4. **Plancherel formulas, Dirac sequences, smooth vectors**

4.1. **Plancherel formulas.** For a compactly supported L^1-function f on G, for brevity let us denote $\pi(fdg)$ just by $\pi(f)$.

Proposition 4.1. *(Plancherel's theorem for compact groups) Let K be a compact group and $f_1, f_2 \in L^2(K)$. Then*

$$(f_1, f_2) = \sum_{\rho \in \mathrm{Irr} K} \dim \rho \cdot \mathrm{Tr}(\pi_\rho(f_1)\pi_\rho(f_2)^\dagger)$$

and this series is absolutely convergent.

Proof. Recall that if e_i is an orthonormal basis of a Hilbert space H and $f_1, f_2 \in H$ then

$$(f_1, f_2) = \sum_i (f_1, e_i)(e_i, f_2)$$

and this series is absolutely convergent. The result now follows by applying this formula to the orthonormal basis provided by the Peter-Weyl theorem:

$$\psi_{\rho ij} = \sqrt{\dim \rho}(\pi_\rho(g)v_{\rho i}, v_{\rho j}),$$

where $\{v_{\rho i}\}$ is an orthonormal basis of ρ. □

Example 4.2. If $K = S^1$, Plancherel's theorem reduces to the usual Parseval equality in Fourier analysis:

$$(f_1, f_2) = \sum_{n \in \mathbb{Z}} c_n(f_1)\overline{c_n(f_2)},$$

where $c_n(f)$ are the Fourier coefficients of f.

Proposition 4.3. *(Plancherel's formula) If K is a compact Lie group and $f \in C^\infty(K)$ then*

$$f(1) = \sum_{\rho \in \mathrm{Irr} K} \dim \rho \cdot \mathrm{Tr}(\pi_\rho(f))$$

and this series is absolutely convergent.

Example 4.4. If $K = S^1$ then this formula says that for $f \in C^\infty(S^1)$

$$f(1) = \sum_{n \in \mathbb{Z}} c_n(f),$$

i.e., the Fourier series of f absolutely converges at 1. Note that for $f \in C(S^1)$ this is false in general![10]

[10] One can show that for an N-dimensional group, the differentiability needed for the Plancherel formula is C^k for $k > N/2$.

Proof. Consider the integral operator A of convolution with the function f:

$$(A\psi)(x) = (f * \psi)(x) = \int_K f(xy^{-1})\psi(y)dy.$$

This operator is trace class, since it has smooth integral kernel $F(x, y) = f(xy^{-1})$, and

$$\mathrm{Tr}(A) = \int_K F(x, x)dx = \int_K f(1)dx = f(1).$$

On the other hand, A is right-invariant, so it preserves the decomposition of $L^2(K)$ into the direct sum of $\rho \otimes \rho^*$ and acts on each such summand as $\pi_\rho(f) \otimes 1$. Thus we also have

$$\mathrm{Tr}(A) = \sum_{\rho \in \mathrm{Irr} K} \dim(\rho) \cdot \mathrm{Tr}(\pi_\rho(f)),$$

as desired. □

4.2. **Dirac sequences.** If G is a locally compact group then multiplication by dg defines an inclusion $C_c(G) \hookrightarrow \mathrm{Meas}_c(G)$ of compactly supported continuous functions into compactly supported measures as a (non-unital) subalgebra. Moreover, if G is a Lie group then we have a nested sequence of subalgebras $C_c^k(G)$, $0 \le k \le \infty$ (compactly supported C^k-functions). The following lemma shows that while these subalgebras are non-unital, they are "almost unital".

Lemma 4.5. *There exists a sequence $\phi_n \in C_c(G)$ such that $\phi_n \to \delta_1$ in the weak topology as $n \to \infty$. Moreover, if G is a Lie group, we can choose $\phi_n \in C_c^\infty(G)$.*

Proof. (sketch) ϕ_n can be constructed as a sequence of "hat" functions supported on a decreasing sequence of balls $B_1 \supset B_2 \supset ...$ whose intersection is $1 \in G$. Such hat functions can be chosen smooth if G is a Lie group. □

Such sequences ϕ_n are called **Dirac sequences**.

Corollary 4.6. *$C_c(G)$ is sequentially dense in* $\mathrm{Meas}_c(G)$. *For Lie groups, $C_c^\infty(G)$ is sequentially dense in* $\mathrm{Meas}_c(G)$.

Proof. By translating a Dirac sequence, for any $g \in G$ we can construct a sequence $\psi_n \to \delta_g$. This implies that $\mathrm{Meas}_c^0(G)$ is contained in the sequential closure of $C_c(G)$ (and of $C_c^\infty(G)$ in the Lie case). So the result follows from Lemma 3.4. □

4.3. **Density of K-finite vectors.**

Corollary 4.7. *Let V be a continuous representation of a compact group K. Then V^{fin} is dense in V.*

Proof. Let $v \in V$, and $\phi_n \to \delta_1$ a continuous Dirac sequence, which exists by Lemma 4.5. Then $\pi(\phi_n)v \to v$ as $n \to \infty$. But $\phi_n \in L^2(K)$, so by the Peter-Weyl theorem, there exists $\psi_n \in L^2(K)^{\text{fin}} = \oplus_\rho \rho^* \otimes \rho$ such that

$$\|\psi_n - \phi_n\|_2 < \frac{1}{n}.$$

Then $\psi_n - \phi_n \to 0$ in $L^2(K)$, hence in $\text{Meas}_c(K)$. So by Corollary 3.8, $\pi(\psi_n - \phi_n)v \to 0$ as $n \to \infty$. It follows that $\pi(\psi_n)v \to v$ as $n \to \infty$. But $\pi(\psi_n)v \in V^{\text{fin}}$. □

Corollary 4.8. *$L^2(K)^{\text{fin}} \subset C(K)$ is a dense subspace. Moreover, if K is a Lie group then $L^2(K)^{\text{fin}} \subset C^k(K)$ is a dense subspace for $0 \le k \le \infty$.*

Proof. The claimed inclusions follow since matrix coefficients of finite-dimensional representations of K are continuous, and moreover C^∞ in the case of Lie groups. The density then follows from Corollary 4.7. □

Corollary 4.9. *If V is an irreducible continuous representation of K then V is finite-dimensional.*

Proof. By Corollary 4.7, V^{fin} is dense in V. Hence $V^{\text{fin}} \neq 0$. Let ρ be a finite-dimensional subrepresentation of V^{fin}. Then ρ is a closed invariant subspace of V. Hence $\rho = V$. □

4.4. **Smooth vectors.** Let G be a Lie group. As we have noted in Subsection 1.2, any continuous *finite-dimensional* representation $\pi : G \to \text{Aut}(V)$ is automatically smooth and thereby defines a representation $\pi_* : \mathfrak{g} \to \text{End}(V)$ of the corresponding Lie algebra, which determines π if G is connected. Moreover, if G is simply connected, this correspondence is an equivalence of categories. This immediately reduces the problem to pure algebra and is the main tool of studying finite-dimensional representations of Lie groups.

We would like to have a similar theory for infinite-dimensional representations. But in the infinite-dimensional setting the above statements don't hold in the literal sense.

Example 4.10. Consider the action of S^1 on $L^2(S^1)$. Then the Lie algebra should act by $\frac{d}{d\theta}$. But this operator does not act on $L^2(S^1)$. The largest subspace of $L^2(S^1)$ preserved by this operator (acting on distributions on S^1) is $C^\infty(S^1)$.

This motivates the notion of a **smooth vector** in a continuous representation of a Lie group. To define this notion, for a manifold X and a topological vector space V, denote by $C^\infty(X,V)$ the space of smooth maps $X \to V$ (where smooth maps are defined in the same way as in the case of finite-dimensional V).

Definition 4.11. Let (V,π) be a continuous representation of a Lie group G. A vector $v \in V$ is called **smooth** if the map $G \to V$ given by $g \mapsto \pi(g)v$ is smooth, i.e., belongs to $C^\infty(G,V)$. The space of smooth vectors is denoted by V^∞.

It is clear that $V^\infty \subset V$ is a G-invariant subspace (although not a closed one).

Example 4.12. For the representation of a compact Lie group K on $V = L^2(K)$, we have $V^\infty = C^\infty(K)$.

Proposition 4.13. *Let (V,π) be a continuous representation of a Lie group G with $\mathfrak{g} = \mathrm{Lie}(G)$. Let $v \in V^\infty$. Then we have a linear map $\pi_{*,v} : \mathfrak{g} \to V^\infty$ given by*

$$\pi_{*,v}(b) = \frac{d}{dt}|_{t=0}\pi(e^{tb})v.$$

This defines a Lie algebra homomorphism $\pi_* : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V^\infty)$ (algebra of all linear endomorphisms of V^∞) given by $\pi_*(b)v := \pi_{*,v}(b)$.

Exercise 4.14. Prove Proposition 4.13.

Proposition 4.15. *(i) V^∞ is dense in V.*
(ii) $V^{\mathrm{fin}} \subset V^\infty$.

Proof. (i) Let $\phi_n \to \delta_1$ be a smooth Dirac sequence. Then $\pi(\phi_n)v \to v$ as $n \to \infty$. But it is easy to see that $\pi(\phi_n)v \in V^\infty$.

(ii) This follows since matrix coefficients of finite-dimensional representations are smooth. $\square$

18.757 Representations of Lie Groups
Fall 2023