7. **Infinitesimal equivalence and globalization**

7.1. **Infinitesimal equivalence.** The functor of Theorem 6.16 is not full, however, since there exist pairs of non-isomorphic $V, W \in \operatorname{Rep} G$ such that $V^{K-\mathrm{fin}} \cong W^{K-\mathrm{fin}}$ as Harish-Chandra modules. Representations $V, W \in \operatorname{Rep} G$ such that $V^{K-\mathrm{fin}} \cong W^{K-\mathrm{fin}}$ as Harish-Chadra modules are called **infinitesimally equivalent**. In other words, infinitesimally equivalent representations with the same underlying Harish-Chandra module M differ by what topology we put on M (namely, the corresponding representation $\widehat{M}$ is the completion of M is this topology). An example of infinitesimally equivalent but non-isomorphic representations are $L^2(\mathbb{RP}^1)$ and $C^\infty(\mathbb{RP}^1)$ as representations of $G = SL_2(\mathbb{R})$ (with G-action on half-densities).

However, we have the following proposition.

Proposition 7.1. *Let V, W be two unitary representations in* $\operatorname{Rep} G$. *If $V^{K-\mathrm{fin}} \cong W^{K-\mathrm{fin}}$ as Harish-Chandra modules, then $V \cong W$ as unitary representations. In other words, infinitesimally equivalent unitary representations in* $\operatorname{Rep} G$ *are isomorphic.*

Proof. Clearly, it suffices to assume that V, W are irreducible. If V is unitary irreducible then $V^{K-\mathrm{fin}}$ has an invariant positive Hermitian inner product $B = B_V$ restricted from V. Moreover, B is the unique invariant Hermitian inner product on $V^{K-\mathrm{fin}}$ up to scaling.[13] Indeed, if B' is another then pick a nonzero $v \in V^{K\text{-fin}}$ and let $\lambda := \frac{B'(v,v)}{B(v,v)}$. Then $B' - \lambda B$ has a nonzero kernel, which is a $(\mathfrak{g}, K)$-submodule of $V^{K-\mathrm{fin}}$. This kernel therefore must be the whole $V^{K-\mathrm{fin}}$, so $B' = \lambda B$.

Thus if $A : V^{K-\mathrm{fin}} \to W^{K-\mathrm{fin}}$ is an isomorphism then it is an isometry with respect to B_V, B_W under suitable normalization of these forms. Then A extends by continuity to a unitary isomorphism $V \to W$ which commutes with K.

It remains to show that A commutes with G. For $v \in V$, $w \in W$, consider the function

$$f_{w,v}(g) := B_W((gA - Ag)v, w) = B_W(gAv, w) - B_V(gv, A^{-1}w),\ g \in G.$$

Our job is to show that $f_{w,v}(g) = 0$. It suffices to check this when $v \in V^{K-\mathrm{fin}}$, as it is dense in V. In this case by Harish-Chandra's analyticity theorem, the function $f_{w,v}(g)$ is analytic on G. Also all its derivatives at 1 vanish since $bA - Ab = 0$ for any $b \in U(\mathfrak{g})$. This implies that $f_{w,v}$ is indeed zero, as desired. □

[13]An invariant inner product on a $(\mathfrak{g}, K)$-module is one that is invariant under both $\mathfrak{g}$ and K, i.e., K-invariant and satisfying the equality $B(av, w) + B(v, aw) = 0$ for all $a \in \mathfrak{g}$.

7.2. **Dixmier's lemma and infinitesimal character.** The following is an infinite-dimensional analog of Schur's lemma.

Lemma 7.2. *(Dixmier) Let A be a countable-dimensional $\mathbb{C}$-algebra and M a simple A-module. Then $\mathrm{End}_A(M) = \mathbb{C}$. In particular, the center Z of A acts on M by a character $\chi : Z \to \mathbb{C}$.*

Note that the condition of countable dimension cannot be dropped. Without it, a counterexample is $A = M = \mathbb{C}(x)$ (the field of rational functions in one variable), then $\mathrm{End}_A(M) = \mathbb{C}(x)$.

Proof. Let $D := \mathrm{End}_A(M)$. By the usual Schur lemma, D is a division algebra. Assume the contrary, that $D \neq \mathbb{C}$. Then for any $x \in D \setminus \mathbb{C}$, D contains the field $\mathbb{C}(x)$ of rational functions of x (as $\mathbb{C}$ has no finite field extensions). But $\mathbb{C}(x)$ has uncountable dimension (contains linearly independent elements $\frac{1}{x-a}$, $a \in \mathbb{C}$), hence so does D. On the other hand, let $v \in M$ be a nonzero vector, then $M = Av$ and the map $D \to M$ given by $T \mapsto Tv$ is injective. Thus M is countable-dimensional, hence so is D, contradiction. $\square$

Now let $\mathfrak{g}$ be a countable-dimensional complex Lie algebra and M a simple $\mathfrak{g}$-module. By Lemma 7.2, the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts on M by a character, $\chi : Z(\mathfrak{g}) \to \mathbb{C}$. This character is called the **infinitesimal character** of M.

In particular, for semisimple groups we obtain

Corollary 7.3. *(Schur's lemma for $(\mathfrak{g}, K)$-modules) Any endomorphism of an irreducible $(\mathfrak{g}, K)$-module M is a scalar. Thus the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts on M by a infinitesimal character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$.*

The character χ is often also called the **infinitesimal character** of M.

Exercise 7.4. Show that the action of $Z(\mathfrak{g})$ on every admissible $(\mathfrak{g}, K)$-module M is locally finite.

7.3. **Harish-Chandra's globalization theorem.**

Theorem 7.5. *(Harish-Chandra's globalization theorem) Every unitary irreducible Harish-Chandra module M for G uniquely integrates (=globalizes) to an irreducible admissible unitary representation of G.*

Proof. As before, fix a positive definite K-invariant inner product on $\mathfrak{g}$, and consider the element $C_{\mathfrak{g}}^{+} := \sum_{j=1}^{\dim \mathfrak{g}} a_j^2 \in U(\mathfrak{g})$, where a_j is an orthonormal basis of $\mathfrak{g}$ under this inner product. If $C_{\mathfrak{g}}$ is the (suitably normalized) quadratic Casimir of $\mathfrak{g}$, then $C_{\mathfrak{g}}^{+} = C_{\mathfrak{g}} + 2C_{\mathfrak{k}}$, where $C_{\mathfrak{k}}$ is the Casimir of $\mathfrak{k} := \mathrm{Lie}K$ corresponding to the restriction of the inner product to $\mathfrak{k}$. If L_ν is the highest weight irreducible representation with highest weight ν then $-C_{\mathfrak{k}}|_{L_\nu} = |\nu+\rho_K|^2 - |\rho_K|^2$, where ρ_K is the half-sum of positive roots of K. Also $C_{\mathfrak{g}}|_M = C_M$ is a scalar. Thus if $M^\nu := M^{L_\nu}$ then

$$-C_{\mathfrak{g}}^+|_{M^\nu} = 2|\nu+\rho_K|^2 - 2|\rho_K|^2 - C_M =: q(\nu).$$

Note that for $v \in M^\nu$ we have

$$\sum_{j=1}^{\dim \mathfrak{g}} \|a_j v\|^2 = -(\sum_{j=0}^{\dim \mathfrak{g}} a_j^2 v, v) = -(C_{\mathfrak{g}}^+ v, v) = q(\nu)\,\|v\|^2;$$

in particular, $q(\nu) \geq 0$ and $q(\nu) \sim 2|\nu|^2$ for large ν. It follows that for any $a \in \mathfrak{g}, v \in M^\nu$,

$$\|av\|^2 \leq q(\nu)\,\|a\|^2\,\|v\|^2.$$

Now, for $a \in M^{\nu_0}$ all components of $a^n v$ belong to M^ν, where $\nu = \nu_0 + \beta_1 + ... + \beta_n$ and β_j are weights of $\mathfrak{g}$ as a K-module. So there exist $R, c = c(\nu_0) > 0$ such that

$$\|a^n v\| \leq (Rn + c)\,\|a\|\,\left\|a^{n-1}v\right\|, n \geq 1.$$

Thus

$$\|a^n v\| \leq (R+c)...(Rn+c)\,\|a\|^n\,\|v\|.$$

So the series

$$e^a v := \sum_{n\geq 0} \frac{a^n v}{n!}$$

converges absolutely in the Hilbert space $\widehat{M}$ in the region $\|a\| < R^{-1}$, and convergence is uniform on compact sets with all derivatives, and defines an analytic function of a. Moreover, it is easy to check that $\|e^a v\| = \|v\|$ (since a is skew-symmetric under the inner product of M). Thus the operator $e^a : M \to \widehat{M}$ extends to a unitary operator on $\widehat{M}$. The formal Campbell-Hausdorff formula then implies that this defines a continuous unitary action π of a neighborhood U of 0 in G on $\widehat{M}$ such that $\pi(xy) = \pi(x)\pi(y)$ if $x, y, xy \in U$. It is well known that this implies that π extends to a unitary representation of the universal cover $\widetilde{G}$ of G on M. Now let $\widetilde{K}$ be the preimage of K in $\widetilde{G}$ (by the polar decomposition, it is also the universal cover of K). Since by definition $\pi|_{\widetilde{K}}$ extends to K, it follows that π actually factors through G. □

Thus, using Harish-Chandra's admissibility theorem, we obtain

Corollary 7.6. *For a semisimple Lie group G, the assignment $V \mapsto V^{K-\mathrm{fin}}$ is an equivalence of categories between unitary representations of G of finite length and unitary Harish-Chandra modules of finite* *length (i.e., Harish-Chandra modules which admit an invariant positive Hermitian inner product).*

However, while irreducible Harish-Chandra modules for any G have been classified, determining which of them are unitary is a very difficult problem which is not yet fully solved.

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