x := \lim_{t\to\infty} e^{ta}x_0 \in \mathbb{P}V$. Then $x \in X$ as X is closed and, and x is fixed by $\mathfrak{a}$, as desired. $\square$

51.4. **Parabolic and Levi subalgebras.** A Lie subalgebra $\mathfrak{p} \supset \mathfrak{b}$ of a reductive Lie algebra $\mathfrak{g}$ containing some Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is called a **parabolic subalgebra** of $\mathfrak{g}$. The corresponding connected Lie subgroup $P \subset G$ is called a **parabolic subgroup**. It is easy to see that $P \subset G$ is necessarily closed (check it!).

Exercise 51.7. Show that parabolic subalgebras $\mathfrak{p}$ containing $\mathfrak{b}_+$ are in bijection with subsets $S \subset \Pi$ of the set of simple roots of $\mathfrak{b}_+$, namely, $\mathfrak{p}$ is sent to the set $S_{\mathfrak{p}}$ of $i \in \Pi$ such that $f_i \in \mathfrak{p}$, and S is sent to the Lie subalgebra $\mathfrak{p}_S$ of $\mathfrak{g}$ generated by $\mathfrak{b}_+$ and $f_i, i \in S$.

Let $P \subset G$ be a parabolic subgroup with Lie algebra $\mathfrak{p}$. Let $\mathfrak{u} \subset \mathfrak{p}$ be the nilpotent radical of $\mathfrak{p}$; for instance, if $\mathfrak{p} \supset \mathfrak{b}_+$ then $\mathfrak{u}$ is the Lie subalgebra spanned by e_α such that $e_{-\alpha} \notin \mathfrak{p}$. It is easy to see that there exists a (non-unique) Lie subalgebra $\mathfrak{l} \subset \mathfrak{p}$ complementary to $\mathfrak{u}$, which therefore projects isomorphically to $\mathfrak{p}/\mathfrak{u}$; indeed, if $\mathfrak{p} \supset \mathfrak{b}_+$ then we can take $\mathfrak{l}$ to be the Lie subalgebra spanned by $\mathfrak{h}$ and $e_\alpha, e_{-\alpha}$ where α runs through positive roots for which $e_{-\alpha} \in \mathfrak{p}$. Such a subalgebra $\mathfrak{l}$ is called a **Levi subalgebra** of $\mathfrak{p}$, and we have $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{u}$, which is $\mathfrak{l} \oplus \mathfrak{u}$ as a vector space.

Let $U = \exp(\mathfrak{u})$. The quotient P/U is a reductive group with Lie algebra $\mathfrak{p}/\mathfrak{u}$. A **Levi subgroup** of P is a subgroup L in P such that $\mathfrak{l} :=$ Lie(L) is a Levi subalgebra of $\mathfrak{p}$; equivalently, L projects isomorphically to P/U, so we have $P = L \ltimes U$, written shortly as $P = LU$. It is not difficult to show that all Levi subgroups of P (or, equivalently, all Levi subalgebras of $\mathfrak{p}$) are conjugate by the action of U (check it!).

For example, L is a maximal torus if and only if P is a Borel subgroup, and $L = G$ if and only if $P = G$.

Example 51.8. Let $n = n_1 + ... + n_k$ where n_i are positive integers. Then the subgroup P of block upper triangular matrices with diagonal blocks of size $n_1, ..., n_k$ is a parabolic subgroup of $GL_n(\mathbb{C})$, and the subgroup L of block diagonal matrices in P is a Levi subgroup. The unipotent radical U of P is the subgroup of block upper triangular matrices with identity matrices on the diagonal.

51.5. **Maximal solvable and maximal nilpotent subalgebras.** Note that $\mathfrak{b}_+$ is a maximal solvable subalgebra of $\mathfrak{g}$; indeed, any bigger parabolic subalgebra contains a negative root vector, hence the corresponding root $\mathfrak{sl}_2$-subalgebra, so it is not solvable. Moreover, B_+ is a maximal solvable subgroup of G: if $P \supset B_+$ then some element $g \in P$

does not normalize $\mathfrak{b}_+$, so Lie(P) has to be larger than $\mathfrak{b}_+$, hence not solvable. Thus any Borel subalgebra (subgroup) is a maximal solvable one. It turns out that the converse also holds.

Proposition 51.9. *Any solvable Lie subalgebra of $\mathfrak{g}$ (respectively, connected solvable subgroup of G) is contained in a Borel subalgebra (subgroup).*

Proof. Let $\mathfrak{a} \subset \mathfrak{g}$ be a solvable Lie subalgebra. By the Borel fixed point theorem, $\mathfrak{a}$ has a fixed point $\mathfrak{b} \in G/B_+$. Thus $\mathfrak{a}$ normalizes $\mathfrak{b}$. Hence $\mathfrak{a} \subset \mathfrak{b}$, as claimed. □

Corollary 51.10. *Any element of $\mathfrak{g}$ is contained in a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$.*

Let us say that a Lie subalgebra $\mathfrak{a} \subset \mathfrak{g}$ is **a nilpotent subalgebra** if it consists of nilpotent elements. Note that this is a stronger condition than just being nilpotent as a Lie algebra; for example, a Cartan subalgebra is a nilpotent Lie algebra (since it is abelian) but it is not a nilpotent subalgebra of $\mathfrak{g}$.

Corollary 51.11. *Any nilpotent subalgebra of $\mathfrak{g}$ is conjugate to a Lie subalgebra of $\mathfrak{n}_+$. Thus $\mathfrak{n}_+$ is a maximal nilpotent subalgebra of $\mathfrak{g}$, and any maximal nilpotent subalgebra of $\mathfrak{g}$ is conjugate to $\mathfrak{n}_+$.*

Proof. By Proposition 51.9 there is $g \in G$ such that $\mathrm{Ad}_g\mathfrak{a} \subset \mathfrak{b}_+$, but since $\mathfrak{a}$ is nilpotent we actually have $\mathrm{Ad}_g\mathfrak{a} \subset \mathfrak{n}_+$. □

A similar result holds for groups, with the same proof:

Corollary 51.12. *Any unipotent subgroup of G is conjugate to a (closed) Lie subgroup of N_+. Thus N_+ is a maximal unipotent subgroup of G, and any maximal unipotent subgroup of G is conjugate to N_+.*

We also have

Proposition 51.13. *The normalizer of $\mathfrak{n}_+$ and N_+ in G is B_+. Thus every maximal nilpotent subalgebra (unipotent subgroup) is contained in a unique Borel subgroup. Hence such subalgebras (subgroups) are parametrized by the flag manifold G/B_+.*

Proof. Clearly B_+ is contained in the normalizer of N_+, so this normalizer is a parabolic subgroup. We have seen that such a subgroup, if larger than B_+, must have a Lie algebra larger that $\mathfrak{b}_+$, so it must be $\mathfrak{p}_S$ for some $S \neq \emptyset$, hence contains some root $\mathfrak{sl}_2$-subalgebra. But the group corresponding to such a subalgebra does not normalize $\mathfrak{n}_+$, a contradiction. □

51.6. **Iwasawa decomposition of a real semisimple linear group.** Let $G_\theta = K^c P_\theta$ be the polar decomposition of a real form of a complex semisimple group G, $\mathfrak{g}_\theta = \mathfrak{k}^c \oplus \mathfrak{p}_\theta$ the additive version, $\mathfrak{a} \subset \mathfrak{p}_\theta$ a maximal abelian subspace. Let $A = \exp(\mathfrak{a}) \subset P_\theta$ be the corresponding abelian subgroup of G_θ. Pick a generic element $a \in \mathfrak{a}$. Let $\mathfrak{z} = \mathfrak{g}_\theta^a$ be the centralizer of a in $\mathfrak{g}_\theta$ and let $\mathfrak{n}_{a,\pm}$ be the (nilpotent) Lie subalgebras of $\mathfrak{g}_\theta$ spanned by eigenvectors of ad$a$ with positive, respectively negative eigenvalues, so that $\mathfrak{g}_\theta = \mathfrak{n}_{a-} \oplus \mathfrak{z} \oplus \mathfrak{n}_{a+}$. Let $N_{a\pm} = \exp(\mathfrak{n}_{a\pm})$.

The following theorem is a generalization of Proposition 51.5.

Theorem 51.14. *(Iwasawa decomposition) The multiplication map $K^c \times A \times N_{a+} \to G_\theta$ is a diffeomorphism.*

Theorem 51.14 is proved in the following exercise.

Exercise 51.15. (i) Let $\mathfrak{m} = \mathfrak{z} \cap \mathfrak{k}^c$. Show that $\mathfrak{z} = \mathfrak{m} \oplus \mathfrak{a}$ (use Proposition 44.11(ii)).

(ii) Given $x \in \mathfrak{p}$, write $x = x_- + x_0 + x_-$, $x_\pm \in \mathfrak{n}_{a\pm}$, $x_0 \in \mathfrak{z}$. Show that $\theta(x_\pm) = -x_\mp$, $\theta(x_0) = -x_0$. Deduce the **additive Iwasawa decomposition** $\mathfrak{g}_\theta = \mathfrak{k}^c \oplus \mathfrak{a} \oplus \mathfrak{n}_{a+}$ (write x as $(x_- - x_+) + x_0 + 2x_+$).

(iii) Show that $\mathfrak{z} \oplus \mathfrak{n}_{a+} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}_{\mathfrak{a}+}$ is a parabolic subalgebra in $\mathfrak{g}_\theta$ with Levi subalgebra $\mathfrak{z}$ (i.e., their complexifications are a parabolic subalgebra in $\mathfrak{g}$ and its Levi subalgebra) and its unipotent radical is $\mathfrak{n}_{a+}$.

(iv) Let M be the centralizer of a in K^c. Show that $\mathbb{P} := MAN_{a+}$ is a subgroup of G_θ and $X := G_\theta/\mathbb{P}$ is a compact homogeneous space.

(v) Show that K^c acts transitively on X, and $X \cong K^c/M$ as a homogeneous space for K^c (generalize the argument in Subsection 51.2). Deduce Theorem 51.14.

51.7. **The Bruhat decomposition.** Let G be a connected complex reductive group, $H \subset G$ a maximal torus, $B = B_+ \supset H$ a Borel subgroup. The Bruhat decomposition is the decomposition of G into double cosets of B.

Let $N(H)$ be the normalizer of H in G and $W = N(H)/H$ be the Weyl group. Given $w \in W$, let $\widetilde{w}$ be a lift of w to $N(H)$ and consider the double coset $B\widetilde{w}B \subset G$. Since any two lifts of w differ by an element of H which is contained in B, the set $B\widetilde{w}B$ does not depend on the choice of $\widetilde{w}$, so we will denote it by BwB.

Proposition 51.16. *The double cosets BwB, $w \in W$ are disjoint.*

Proof. Let $w_1, w_2 \in N(H)$ be such that $Bw_1B = Bw_2B$. Then there exist elements $b_1, b_2 \in B$ such that $b_1w_1 = w_2b_2$. Let us apply this identity to a highest weight vector v_λ of an irreducible representation

L_λ of G, where $\lambda \in P_+$ is regular. We have $w_2 b_2 v_\lambda = C v_{w_2\lambda}$ for some $C \in \mathbb{C}^\times$, where $v_{w_2\lambda}$ is an extremal vector of weight $w_2\lambda$. On the other hand, $b_1 w_1 v_\lambda = C' b_1 v_{w_1\lambda}$ for some $C' \in \mathbb{C}^\times$. Thus $C v_{w_2\lambda} = C' b_1 v_{w_1\lambda}$. But $b_1 v_{w_1\lambda}$ equals $C'' v_{w_1\lambda}$ plus terms of weight $> w_1\lambda$, where $C'' \in \mathbb{C}^\times$. It follows that $w_1\lambda = w_2\lambda$, hence $w_1 = w_2 h$, $h \in H$. $\square$

Theorem 51.17. *(Bruhat decomposition) The union of the double cosets BwB, $w \in W$ is the entire group G. Thus they define a partition of G into double cosets of B.*

Theorem 51.17 can be reformulated as a classification of B-orbits on the flag manifold G/B. Namely, given $w \in W$, the set BwB/B is an orbit of B on G/B, which we will denote by C_w. By Theorem 51.16, C_w are disjoint, and Theorem 51.17 is equivalent to

Theorem 51.18. *(Schubert decomposition) $C_w, w \in W$ give the partition of G/B into B-orbits.*

The sets BwB are called **Bruhat cells** and the sets C_w are called **Schubert cells.**[43]

Note that for type A_{n-1} ($G = SL_n(\mathbb{C})$ or its quotient), we have already proved Theorem 51.18 in Subsection 47.3, where we decomposed the flag manifold $\mathcal{F}_n$ into Schubert cells labeled by permutations.

A proof of Theorem 51.18 can be found, for example, in the textbook [CG]. It is also sketched in the following exercise.

Exercise 51.19. (i) Let $B = B_+$ and $w \in W$. Consider the multiplication map $\mu_{i,w} : Bs_iB \times_B C_w \to G/B$. Show that if $\ell(s_i w) = \ell(w) + 1$ then $\mu_{i,w}$ is an isomorphism onto $C_{s_i w}$, while if $\ell(s_i w) = \ell(w) - 1$ then the image of $\mu_{i,w}$ consists of C_w and $C_{s_i w}$.

Hint: Reduce to the SL_2-case.

(ii) For $i \in \Pi$ let P_i be the **minimal parabolic** subgroup of G generated by B and the 1-parameter subgroup $\exp(tf_i)$. Show that $P_i/B = C_{s_i} \cup C_1 \cong \mathbb{CP}^1 \subset G/B$ (where C_1 is a point and $C_{s_i} \cong \mathbb{C}$).

(iii) Let $w = s_{i_1}...s_{i_l}$ be a reduced decomposition of $w \in W$ (so $l = \ell(w)$); denote this decomposition by $\overline{w}$. The product $\prod_{k=1}^{l} P_{i_k}$ carries a free action of B^l via

$$(b_1, ..., b_l) \circ (p_1, ..., p_l) := (p_1 b_1^{-1}, b_1 p_2 b_2^{-1}, ..., b_{l-1} p_l b_l^{-1}).$$

Define the **Bott-Samelson variety** $X_{\overline{w}} := (\prod_{k=1}^{l} P_{i_k})/B^l$. Use (ii) to show that if $\overline{w} = s_i \overline{u}$ then $X_{\overline{w}}$ fibers over $\mathbb{CP}^1$ with fiber $X_{\overline{u}}$. Deduce that $X_{\overline{w}}$ is a smooth projective variety of dimension $\ell(w)$.

[43]We note that Bruhat cells, unlike Schubert cells, are not literally cells in the topological sense – they are not homeomorphic to an affine space, but are homeomorphic to the product of an affine space and a torus.

(iv) Define the **Bott-Samelson map**

$$\mu_{\overline{w}} : X_{\overline{w}} \to G/B$$

given by multiplication. Use (i) to show that the image of $\mu_{\overline{w}}$ is the **Schubert variety** $\overline{C}_w$, the closure of C_w in G/B. Moreover, show that $\overline{C_w} \setminus C_w$ is the union of C_u over some $u \in W$ with $\ell(u) < \ell(w)$.

(v) Apply (iv) to the maximal element $w = w_0 \in W$. In this case, show that $\mu_{\overline{w}}$ is surjective, and deduce Theorem 51.18.

Let us derive some corollaries of Theorem 51.18.

Corollary 51.20. *(i) Any pair of Borel subgroups of G is conjugate to the pair $(B, w(B))$ for a unique $w \in W$. In particular, any two Borel subgroups of G share a maximal torus.*

(ii) The cell C_w is isomorphic to $\mathbb{C}^{\ell(w)}$.

Proof. (i) Let (B_1, B_2) be a pair of Borel subgroups in G. Then we can conjugate B_1 to B, and B_2 will be conjugated to some Borel subgroup B_3. This subgroup is conjugate to B, i.e., is of the form gBg^{-1} for some $g \in G$. By Bruhat decomposition, we can write g as $g = b_1\widetilde{w}b_2$, $b_1, b_2 \in B$, $\widetilde{w} \in N(H)$. So conjugating by b_1^{-1}, we will bring our pair to the required form $(B, w(B))$, where w is the image of $\widetilde{w}$ in W. Uniqueness follows from Proposition 51.16.

(ii) By (i) we have $C_w \cong B/(B \cap w(B))$. Since $B = NH$, where $N = [B, B]$ and $B \cap w(B) \supset H$, we get $C_w = N/(N \cap w(B)) = N/(N \cap w(N))$. This is a complex affine space of dimension equal to the number of positive roots mapped to negative roots by w, i.e., $\ell(w)$. □

Corollary 51.21. *The Poincaré polynomial of the flag manifold G/B is*

$$\sum_{i \geq 0} b_{2i}(G/B)q^i = \sum_{w \in W} q^{\ell(w)}.$$

Remark 51.22. Similarly to the type A case, one can show that this polynomial can also be written as $\prod_{i=1}^{r}[m_i + 1]_q$, where m_i are the exponents of G, but we will not give a proof of this identity.

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