5. **Admissible representations and $(\mathfrak{g}, K)$-modules**

5.1. **Admissible representations.** Now let G be a Lie group and $K \subset G$ a compact subgroup. For a continuous representation V of G, denote by $V^{K\text{-fin}}$ the space $(V|_K)^{\text{fin}}$. In general $V^{K\text{-fin}}$ is not contained in V^∞; for example, if $K = 1$ then $V^{K\text{-fin}} = V$. However, this inclusion holds if K is sufficiently large and V is sufficiently small.

Definition 5.1. V is said to be K**-admissible** (or of **finite** K**-type**) if for every finite-dimensional irreducible representation ρ of K, the space $\mathrm{Hom}_K(\rho, V)$ is finite-dimensional.

Example 5.2. Let G be a connected Lie group and $V = L^2(G/B)$ where B is a closed subgroup of G (half-densities on G/B). Then V is K-admissible iff K acts transitively on G/B, i.e., $KB = G$. In this case setting $T = K \cap B$, we have $G/B = K/T$, so $V = L^2(K/T)$ and $\mathrm{Hom}_K(\rho, V) \cong (\rho^T)^*$.[11]

Example 5.3. For $G = SL_2(\mathbb{C})$ and $K = SU(2)$, the unitary representation of G on the space $V = L^2(\mathbb{CP}^1)$ of square-integrable half-densities on $\mathbb{CP}^1$ is K-admissible. Indeed, taking ρ_n to be the representation of $SU(2)$ with highest weight n, we have $\dim \mathrm{Hom}(\rho_n, V) = 0$ for odd n and 1 for even n.

More generally, for a real number s we may consider the representation V_s of square integrable $\frac{1}{2}+is$-densities on $\mathbb{CP}^1$; this space is canonically defined since for a $\frac{1}{2}+is$-density f, the complex conjugate $\overline{f}$ is a $\frac{1}{2}-is$-density, so $|f|^2 = f\overline{f}$ is a density and can be integrated canonically over $\mathbb{CP}^1$. This representation has the same K-multiplicities as $V = V_0$.

Similarly, for $G = SL_2(\mathbb{R})$, $K = SO(2)$, we have a unitary K-admissible representation $V = L^2(\mathbb{RP}^1)$ (half-densities) and more generally V_s ($\frac{1}{2}+is$-densities). For the K-multiplicities we have equalities $\dim \mathrm{Hom}(\chi_n, V_s) = 1$ for odd n and 0 for even n, where $\chi_n(\theta) = e^{in\theta}$.

We will see that the representations V_s in both cases are irreducible and V_s, V_t are isomorphic iff $s = \pm t$. The family of representations V_s is called the **unitary spherical principal series**.

[11] Note that here we don't have to distinguish between half-densities and functions on K/T since K/T always has a K-invariant volume form as K is compact.

Note that this family makes sense also when s is a complex number which is not necessarily real. In this case V_s is not necessarily unitary but still a continuous representation on square integrable $\frac{1}{2}+is$-densities. The space of such densities is canonically defined as a topological vector space, although its Hilbert norm is not canonically defined unless s is real (however, we will see that for some non-real s, corresponding to so-called **complementary series**, this representation is still unitary, even though the inner product is not given by the standard formula). The family V_s with arbitrary complex s is called the **spherical principal series**.

Explicitly, the action of G on V_s looks as follows (realizing elements of V_s as functions on $\mathbb{R}$ or $\mathbb{C}$, removing the point at infinity):

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} f\right)(z) = f\left(\frac{az+b}{cz+d}\right)|cz+d|^{-m(1+2is)},$$

where $m = 1$ in the real case and $m = 2$ in the complex case.

Proposition 5.4. *If V is K-admissible then $V^{K\text{-fin}} \subset V^\infty$, and it is a $\mathfrak{g}$-submodule (although not in general a G-submodule).*

Proof. For a finite-dimensional irreducible representation ρ of K, let $V^\rho := \mathrm{Hom}(\rho, V) \otimes \rho$ be the isotypic component of ρ.

We claim that for any continuous representation V the space $V^\infty \cap V^\rho$ is dense in V^ρ. Indeed, let $\psi_\rho \in L^2(K)^{\mathrm{fin}}$ be the character of ρ given by

$$\psi_\rho = \sum_i \psi_{\rho ii}.$$

Let ξ_ρ be the pushforward of $\psi_\rho dx$ from K to G (a measure on G supported on K). Then $\pi(\xi_\rho)$ is the projector to V^ρ annihilating $\overline{\oplus_{\eta\neq\rho} V^\eta}$. Let $\phi_n \to \delta_1$ be a smooth Dirac sequence on G. Then for $v \in V^\rho$,

$$\pi(\xi_\rho * \phi_n)v = \pi(\xi_\rho)\pi(\phi_n)v \to \pi(\xi_\rho)v = v$$

as $n \to \infty$. However, $\xi_\rho * \phi_n$ is smooth, so $\pi(\xi_\rho * \phi_n)v \in V^\infty \cap V^\rho$.

Thus if V^ρ is finite-dimensional (which happens for K-admissible V) then $V^\infty \cap V^\rho = V^\rho$, so $V^\rho \subset V^\infty$. Hence $V^{K\text{-fin}} \subset V^\infty$.

Finally, it is clear that for $b \in \mathfrak{g}$ and $v \in V^\rho$, the vector bv generates a K-submodule of a multiple of $\mathfrak{g} \otimes \rho$, so $bv \in V^{K\text{-fin}}$. It follows that $V^{K\text{-fin}}$ is a $\mathfrak{g}$-submodule. □

Example 5.5. If $G = SL_2(\mathbb{R})$, $K = SO(2)$, $V = V_s = L^2(S^1)$ is a spherical principal series representation, then $V^{K\text{-fin}}$ is the space of trigonometric polynomials. Note that this space is *not* invariant under the action of G. However, the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{R})$ does act on this space.

Exercise 5.6. Compute this Lie algebra action in the basis $v_n = e^{in\theta}$ and write it as first order differential operators in the angle θ. (Pick generators e, h, f in $\mathfrak{g}_{\mathbb{C}}$ so that h acts diagonally in the basis v_i).

5.2. $(\mathfrak{g}, K)$**-modules.** This motivates the following definition. Let K be a compact connected Lie group and $\mathfrak{k} = \mathrm{Lie}K$. Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra containing $\mathfrak{k}$, and suppose the adjoint action of $\mathfrak{k}$ on $\mathfrak{g}$ integrates to an action of K. In this case we say that $(\mathfrak{g}, K)$ is a **Harish-Chandra pair**.

Definition 5.7. Let $(\mathfrak{g}, K)$ be a Harish-Chandra pair.

(i) A $(\mathfrak{g}, K)$-**module** is a vector space M with actions of K and $\mathfrak{g}$ such that

- M is a direct sum of finite-dimensional continuous K-modules;
- the two actions of $\mathfrak{k}$ on M (coming from the actions of $\mathfrak{g}$ and K) coincide.

(ii) Such a module is said to be **admissible** if for every $\rho \in \mathrm{Irr}K$ we have $\dim \mathrm{Hom}_K(\rho, M) < \infty$.

(iii) An admissible $(\mathfrak{g}, K)$-module which is finitely generated over $U(\mathfrak{g})$ is called a **Harish-Chandra module**.

Exercise 5.8. Show that if M is a $(\mathfrak{g}, K)$-module then for every $g \in K, a \in \mathfrak{g}, v \in M$ we have

$$gav = \mathrm{Ad}(g)(a)gv,$$

where Ad denotes the K-action on $\mathfrak{g}$.

In fact, a $(\mathfrak{g}, K)$-module is a purely algebraic object, since finite-dimensional K-modules can be described as algebraic representations of the complex reductive group $K_{\mathbb{C}}$. Moreover, we can represent them even more algebraically in terms of the action of $\mathfrak{k}$. Namely, let us say that a finite-dimensional representation of $\mathfrak{k}$ is **integrable** to K if it corresponds to a representation of K (note that this is automatic if K is simply connected). Then $(\mathfrak{g}, K)$-modules are simply $\mathfrak{g}$-modules which are locally integrable to K when restricted to $\mathfrak{k}$ (i.e., sum of integrable modules). So if K is simply connected (in which case $\mathfrak{k}$ is semisimple) then a $(\mathfrak{g}, K)$-module is the same thing as a $\mathfrak{g}$-module which is locally finite when restricted to $\mathfrak{k}$ (i.e., sum of finite-dimensional modules).

Thus $(\mathfrak{g}, K)$-modules form an abelian category closed under extensions (and this category can be defined over any algebraically closed field of characteristic zero). The same applies to admissible $(\mathfrak{g}, K)$-modules and to Harish-Chandra modules (the latter is closed under taking kernels of morphisms because the algebra $U(\mathfrak{g})$ is Noetherian, as so is its associated graded $S\mathfrak{g}$ by the Hilbert basis theorem).

Example 5.9. Let G be a connected complex semisimple Lie group. Then its maximal compact subgroup is the compact form $K = G_c$. Thus a $(\mathfrak{g}, K)$-module is a $\mathfrak{g}$-module M which is locally finite for $\mathfrak{g}_c \subset \mathfrak{g}$, where $\mathfrak{g}_c = \mathrm{Lie}G_c$. Note that the action of $\mathfrak{g}$ here is only **real linear**. Thus we may pass to complexifications: $(\mathfrak{g}_c)_{\mathbb{C}} = \mathfrak{g}$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \mathfrak{g}$, and $\mathfrak{g}$ sits inside $\mathfrak{g} \oplus \mathfrak{g}$ as the diagonal. Thus a $(\mathfrak{g}, K)$-module is simply a $\mathfrak{g} \oplus \mathfrak{g}$-module which is locally finite for the diagonal copy of $\mathfrak{g}$. This is the same as a $\mathfrak{g}$-bimodule[12] with locally finite adjoint action

$$\mathrm{ad}(b)m := [b, m] = bm - mb.$$

For example, if I is any two-sided ideal in $U(\mathfrak{g})$ then $U(\mathfrak{g})/I$ is a $(\mathfrak{g}, K)$-module.

Thus we obtain the following proposition.

Proposition 5.10. *If V is a K-admissible continuous representation of G then $V^{K\text{-}fin}$ is an admissible $(\mathfrak{g}, K)$-module.*

Exercise 5.11. Show that for any continuous representation V of G, the intersection $V^\infty \cap V^{K\text{-fin}}$ is a $(\mathfrak{g}, K)$-module (not necessarily admissible).

Exercise 5.12. Show that if V is an admissible representation of G and L a finite-dimensional (continuous) representation of G then $V \otimes L$ is also admissible. Prove the same statement for $(\mathfrak{g}, K)$-modules.

5.3. **Harish-Chandra's admissibility theorem.** We will now restrict our attention to **semisimple** Lie groups G. By this we will mean a connected linear real Lie group G with semisimple Lie algebra $\mathfrak{g}$. "Linear" means that it has a faithful finite-dimensional representation, i.e., is isomorphic to a closed subgroup of $GL_n(\mathbb{C})$. In other words, G is the connected component of the identity in $\mathbf{G}(\mathbb{R})$, where $\mathbf{G}$ is a semisimple algebraic group defined over $\mathbb{R}$. Typical examples of such groups include $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$ (in the latter case $\mathbf{G} = SL_n \times SL_n$ and the real structure defined by the involution permuting the two factors).

A fundamental result about the structure of semisimple Lie groups is

Theorem 5.13. *(E. Cartan) Every semisimple Lie group G has a maximal compact subgroup $K \subset G$ which is unique up to conjugation.*

[12]Indeed, every $\mathfrak{g} \oplus \mathfrak{g}$-module M with action $(a, b, v) \mapsto (a, b) \circ v$, $a, b \in \mathfrak{g}$, $v \in M$ is a $\mathfrak{g}$-bimodule with $av = (a, 0) \circ v$ and $vb = (0, -b) \circ v$, and vice versa.

Example 5.14. For $G = SL_n(\mathbb{R})$ we have $K = SO(n)$ and for $G = SL_n(\mathbb{C})$ we have $G = SU(n)$.

We will say that a continuous representation V of G is **admissible** if it is K-admissible with respect to a maximal compact subgroup $K \subset G$ (does not matter which since they are all conjugate).

Theorem 5.15. *(Harish-Chandra's admissibility theorem,* [HC2]*) Every irreducible unitary representation of a semisimple Lie group is admissible.*

We will not give a proof (see [HC2],[Ga]).

Remark 5.16. 1. This theorem extends straightforwardly to the more general case of real reductive Lie groups.

2. Let $G = \widetilde{SL_2}(\mathbb{R})$ be the universal covering of $SL_2(\mathbb{R})$. Then G is not linear (why?) and so it is **not** viewed as a semisimple Lie group according to our definition. In fact, Harish-Chandra's theorem does not hold as stated for this group, since it has no nontrivial compact subgroups. This happens because when we take the universal cover, the maximal compact subgroup $SO(2) = S^1$ gets replaced by the non-compact group $\mathbb{R}$. However, if we take for K the universal cover of $SO(2)$ (even though it is not compact) then Harish-Chandra's theorem extends straightforwardly to this case.

Exercise 5.17. Let M be an admissible $(\mathfrak{g}, K)$-module and

$$M^\vee := \oplus_{V \in \mathrm{Irr} K}(\mathrm{Hom}(V, M) \otimes V)^* \subset M^*$$

be the restricted dual to M. Show that $M^\vee$ has a natural structure of an admissible $(\mathfrak{g}, K)$-module, and $(M^\vee)^\vee \cong M$.