3. **Algebras of measures on locally compact groups**

3.1. **The space of measures.** Let X be a locally compact second countable Hausdorff topological space. It is well known that such a space is metrizable, so let us fix a metric d defining the topology on X.

As we have seen in Example 1.1, the space $C(X)$ of continuous functions on X is a separable Fréchet space. So let us consider the topological dual space, $C(X)^*$, of continuous linear functionals on $C(X)$. This space is denoted by $\mathrm{Meas}_c(X)$; its elements are called (complex-valued) **compactly supported (Radon) measures on** X. We will often use the standard notation from measure theory: for $f \in C(X)$ and $\mu \in \mathrm{Meas}_c(X)$,

$$\mu(f) = \int_X f(x) d\mu(x).$$

We equip $\mathrm{Meas}_c(X)$ with the **weak topology**,[9] in which $\mu_n \to \mu$ iff $\mu_n(f) \to \mu(f)$ for any $f \in C(X)$ (this topology is commonly called the weak* topology, but we will drop the *). Namely, the weak topology is defined by the family of seminorms $\mu \mapsto |\mu(f)|$, $f \in C(X)$, so $\mathrm{Meas}_c(X)$ is Hausdorff and locally convex. We will also see that $\mathrm{Meas}_c(X)$ is separable and sequentially complete, but, as shown by the example below, in general it is **not** first countable (so in particular not second countable or metrizable), nor complete, so it is not a Fréchet space.

Example 3.1. Let $X = \mathbb{N}$ with discrete topology. Then $C(X)$ is the space of complex sequences $\mathbf{a} = \{a_n, n \in \mathbb{N}\}$ with topology defined by the seminorms $\nu_n(\mathbf{a}) := |a_n|, n \in \mathbb{N}$ (i.e., topology of termwise convergence). So $C(X)^* = \mathrm{Meas}_c(X)$ is the space of *eventually vanishing* complex sequences $\mathbf{f} = \{f_n, n \in \mathbb{N}\}$ (acting on $C(X)$ by $\mathbf{f}(\mathbf{a}) = \sum_{n\in\mathbb{N}} f_n a_n$) with topology having base of neighborhoods of zero consisting of finite intersections of the sets $U_{\mathbf{a},\varepsilon} = \{\mathbf{f} \in C(X)^* : |\mathbf{f}(\mathbf{a})| < \varepsilon\}$, $\mathbf{a} \in C(X)$, $\varepsilon > 0$. This space has a basis $\{\mathbf{e}_m, m \geq 0\}$ given by $(\mathbf{e}_m)_n := \delta_{mn}$, i.e., it is countable-dimensional.

We have $\mathbf{f}_n \to 0$ in $C(X)^*$ iff all $\mathbf{f}_n$ are supported on some finite set $S \subset \mathbb{N}$ and for all $j \in S$, $f_{nj} \to 0$. This implies that $C(X)^*$ is not first countable (hence all the more not second countable and not metrizable). Indeed if $W_m, m \geq 1$ were a basis of neighborhoods of zero then by replacing W_m by $W_1 \cap ... \cap W_m$ we can ensure that $W_1 \supset W_2 \supset ....$ Assuming this is the case, pick $N_m \geq 1$ such that the sequence $\mathbf{a}_m := \frac{\mathbf{e}_m}{N_m}$ belongs to W_m (it exists since for each m, $\frac{\mathbf{e}_m}{N} \to 0$ in

[9] If X is compact then $C(X)$ is a Banach space and thus so is $\mathrm{Meas}_c(X)$, in the corresponding norm topology. However, this norm topology is stronger than the weak topology and is not relevant here.

$C(X)^*$ as $N \to \infty$). Then the sequence $\{\mathbf{a}_m, m \geq 1\}$ does not converge to 0, and yet for each m, $\mathbf{a}_j \in W_m$ for all $j \geq m$, contradiction.

Also $C(X)^*$ is not complete, as it is a dense subspace of the space $C(X)^*_{\rm alg}$ of all (not necessarily continuous) linear functionals on $C(X)$ (uncountable-dimensional, hence bigger than $C(X)^*$), from which it inherits the weak topology. On the other hand, it is sequentially complete. Indeed, if $\{\mathbf{f}_n\}$ is a Cauchy sequence in $\mathbb{C}(X)^*$ then $\mathbf{f}_n - \mathbf{f}_{n+1}$ goes to 0 as $n \to \infty$, so for some N and $n \geq N$, $\mathbf{f}_n - \mathbf{f}_{n+1}$ is supported on some finite set $S \subset \mathbb{N}$. Hence for all n, $\mathbf{f}_n$ is supported on the union of S and the supports of $\mathbf{f}_i$, $1 \leq i \leq N$, which is a finite set. Hence it converges (as it is Cauchy). Also, the countable set $C(X)^*_{\rm rat}$ of eventually vanishing sequences of Gaussian rationals is dense in $C(X)^*$, so $C(X)^*$ is separable.

Thus we see that $C(X)^* = \bigcup_{i\geq 1} C(K_i)^*$ as a vector space, or, without making any choices, $C(X)^* = \varinjlim_{K\subset X} C(K)^*$, where K runs over compact subsets of X.

Pick a representation of X as a nested union of compact subsets $K_i, i \geq 1$. We claim that for any $\mu \in C(X)^*$ there exists i such that if $f \in C(X)$ satisfies $f|_{K_i} = 0$ then $\mu(f) = 0$. Indeed, if not then for each i there is $f_i \in C(X)$ with $f_i|_{K_i} = 0$ but $\mu(f_i) = 1$. Then the series $\sum_i f_i$ converges in $C(X)$ (as it terminates on each K_i, and every compact subset of X is contained in some K_i) while the series $\mu(\sum_i f_i) = \sum_i \mu(f_i) = \sum_i 1$ diverges, a contradiction. Thus we see that $C(X)^* = \bigcup_{i\geq 1} C(K_i)^*$ as a vector space, or, without making any choices, $C(X)^* = \varinjlim_{K\subset X} C(K)^*$, where K runs over compact subsets of X.

Lemma 3.2. *(i) If a sequence $\{\mu_n, n \geq 1\} \in C(X)^*$ is Cauchy then there is a compact subset $K \subset X$ such that $\mu_n \in C(K)^* \subset C(X)^*$ for all n.*

(ii) $C(X)^*$ is sequentially complete.

Proof. (i) Otherwise for each $j \geq 1$ there exists the largest positive integer $N_j \geq 0$ such that if $f \in C(X)$ and $f|_{K_j} = 0$ then $\mu_1(f) = ... = \mu_{N_j-1}(f) = 0$. The numbers N_j form a nondecreasing sequence, and since $C(X)^* = \bigcup_{i\geq 1} C(K_i)^*$, we have $N_j \to \infty$. So let $p(j) \geq j$ be the largest i for which $N_i = N_j$. By assumption, for every $j \geq 1$ there is $f_j \in C(X)$ with $f_i|_{K_j} = 0$ and $\mu_{N_j}(f_j) \neq 0$. Then we can arrange that $\mu_{N_j}(f_1 + ... + f_j) = j$, and $\mu_{N_j}(f_i) = 0$ if $i > p(j)$. Now, the series $f := \sum_{i\geq 1} f_i$ converges in $C(X)$, and we have

$$\mu_{N_j}(f) = \mu_{N_j}(f_1 + ... + f_{p(j)}) = \mu_{N_{p(j)}}(f_1 + ... + f_{p(j)}) = p(j).$$

On the other hand, since μ_n is Cauchy, we get

$$p(j+1)-p(j)=\mu_{N_{j+1}}(f)-\mu_{N_j}(f)\to 0,\ j\to\infty,$$

a contradiction since $p(j)\ge j$.

(ii) Let $\{\mu_n, n\ge 1\}$ be a Cauchy sequence in $C(X)^*$. By (i), $\mu_n\in C(K)^*$ for some compact $K\subset X$, so we may assume that X is compact. Since μ_n is Cauchy, so is $\mu_n(f)$ for any $f\in C(X)$. Thus μ_n weakly converges to some linear functional $\mu: C(X)\to\mathbb{C}$ given by $\mu(f):=\lim_{n\to\infty}\mu_n(f)$, and our job is to show, that μ is continuous. Since $\mu_n(f)$ is convergent, it is bounded, so by the uniform boundedness principle, the sequence $\|\mu_n\|$ is bounded above by some constant C, i.e., $|\mu_n(f)|\le C\|f\|$. But then $|\mu(f)|\le C\|f\|$, so $\|\mu\|\le C$, as desired. $\square$

3.2. **Support of a measure.** Define the **support** of $\mu\in C(X)^*$, denoted $\mathrm{supp}\mu$, to be the set of all $x\in X$ such that for any neighborhood U of x in X there exists $f\in C(X)$ vanishing outside U with $\mu(f)\neq 0$. Thus the complement $(\mathrm{supp}\mu)^c$ is the set of $x\in X$ which admit a neighborhood U such that for every $f\in C(X)$ vanishing outside U we have $\mu(f)=0$. In this case, $U\subset(\mathrm{supp}\mu)^c$, so $(\mathrm{supp}\mu)^c$ is open, hence $\mathrm{supp}\mu$ is closed. Moreover, since $C(X)^*=\varinjlim_{K\subset X}C(K)^*$, $\mathrm{supp}\mu$ is contained in some compact subset $K\subset X$, so it is itself compact.

Proposition 3.3. *If $f\in C(X)$ and $f|_{\mathrm{supp}\mu}=0$ then $\mu(f)=0$.*

Proof. For every $z\in(\mathrm{supp}\mu)^c$ there is a neighborhood $U_z\subset(\mathrm{supp}\mu)^c$ such that for any $\phi\in C(X)$ vanishing outside U_z, $\mu(\phi)=0$. These neighborhoods form an open cover of $(\mathrm{supp}\mu)^c$. Since $(\mathrm{supp}\mu)^c$ is second countable, this cover has a countable subcover $\{U_i, i\in\mathbb{N}\}$. Let $\{\phi_i, i\in\mathbb{N}\}$ be a continuous partition of unity subordinate to this cover. Then $\mu(\phi_i f)=0$ for all i, so $\mu(f)=\mu(\sum_i\phi_i f)=\sum_i\mu(\phi_i f)=0$, as claimed. $\square$

3.3. **Finitely supported measures.** A basic example of an element of $\mathrm{Meas}_c(X)$ is a **Dirac measure** δ_a, $a\in X$, such that $\delta_a(f)=f(a)$. Thus if $a_n\to a$ in X as $n\to\infty$ then $\delta_{a_n}\to\delta_a$ in the weak topology. A finite linear combination of Dirac measures is called a **finitely supported** measure, since such measures are exactly the measures with finite support. The subspace of finitely supported measures is denoted $\mathrm{Meas}_c^0(X)$.

Lemma 3.4. $\mathrm{Meas}_c^0(X)$ *is a sequentially dense (in particular, dense) subspace in* $\mathrm{Meas}_c(X)$*, i.e., every element $\mu\in\mathrm{Meas}_c(X)$ is the limit of a sequence $\mu_n\in\mathrm{Meas}_c^0(X)$ in the weak topology.*

Proof. By replacing X with suppμ, we may assume that X is compact. For every $n \geq 1$, let X_n be a finite subset of X such that the open balls $B(x, \frac{1}{n})$ around $x \in X_n$ cover X. Let $\{\phi_{nx}, x \in X_n\}$ be a continuous partition of unity subordinate to this cover, and let

$$\mu_n := \sum_{x \in X_n} \mu(\phi_{nx})\delta_x \in \mathrm{Meas}_c^0(X).$$

We claim that $\mu_n \to \mu$ in the weak topology.

Indeed, let $f \in C(X)$. Then

$$|\mu_n(f) - \mu(f)| = |\mu(\sum_{x \in X_n} \phi_{nx}(f - f(x))| \leq \|\mu\| \sup_{y \in X} \sum_{x \in X_n} \phi_{nx}(y)|f(y) - f(x)|.$$

But f is uniformly continuous, so for every $\varepsilon > 0$ there is N such that if $d(x,y) < \frac{1}{N}$ then $|f(x) - f(y)| < \varepsilon$. So for $n \geq N$, whenever $\phi_{nx}(y) \neq 0$, we have $|f(y) - f(x)| < \varepsilon$. Thus we get

$$|\mu_n(f) - \mu(f)| \leq \varepsilon \|\mu\| \sup_{y \in X} \sum_{x \in X_n} \phi_{nx}(y) = \varepsilon \|\mu\|,$$

which implies the desired statement. □

Note that since X is separable, so is $\mathrm{Meas}_c^0(X)$ (given a countable dense subset $T \subset X$, finitely supported measures with support in T and Gaussian rational coefficients form a countable, sequentially dense subset $E_T \subset \mathrm{Meas}_c^0(X)$). Thus we get that $\mathrm{Meas}_c(X)$ is separable; moreover, since E_T is *sequentially* dense in $\mathrm{Meas}_c(X)$, the latter is *sequentially separable.*

Corollary 3.5. *If X, Y are locally compact second countable Hausdorff spaces then the natural bilinear map*

$$\boxtimes : \mathrm{Meas}_c^0(X) \times \mathrm{Meas}_c^0(Y) \to \mathrm{Meas}_c(X \times Y)$$

uniquely extends to a bilinear map

$$\boxtimes : \mathrm{Meas}_c(X) \times \mathrm{Meas}_c(Y) \to \mathrm{Meas}_c(X \times Y)$$

which is continuous in each variable.

Proof. It is clear that $\boxtimes$ is continuous in each variable, so the result follows from the facts that $\mathrm{Meas}_c^0(X)$ is sequentially dense in $\mathrm{Meas}_c(X)$ and that $\mathrm{Meas}_c(X)$ is sequentially complete. □

Remark 3.6. Here is another proof of Corollary 3.5. We may assume that X, Y are compact. Given $\mu \in C(X)^*, \nu \in C(Y)^*$, define a linear functional $\mu \boxtimes \nu$ on $C(X) \otimes C(Y) \subset C(X \times Y)$ by

$$(\mu \boxtimes \nu)(f \otimes g) := \mu(f)\nu(g).$$

We claim that $\|\mu \boxtimes \nu\| \le \|\mu\| \, \|\nu\|$ (in fact, the opposite inequality is obvious, so we have an equality). Thus our job is to show that for $f_i \in C(X), g_i \in C(Y)$, $1 \le i \le n$, we have

$$|\sum_i \mu(f_i)\nu(g_i)| \le \|\mu\| \, \|\nu\| \max_{x\in X, y \in Y} |\sum_i f_i(x)g_i(y)|$$

i.e., that

$$|\nu(\sum_i \mu(f_i)g_i)| \le \|\mu\| \, \|\nu\| \max_{x\in X, y \in Y} |\sum_i f_i(x)g_i(y)|.$$

To this end, it suffices to show that

$$\max_{y\in Y} |\sum_i \mu(f_i)g_i(y)| \le \|\mu\| \max_{x\in X, y \in Y} |\sum_i f_i(x)g_i(y)|,$$

which would follow from the inequality

$$|\sum_i \mu(f_i)g_i(y)| \le \|\mu\| \max_{x\in X} |\sum_i f_i(x)g_i(y)|.$$

for all $y \in Y$. But this is just the inequality $|\mu(F_y)| \le \|\mu\| \max_{x\in X} |F_y(x)|$ applied to $F_y(x) := \sum_i g_i(y)f_i(x)$.

Now note that by the Stone-Weierstrass theorem, $C(X) \otimes C(Y)$ is dense in $C(X \times Y)$, so $\mu \boxtimes \nu$ extends continuously to $C(X \times Y)$.

3.4. **The algebra of measures on a locally compact group.** Now let G be a locally compact group. In this case $\mathrm{Meas}_c^0(G) = \mathbb{C}G$ is the group algebra of G as an abstract group. Namely, the algebra structure is given by $\delta_x\delta_y = \delta_{xy}$. This multiplication is continuous in the weak topology, hence uniquely extends to $\mathrm{Meas}_c(G)$, since the latter is sequentially complete and $\mathrm{Meas}_c^0(G)$ is sequentially dense in $\mathrm{Meas}_c(G)$. Thus $\mathrm{Meas}_c(G)$ is a topological unital associative algebra with unit δ_1. The multiplication in this algebra may be written as

$$(\mu_1 * \mu_2)(f) = (\mu_1 \boxtimes \mu_2, \Delta(f)) = \int_{G\times G} f(xy) d\mu_1(x) d\mu_2(y),$$

where $\Delta : C(G) \to C(G \times G)$ is given by $\Delta(f)(x,y) := f(xy)$. This multiplication is called the **convolution product**.

Moreover, if dg is a right-invariant Haar measure on G then any compactly supported continuous function (or, more generally, L^1-function) ϕ on G gives rise to a measure $\mu = \phi dg \in \mathrm{Meas}_c(G)$. For such measures $\mu_1 = \phi_1 dg, \mu_2 = \phi_2 dg$ we have

$$(\mu_1 * \mu_2)(f) = \int_{G\times G} f(xy)\phi_1(x)\phi_2(y) dx dy = \int_{G\times G} f(z)\phi_1(zy^{-1})\phi_2(y) dz dy.$$

Thus $\mu_1 * \mu_2 = \phi dg$ where

$$\phi(z) = \int_G \phi_1(zy^{-1})\phi_2(y)dy.$$

This is called the **convolution of functions**.

Now let V be a continuous representation of G with the associated homomorphism $\pi : G \to \mathrm{Aut}(V)$. This map π extends by linearity to a homomorphism $\pi : \mathbb{C}G = \mathrm{Meas}_c^0(G) \to \mathrm{End}(V)$.

Let us equip $\mathbb{C}G$ with weak topology and introduce the corresponding product topology on $\mathbb{C}G \times V$.

Lemma 3.7. *The map $\mathbb{C}G \times V \to V$ given by $g \mapsto \pi(g)v$ is continuous. Thus π is continuous in the weak topology of $\mathbb{C}G$ and strong topology of* $\mathrm{End}(V)$.

Proof. We need to show that for any seminorm λ (from the family defining the topology of V) there exists a neighborhood U of 0 in the space $\mathbb{C}G \times V$ such that for $(\mu, v) \in U$ we have $\lambda(\pi(\mu)v) < 1$. Let $\mu = \sum_{i=1}^n c_i\delta_{x_i}$, then this inequality takes the form

$$\sum_{i=1}^n \lambda(c_i\pi(x_i)v) < 1. \tag{1}$$

Since λ is a seminorm, (1) would follow from the inequality

$$\sum_{i=1}^n |c_i|\lambda(\pi(x_i)v) < 1. \tag{2}$$

We define $|\mu| = \sum_{i=1}^n |c_i|\delta_{x_i}$ and $f_v(x) := \lambda(\pi(x)v)$, $f_v \in C(X)$. Then (2) takes the form

$$|\mu|(f_v) < 1. \tag{3}$$

Clearly, the map $(\mu, v) \mapsto |\mu|(f_v)$ is continuous, so we may take U to be defined by (3). $\square$

Corollary 3.8. *If (V, π) is a continuous representation of G then π the action $G \times V \to V$ uniquely extends to a continuous bilinear map $\mathrm{Meas}_c(G) \times V \to V$, which gives rise to a continuous unital algebra homomorphism*

$$\pi : \mathrm{Meas}_c(G) \to \mathrm{End}(V).$$

Proof. We need to show that for every $v \in V$ the map $\mu \mapsto \pi(\mu)v$ uniquely extends by continuity from $\mathrm{Meas}_c^0(G)$ to $\mathrm{Meas}_c(G)$. This follows from Lemmas 3.4 and 3.7 since V is complete. $\square$

18.757 Representations of Lie Groups
Fall 2023