6. **Weakly analytic vectors**

6.1. **Weakly analytic vectors and Harish-Chandra's analyticity.** Let G be a Lie group and V a continuous representation of G.

Definition 6.1. A vector $v \in V$ is called **weakly analytic** if for each $h \in V^*$ the matrix coefficient $h(gv)$ is a real analytic function of g.

Example 6.2. Let $V = L^2(S^1)$ and $G = S^1$ act by rotations. So if $v(x) = \sum_{n\in\mathbb{Z}} v_n e^{inx}$ and $h(x) = \sum_{n\in\mathbb{Z}} h_n e^{-inx}$ then for $g = e^{i\theta}$ we have

$$h(g(\theta)v) = \sum_{n\in\mathbb{Z}} h_n v_n e^{in\theta}.$$

Thus v is a weakly analytic vector iff the sequence $h_n v_n$ decays exponentially for any ℓ_2-sequence $\{h_n\}$, which is equivalent to saying that v_n decays exponentially, i.e., $v(\theta)$ is analytic.

Theorem 6.3. *(Harish-Chandra's analyticity theorem) If V is an admissible representation of a semisimple Lie group G with maximal compact subgroup K then every $v \in V^{K\text{-fin}}$ is a weakly analytic vector.*

Theorem 6.3 is proved in the next two subsections.

6.2. **Elliptic regularity.** The proof of Theorem 6.3 is based on the **analytic elliptic regularity theorem**, which is a fundamental result in analysis (see [Ca]). To state it, let X be a smooth manifold, and $D(X)$ the algebra of (real) differential operators on X. This algebra has a filtration by order: $D_0(X) = C^\infty(X) \subset D_1(X) \subset ...$, such that

$$D_n(X) = \{D \in \mathrm{End}_{\mathbb{C}} C^\infty(X) : [D, f] \in D_{n-1}(X) \forall f \in C^\infty(X)\},\ n \geq 1,$$

and $\mathrm{gr}D(X) = \oplus_{n\geq 0}\Gamma(X, S^nTX)$, where Γ takes sections of the vector bundle. Thus for every differential operator D on X of order n we have its **symbol** $\sigma(D) \in \mathrm{gr}_n D(X) = \Gamma(X, S^nTX)$. For every $x \in X$, $\sigma(D)_x$ is thus a homogeneous polynomial of degree n on T_x^*X.

Definition 6.4. We say that D is **elliptic** at x if $\sigma(D)_x(p) \neq 0$ for nonzero $p \in T_x^*X$. We say that D is **elliptic** (on X) if it is elliptic at all points $x \in X$.

Example 6.5. 1. If $\dim X = 1$ then any differential operator with nonvanishing symbol is elliptic.

2. Fix a Riemannian metric on X and let Δ be the corresponding Laplace operator, $\Delta f = \mathrm{div}(\mathrm{grad} f)$. Then Δ is elliptic.

3. If D is elliptic then for any nonzero polynomial $P \in \mathbb{R}[t]$ the operator $P(D)$ is elliptic.

Note that ellipticity is an open condition, since it is equivalent to non-vanishing of $\sigma(D)_x$ on the unit sphere in T_x^*X (under some inner product). Thus the set of $x \in X$ on which a given operator D is elliptic is open in X.

Theorem 6.6. *(Elliptic regularity) Suppose D is an elliptic operator with real analytic coefficients on an open set $U \subset \mathbb{R}^N$, and $f(x)$ is a smooth solution of the PDE*

$$Df = 0$$

on U. Then f is real analytic on U.

Corollary 6.7. *Let X be a real analytic manifold and D an elliptic operator on X with analytic coefficients. Then every smooth solution of the equation $Df = 0$ on X is actually real analytic.*

Remark 6.8. 1. This is, in fact, true much more generally, when f is a weak (i.e., distributional) solution of the equation $Df = 0$. Also the equation $Df = 0$ can be replaced by a more general inhomogeneous equation $Df = g$, where g is analytic.

2. If D is not elliptic, there is an obvious counterexample: the equation $\frac{\partial^2 f}{\partial x \partial y} = 0$ on $\mathbb{R}^2$ has smooth non-analytic solutions of the form $f(x) + g(y)$, $f, g \in C^\infty(\mathbb{R})$.

Example 6.9. 1. For $N = 1$ this theorem just says that a solution of an ODE

$$f^{(n)}(x) + a_1(x)f^{(n-1)}(x) + ... + a_n(x)f(x) = 0$$

with real analytic coefficients is itself real analytic, a classical fact about ODE.

2. Let $N = 2$ and $D = \Delta$ be the Laplace operator on $U \subset \mathbb{R}^2$ with respect to some Riemannian metric with real analytic coefficients. This metric defines a conformal structure with real analytic local complex coordinate z. Then every harmonic function f (i.e., one satisfying $\Delta f = 0$) is a real part of a holomorphic function of z, hence is real analytic, which proves elliptic regularity in this special case.

3. Suppose f, g are Schwartz functions on $\mathbb{R}^n$ and $D = Q(\partial)$ is an elliptic operator with constant coefficients, where Q is a real polynomial (so the leading term of Q is nonvanishing for nonzero vectors). Then elliptic regularity says that if g is analytic, so is f. This can be easily proved using Fourier transform. Indeed, for Fourier transforms we get $Q(p)\widehat{f}(p) = \widehat{g}(p)$. Thus $\widehat{g}(p) = \frac{\widehat{f}(p)}{Q(p)}$, so this must be a smooth function. Note that $|Q(p)| \to \infty$ as $p \to \infty$ because Q has non-vanishing leading

term. So, since g is analytic, $\widehat{g}$ decays exponentially at infinity, hence so does $\widehat{f}$. Thus f is analytic.

6.3. **Proof of Harish-Chandra's analyticity Theorem.** We are now ready to prove Theorem 6.3. Let $\mathfrak{g} = \mathrm{Lie}G$ and $b \in U(\mathfrak{g})$. Then we have a linear operator $\pi_*(b) : V^\infty \to V^\infty$, which we will write just as b for short. Moreover, if $b \in U(\mathfrak{g})^K$ then it preserves the subspace $V^\rho \subset V^\infty$ for each irreducible representation ρ of K. Therefore, since all V^ρ are finite-dimensional, for any $v \in V^{K\text{-fin}}$ there exists a nonzero polynomial $P \in \mathbb{R}[t]$ such that $P(b)v = 0$ (e.g., we can take P to be the product of the characteristic polynomial of b on Kv by its complex conjugate).

Now recall that $U(\mathfrak{g})$ can be thought of as the algebra of left-invariant real differential operators on G. Let $\psi_{h,v}(g) := h(gv)$ be the matrix coefficient function. We know that this function is smooth, and we have

$$(P(b)\psi_{h,v})(g) = h(gP(b)v) = 0.$$

Thus if b is an elliptic differential operator on G, it will follow from Corollary 6.7 that $\psi_{h,v}$ is real analytic, as desired.

It remains to find $b \in U(\mathfrak{g})^K$ which defines an elliptic operator on G. For this purpose fix a left-invariant Riemannian metric on G, and make it K-invariant (under right, or, equivalently, adjoint action) by averaging over K. Then the Laplace operator Δ corresponding to this metric is elliptic and given by some element $\Delta \in U(\mathfrak{g})^K$, so we may take $b = \Delta$. This proves Theorem 6.3.

Remark 6.10. If G is simple, there exists a unique up to scaling two-sided invariant metric on G. This metric, however, is pseudo-Riemannian rather than Riemannian if G is not compact. Thus the corresponding Laplace operator is hyperbolic rather than elliptic, so not suitable for our purposes.

6.4. **Applications of weakly analytic vectors.**

Corollary 6.11. *The action of G on V is completely determined by the corresponding $(\mathfrak{g}, K)$-module $V^{K\text{-fin}}$.*

Proof. Since $V^{K\text{-fin}}$ is dense in V, it suffices to specify gv for $v \in V^{K\text{-fin}}$. For this it suffices to specify $h(gv)$ for all $h \in V^*$. By Theorem 6.3 and the analytic continuation principle, this is determined by the derivatives of all orders of $h(gv)$ at $g = 1$. But these have the form $h(bv)$ where $b \in U(\mathfrak{g})$, so are determined by bv. □

Corollary 6.12. *Let $W \subset V^{K\text{-fin}}$ be a sub-$(\mathfrak{g}, K)$-module. Then the closure $\overline{W} \subset V$ is G-invariant.*

Proof. Let $w \in W$, $g \in G$. It suffices to show that $gw \in \overline{W}$. If not, then the space $W' := \overline{W} \oplus \mathbb{C}gw$ is a closed subspace of V containing $\overline{W}$ as a subspace of codimension 1. So there exists a unique continuous linear functional $h : W' \to \mathbb{C}$ such that $h(gw) = 1$ and $h|_{\overline{W}} = 0$. By the Hahn-Banach theorem, h can be extended to an element of V^*. Thus to get a contradiction, it is enough to show that for every $h \in V^*$ that vanishes on $\overline{W}$, we have $h(gw) = 0$. But by Theorem 6.3, this function is analytic in g. So it suffices to check that its derivatives at $g = 1$ vanish. But these derivatives are of the form $h(bw)$ for $b \in U(\mathfrak{g})$, so vanish since $bw \in W$. □

Corollary 6.13. *Let V be an admissible representation of G. There is a bijection between subrepresentations of V and $(\mathfrak{g}, K)$-submodules of $V^{K\text{-fin}}$, given by $\alpha : U \subset V \mapsto U^{K\text{-fin}}$. The inverse is given by $\beta : W \mapsto \overline{W}$.*

Proof. Since $U^{K\text{-fin}}$ is dense in U, we have $\beta \circ \alpha = \mathrm{Id}$. To show that $\alpha \circ \beta = \mathrm{Id}$, we need to show that $\overline{W}^{K\text{-fin}} = W$. Clearly $\overline{W}^{K\text{-fin}}$ contains W, so we just need to prove the opposite inclusion. Let $w \in \overline{W}^{\rho}$, then we have a sequence $w_n \to w$, $w_n \in W$. Now apply the projector ξ_ρ:

$$w'_n := \pi(\xi_\rho)w_n \to \pi(\xi_\rho)w = w, \; n \to \infty,$$

and $w'_n \in W^\rho$. Thus $w \in \overline{W^\rho} = W^\rho$, since W^ρ is finite-dimensional. Hence $\overline{W}^{K\text{-fin}} = W$. □

Corollary 6.14. *If V is irreducible then $V^{K\text{-fin}}$ is an irreducible $(\mathfrak{g}, K)$-module, and vice versa.*

Corollary 6.15. *If V is of finite length then $V^{K\text{-fin}}$ is a Harish-Chandra module.*

Proof. By Corollary 6.13, $V^{K\text{-fin}}$ is a finite length $(\mathfrak{g}, K)$-module. But any finite length $(\mathfrak{g}, K)$-module is finitely generated over $U(\mathfrak{g})$, hence a Harish-Chandra module. □

Let $\mathrm{Rep}\, G$ denote the category of admissible representations of G of finite length, and $\mathcal{HC}_G$ the category of Harish-Chandra modules for G. Thus we obtain

Theorem 6.16. *The assignment $V \mapsto V^{K\text{-fin}}$ defines an exact, faithful functor $\mathrm{Rep}\, G \to \mathcal{HC}_G$, which maps irreducibles to irreducibles.*

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