1. **Continuous representations of topological groups**

This course will be about representations of Lie groups, with a focus on non-compact groups. While irreducible representations of compact groups are all finite-dimensional, this is not so for non-compact groups, whose most interesting irreducible representations are infinite-dimensional. Thus to have a sensible representation theory of non-compact Lie groups, we need to consider their **continuous** representations on **topological vector spaces**.

1.1. **Topological vector spaces.** All representations we'll consider will be over the field $\mathbb{C}$, which is equipped with its usual topology. Recall that a **topological vector space** over $\mathbb{C}$ is a complex vector space V with a topology in which addition $V \times V \to V$ and scalar multiplication $\mathbb{C} \times V \to V$ are continuous. The topological vector spaces V we'll consider will always be assumed to have the following properties:

• **Hausdorff:** any two distinct points of V have disjoint neighborhoods.

• **locally convex:** $0 \in V$ (hence every point) has a base of convex neighborhoods.[1] Equivalently, the topology on V is defined by a family of **seminorms**[2] $\{\nu_\alpha, \alpha \in A\}$: a base of neighborhoods of 0 is formed by finite intersections of the sets $U_{\alpha,\varepsilon} := \{v \in V | \nu_\alpha(v) < \varepsilon\}$, $\alpha \in A$, $\varepsilon > 0$. I.e., it is the weakest of the topologies in which all ν_α are continuous.

• **sequentially complete:** every Cauchy sequence[3] is convergent.

Also, unless specified otherwise, we will assume that V is

• **first countable**: $0 \in V$ (equivalently, every point of V) has a countable base of neighborhoods. By the Birkhoff-Kakutani theorem, this is equivalent to V being **metrizable** (topology defined by a metric), and moreover this metric can be chosen translation invariant: $d(x, y) = D(x - y)$ for some function $D : V \to \mathbb{R}_{\geq 0}$.

In this case V is called a **Fréchet space**. For example, every **Banach spac**e (a complete normed space), in particular, **Hilbert space** is a Fréchet space.

Recall that a Hausdorff topological vector space V is said to be **complete** if whenever V is realized as a dense subspace of a Hausdorff

[1] Recall that a set $X \subset V$ is **convex** if for any $x, y \in X$ and $t \in [0, 1]$ we have $tx + (1 - t)y \in X$.

[2] Recall that a **seminorm** on V is a function $\nu : V \to \mathbb{R}_{\geq 0}$ such that $\nu(x + y) \leq \nu(x) + \nu(y)$ and $\nu(\lambda x) = |\lambda|\nu(x)$ for $x, y \in V$, $\lambda \in \mathbb{C}$. A seminorm is a **norm** iff $\nu(x) = 0$ implies $x = 0$.

[3] Recall that a sequence $a_n \in V$ is **Cauchy** if for any neighborhood U of $0 \in V$ there exists N such that for $n, m \geq N$ we have $a_n - a_m \in U$.

topological vector space $\overline{V}$ with induced topology, we have $V = \overline{V}$. Every complete space is sequentially complete, and the converse holds for metrizable spaces (albeit not in general). Thus a Fréchet space can be defined as a locally convex complete metrizable topological vector space.

Alternatively, a Fréchet space may be defined as a complete topological vector space with topology defined by a *countable* system of seminorms $\nu_n : V \to \mathbb{R}$, $n \geq 1$. Thus, a sequence $x_m \in V$ goes to zero iff $\nu_n(x_m)$ goes to zero for all n. Note that the Hausdorff property is then equivalent to the requirement that any vector $x \in V$ with $\nu_n(x) = 0$ for all n is zero.

A translation-invariant metric on a Fréchet space may be defined by the formula

$$d(x,y) = D(x-y),\ \ D(x) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\nu_n(x)}{1+\nu_n(x)}.$$

Note however that D is not a norm, as it is not homogeneous: for $\lambda \in \mathbb{C}$, $D(\lambda x) \neq |\lambda| D(x)$. If we had a finite collection of seminorms, we could define a norm simply by $D(x) := \sum_n \nu_n(x)$, but if there are infinitely many, this sum may not converge, and we have to sacrifice the homogeneity property for convergence. In fact, the examples below show that there are important Fréchet spaces that are not Banach (i.e., do not admit a single norm defining the topology). We also note that the same Fréchet space structure on V can be defined by different systems of seminorms ν_n, and there is also nothing canonical about the formula for D (e.g., we can replace $\frac{1}{2^n}$ by any sequence $a_n > 0$ with $\sum_n a_n < \infty$), so ν_n or D are not part of the data of a Fréchet space.

Finally, unless specified otherwise, we will assume that V is

• **second countable:** admits a countable base. For metrizable spaces, this is equivalent to being **separable** (having a dense countable subset).

Example 1.1. 1. Let X be a locally compact second countable Hausdorff topological space (e.g., a manifold). Then it is easy to see that X can be represented as a countable nested union of compact subsets: $X = \bigcup_{n \geq 1} K_n$, $K_1 \subset K_2 \subset ....$ Let $C(X)$ be the space of continuous complex-valued functions on X. We can then define seminorms ν_n by

$$\nu_n(f) = \max_{x \in K_n} |f(x)|.$$

(this is well defined since K_n are compact). This makes $C(X)$ into a Fréchet space, and this structure is independent on the choice of the sequence K_n. The convergence in $C(X)$ is uniform convergence on compact sets.

By the Tietze extension theorem, if $K \subset L$ are compact Hausdorff spaces then the restriction map $C(L) \to C(K)$ is surjective. So $C(X) = \varprojlim_{n\to\infty} C(K_n)$ as a vector space. Alternatively, without making any choices, we may write $C(X) = \varprojlim_{K\subset X} C(K)$, where K runs over compact subsets of X.

2. If X is a manifold and $0 \le k \le \infty$, we can similarly define a Fréchet space structure on the space $C^k(X)$ of k times continuously differentiable functions on X. Namely, cover X by countably many closed balls K_n, each equipped with a local coordinate system, and set

$$\nu_{n,m}(f) = \max_{x\in K_n} \|d^m f(x)\|\,,\ 0 \le m \le k$$

where $d^m f(x)$ is the m-th differential of f at x, comprising the m-th mixed partial derivatives of f at x with respect to the local coordinates (these are labeled by two indices rather than one, but it does not matter since this collection is still countable). The convergence in $C^k(X)$ is uniform convergence with all derivatives up to k-th order on compact sets.

These spaces are not Banach unless X is compact. Moreover, $C^\infty(X)$ is not Banach even for compact X (of positive dimension). For example, for $C^\infty(S^1)$ we may take,

$$\nu_m(f) = \sum_{i=0}^{m} \max_{x\in S^1} |f^{(i)}(x)|,$$

but this is still an infinite collection. Note that these are all norms, not just seminorms, but each of them taken separately does not define the correct topology on $C^\infty(S^1)$ (namely, ν_m defines the incomplete topology induced by embedding $C^\infty(S^1)$ as a dense subspace into the Banach space $C^m(S^1)$ with norm ν_m).

3. The Schwartz space $\mathcal{S}(\mathbb{R}) \subset C^\infty(\mathbb{R})$ is the space of functions f with

$$\nu_{m,n}(f) := \sup_{x\in\mathbb{R}} |x^n \partial^m f(x)| < \infty,\ m,n \ge 0.$$

This system of seminorms can then be used to give $\mathcal{S}(\mathbb{R})$ the structure of a (non-Banach) Fréchet space. The same definition can be used for the Schwartz space $\mathcal{S}(\mathbb{R}^N)$, by taking $n = (n_1, ..., n_N)$, $m = (m_1, ..., m_N)$, $x = (x_1, ..., x_N)$, $\partial = (\partial_1, ..., \partial_N)$, and

$$x^n := \prod_i x_i^{n_i},\ \partial^m := \prod_i \partial_i^{m_i}.$$

It is well known that all these spaces are separable (check it!).

1.2. **Continuous representations.** Let G be a locally compact topological group, for example, a Lie group.[4]

Definition 1.2. A **continuous representation** of G is a topological vector space V with a *continuous* linear action $a : G \times V \to V$.[5]

In particular, a continuous representation gives a homomorphism $\pi : G \to \mathrm{Aut}(V)$ from G to the group of continuous automorphisms of V (i.e., continuous linear maps $V \to V$ with continuous inverse).[6]

Definition 1.3. A continuous representation is called **unitary** if V is a Hilbert space and for all $g \in G$, the operator $\pi(g) : V \to V$ is unitary; in other words, π lands in the unitary group $U(V) \subset \mathrm{Aut}(V)$.

Exercise 1.4. Let $1 \le p < \infty$ and $L^p(\mathbb{R})$ be the Banach space of measurable functions $f : \mathbb{R} \to \mathbb{R}$ with

$$\|f\|_p = \left(\int_{\mathbb{R}} |f(x)|^p dx\right)^{\frac{1}{p}} < \infty$$

(modulo functions vanishing outside a set of measure zero), with norm $f \mapsto \|f\|_p$. The Lie group $\mathbb{R}$ acts on $L^p(\mathbb{R})$ by translation.

(i) Show that this is a continuous representation, which is unitary for $p = 2$ (use approximation of L^p functions by continuous functions with compact support).

(ii) Prove the same for the Fréchet spaces $C^k(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$.

Let G be a locally compact group, for example a Lie group. In this case G is known to have a unique up to scaling right-invariant **Haar measure** dx. For Lie groups, this measure is easy to construct by spreading a nonzero element of $\wedge^n \mathfrak{g}^*$, $\mathfrak{g} = \mathrm{Lie}(G)$, $n = \dim \mathfrak{g}$, over the group G by right translations. Thus we can define the Banach space $L^p(G)$ similarly to the case $G = \mathbb{R}$. It is easy to generalize Exercise 1.4 to show that the translation action of G on $L^p(G)$ and $C^k(G)$ is continuous, with $L^2(G)$ unitary.

[4]Topological groups will always be assumed Hausdorff and second countable. Important examples of locally compact topological groups include groups $\mathbf{G}(F)$, where F is a local field and $\mathbf{G}$ is an algebraic group defined over F. If F is archimedean ($\mathbb{R}$ or $\mathbb{C}$) then $\mathbf{G}(F)$ is a real, respectively complex, Lie group. Another example important in number theory is $\mathbf{G}(\mathbb{A}_k)$, where k is a global field, $\mathbb{A}_k$ is its ring of adéles, and $\mathbf{G}$ is an algebraic group over k.

[5]It is easy to see that it suffices to check this property at points $(1, v)$ for $v \in V$.

[6]Note that by the open mapping theorem, in a Fréchet space any invertible continuous operator has a continuous inverse.

Example 1.5. Let X be a manifold with a right action of a Lie group G. We'd like to say that we have a unitary representation of G on $L^2(X)$ via $(gf)(x) = f(xg)$. But for this purpose we need to fix a G-invariant measure on X, and such a nonzero measure does not always exist (e.g., $G = SL_2(\mathbb{R})$, $X = \mathbb{RP}^1 = S^1$).

The way out is to use **half-densities** on X rather than functions. Namely, recall that if $\dim X = m$ then the canonical line bundle $K_X := \wedge^m T^*X$ has structure group $\mathbb{R}^\times$. Consider the character $\mathbb{R}^\times \to \mathbb{R}^{>0}$ given by $t \mapsto |t|^s$, $s \in \mathbb{R}$, and denote the associated line bundle $|K|^s$. This is called the bundle of s-**densities** on X (in particular, **densities** for $s = 1$ and **half-densities** for $s = \frac{1}{2}$). Thus in local coordinates s-densities are ordinary functions, but when we change coordinates by $x \mapsto x' = x'(x)$, these functions change as

$$f = f' |\det(\tfrac{\partial x'}{\partial x})|^s.$$

The benefit of half-densities is that for any half-density f, the expression $|f|^2$ is naturally a density on X, which canonically defines a measure that can be integrated over X. As a result, the space $L^2(X)$ of half-densities f on X with

$$\|f\|_2 = \sqrt{\int_X |f|^2} < \infty$$

is a Hilbert space attached canonically to X (without choosing any additional structures), and any diffeomorphism $g : X \to X$ defines a unitary operator on $L^2(X)$. Thus similarly to Exercise 1.4, $L^2(X)$ is a unitary representation of G. Note that if X has a G-invariant measure, this is the same as a representation of G on L^2-functions on X.

In particular, we see that we have a unitary representation of $G \times G$ on $L^2(G)$ by left and right translation even though the right-invariant Haar measure is not always left-invariant.

If V is finite-dimensional, $\mathrm{Aut}(V) = GL(V)$ is just the group of invertible matrices, and the continuity condition for representations of G is just that the map $\pi : G \to \mathrm{Aut}(V)$ is continuous in the usual topology. Then it is well known that this map is smooth and is determined by the corresponding Lie algebra map $\mathfrak{g} \to \mathrm{End}(V) = \mathfrak{gl}(V)$, and this correspondence is a bijection if G is simply connected. In this way the theory of finite-dimensional continuous representations of connected Lie groups is immediately reduced to pure algebra.

On the other hand, for infinite-dimensional representations the situation is more tricky, as there are several natural topologies on $\mathrm{Aut}(V)$.

One of them is the **strong topology** of $\mathrm{End}(V)$ (continuous endomorphisms of V), in which $T_n \to T$ iff for all $v \in V$ we have $T_n v \to Tv$. It is clear that if (V, π) is a continuous representation of G then the map $\pi : G \to \mathrm{Aut}(V)$ is continuous in the strong topology, but the converse is not true, in general. However, the converse holds for Banach spaces (in particular, for unitary representations).

Proposition 1.6. *If V is a Banach space then a representation (V, π) of G is continuous if and only if the map $\pi : G \to \mathrm{Aut}(V)$ is continuous in the strong topology.*

Proof. Recall the **uniform boundedness principle**: If T_n is a sequence of bounded operators from a Banach space V to a normed space and for any $v \in V$ the sequence $T_n v$ is bounded then the sequence $\|T_n\|$ is bounded.

Now assume that π is continuous in the strong topology. Let $g_n \in G$, $g_n \to 1$, and $v_n \to v \in V$. Since G is second countable, our job is to show that $\pi(g_n)v_n \to v$. We know that $\pi(g_n)v \to v$, as $\pi(g_n) \to 1$ in the strong topology. So it suffices to show that $\pi(g_n)(v_n - v) \to 0$. As $v_n - v \to 0$, it suffices to show that the sequence $\|\pi(g_n)\|$ is bounded. But this follows from the uniform boundedness principle. $\square$

Remark 1.7. 1. Another topology on $\mathrm{End}(V)$ for a Banach space V is the **norm topology**, defined by the operator norm. It is stronger than the strong topology, and a continuous representation $\pi : G \to \mathrm{Aut}(V)$ does **not** have to be continuous in this topology. For example, the action of $\mathbb{R}$ on $L^2(\mathbb{R})$ is not. Indeed, denoting by T_a the operator $\pi(a)$ given by $(T_a f)(x) = f(x+a)$, we have $\|T_a - 1\| = 2$ for all $a \neq 0$ (show it!).

2. If $\dim V = \infty$ then $\mathrm{Aut}(V)$ is **not** a topological group with respect to strong topology (multiplication is not continuous).

1.3. **Subrepresentations, irreducible representations.**

Definition 1.8. A **subrepresentation** of a continuous representation V of G is a *closed* G-invariant subspace of V. We say that V is **irreducible** if its only subrepresentations are 0 and V.

Example 1.9. The translation representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ is not irreducible, although this is not completely obvious. To see this, we apply Fourier transform, which is a unitary automorphism of $L^2(\mathbb{R})$. The Fourier transform maps the operator T_a to the operator of multiplication by e^{iax}. But it is easy to construct closed subspaces of $L^2(\mathbb{R})$ invariant under multiplication by e^{iax}: take any measurable subset $X \subset \mathbb{R}$ and the subspace $L^2(X) \subset L^2(\mathbb{R})$ of functions that essentially vanish outside X (e.g., one can take $X = [0, +\infty)$).

Example 1.10. Here is the most basic example of an irreducible infinite-dimensional representation of a Lie group. Let G be the **Heisenberg** group, i.e., the group of upper triangular unipotent real 3-by-3 matrices. It can be realized as the Euclidean space $\mathbb{R}^3$ (with coordinates x, y, z being the above-diagonal matrix entries), with multiplication law

$$(a, b, c)(a', b', c') = (a + a', b + b', c + c' + ab').$$

Then we can define a unitary representation of G on $V = L^2(\mathbb{R})$ by setting $\pi(a, 0, 0) = e^{iax}$ (multiplication operator) and $\pi(0, b, 0) = T_b$ (shift by b).

Exercise 1.11. (i) Show that this gives rise to a well defined unitary representation of G, and compute $\pi(a, b, c)$ for general (a, b, c).

(ii) Show that V is irreducible.

Hint. Suppose $W \subset V$ is a proper subrepresentation, and denote by $P : V \to V$ the orthogonal projector to W. We can write P as an integral operator with Schwartz kernel[7] $K(x, y)$, a distribution on $\mathbb{R}^2$. Show that K is translation invariant, i.e., $K(x + a, y + a) = K(x, y)$, and deduce $K(x, y) = k(x - y)$ for some distribution $k(x)$ on $\mathbb{R}$.[8] Show that $(e^{iax} - 1)k(x) = 0$ for all $a \in \mathbb{R}$. Deduce that P is a scalar operator. Conclude that $P = 0$, so $W = 0$.

[7]Recall that every smooth function $\phi(x, y)$ on $\mathbb{R}^2$ with compact support defines a trace class operator T_ϕ with kernel $\phi(y, x)$, i.e.,

$$(T_\phi f)(x) = \int_{\mathbb{R}} \phi(y, x) f(y) dy.$$

Then the Schwartz kernel K of a continuous endomorphism A of $L^2(\mathbb{R})$ is defined by the formula $(K, \phi) = \mathrm{Tr}(AT_\phi)$ (which is well defined since the operator AT_ϕ is trace class).

[8]This means that $(K, \phi) = (k, \widetilde{\phi})$, where $\widetilde{\phi}(x) := \int_{\mathbb{R}} \phi(x + y, y) dy$.