## 8. **Highest weight modules and Verma modules**

8.1. **$\mathfrak{g}$-modules with a weight decomposition.** Let us recall basic results on highest weight modules and Verma modules for a complex semisimple Lie algebra $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a triangular decomposition and $\lambda \in \mathfrak{h}^*$ be a weight. We have $\mathfrak{n}_\pm = \oplus_{\alpha \in R_\pm} \mathfrak{g}_\alpha$, where $R_\pm$ are the sets of positive and negative roots. Let $Q \subset \mathfrak{h}^*$ be the root lattice of $\mathfrak{g}$ spanned by its roots. Let $e_i, f_i, h_i, i = 1, ..., r$ be the Chevalley generators of $\mathfrak{g}$. Let $P \subset \mathfrak{h}^*$ be the weight lattice, consisting of $\lambda \in \mathfrak{h}^*$ with $\lambda(h_i) \in \mathbb{Z}$ for all $i$ and $P_+ \subset P$ be the set of dominant integral weights, defined by the condition $\lambda(h_i) \in \mathbb{Z}_{\geq 0}$ for all $i$. Finally, let $Q_+ \subset Q$ be the set of sums of positive roots.

**Definition 8.1.** Let $V$ a representation of $\mathfrak{g}$ (possibly infinite-dimensional). Then a vector $v \in V$ is said to have **weight** $\lambda$ if $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$. The subspace of such vectors is denoted by $V[\lambda]$. If $V[\lambda] \neq 0$, we say that $\lambda$ is a weight of $V$, and the set of weights of $V$ is denoted by $P(V)$.

It is easy to see that $\mathfrak{g}_\alpha V[\lambda] \subset V[\lambda + \alpha]$.

Let $V' \subset V$ be the span of all weight vectors in $V$. Then it is clear that $V' = \oplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$.

**Definition 8.2.** We say that $V$ **has a weight decomposition** (with respect to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$), or is **$\mathfrak{h}$-semisimple** if $V' = V$, i.e., if $V = \oplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$.

Note that not every representation of $\mathfrak{g}$ has a weight decomposition (e.g., for $V = U(\mathfrak{g})$ with $\mathfrak{g}$ acting by left multiplication all weight subspaces are zero).

**Definition 8.3.** A vector $v$ in $V[\lambda]$ is called a **singular (or highest weight) vector of weight** $\lambda$ if $e_i v = 0$ for all $i$, i.e., if $\mathfrak{n}_+ v = 0$. A representation $V$ of $\mathfrak{g}$ is a **highest weight representation with highest weight** $\lambda$ if it is generated by such a nonzero vector.

8.2. **Verma modules.** The **Verma module** $M_\lambda$ is defined as "the largest highest weight module with highest weight $\lambda$". Namely, it is generated by a single highest weight vector $v_\lambda$ with **defining relations** $hv = \lambda(h)v$ for $h \in \mathfrak{h}$ and $e_i v = 0$. More formally, we make the following definition.

**Definition 8.4.** Let $I_\lambda \in U(\mathfrak{g})$ be the left ideal generated by the elements $h - \lambda(h), h \in \mathfrak{h}$ and $e_i$, $i = 1, ..., r$. Then the **Verma module** $M_\lambda$ is the quotient $U(\mathfrak{g})/I_\lambda$.

In this realization, the highest weight vector $v_\lambda$ is just the class of the unit 1 of $U(\mathfrak{g})$.

**Proposition 8.5.** *The map $\phi : U(\mathfrak{n}_-) \to M_\lambda$ given by $\phi(x) = xv_\lambda$ is an isomorphism of left $U(\mathfrak{n}_-)$-modules.*

*Proof.* By the PBW theorem, the multiplication map

$$\xi : U(\mathfrak{n}_-) \otimes U(\mathfrak{h} \oplus \mathfrak{n}_+) \to U(\mathfrak{g})$$

is a linear isomorphism. It is easy to see that $\xi^{-1}(I_\lambda) = U(\mathfrak{n}_-) \otimes K_\lambda$, where

$$K_\lambda := \sum_i U(\mathfrak{h} \oplus \mathfrak{n}_+)(h_i - \lambda(h_i)) + \sum_i U(\mathfrak{h} \oplus \mathfrak{n}_+)e_i$$

is the kernel of the homomorphism $\chi_\lambda : U(\mathfrak{h} \oplus \mathfrak{n}_+) \to \mathbb{C}$ given by $\chi_\lambda(h) = \lambda(h)$, $h \in \mathfrak{h}$, $\chi_\lambda(e_i) = 0$. Thus, we have a natural isomorphism of left $U(\mathfrak{n}_-)$-modules

$$U(\mathfrak{n}_-) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h} \oplus \mathfrak{n}_+)/K_\lambda \to M_\lambda,$$

as claimed. □

**Remark 8.6.** The definition of $M_\lambda$ means that it is the **induced module** $U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h} \oplus \mathfrak{n}_+$ on which it acts via $\chi_\lambda$.

**Corollary 8.7.** *$M_\lambda$ has a weight decomposition with $P(M_\lambda) = \lambda - Q_+$, $\dim M_\lambda[\lambda] = 1$, and weight subspaces of $M_\lambda$ are finite-dimensional.*

**Proposition 8.8.** *(i) If $V$ is a representation of $\mathfrak{g}$ and $v \in V$ is a vector such that $hv = \lambda(h)v$ for $h \in h$ and $e_iv = 0$ then there is a unique homomorphism $\eta : M_\lambda \to V$ such that $\eta(v_\lambda) = v$. In particular, if $V$ is generated by such $v \neq 0$ (i.e., $V$ is a highest weight representation with highest weight vector $v$) then $V$ is a quotient of $M_\lambda$.*

*(ii) Every highest weight representation has a weight decomposition into finite-dimensional weight subspaces.*

*(iii) Every highest weight representation $V$ has a unique highest weight generator, up to scaling.*

*Proof.* (i) Uniqueness follows from the fact that $v_\lambda$ generates $M_\lambda$. To construct $\eta$, note that we have a natural map of $\mathfrak{g}$-modules $\widetilde{\eta} : U(\mathfrak{g}) \to V$ given by $\widetilde{\eta}(x) = xv$. Moreover, $\widetilde{\eta}|_{I_\lambda} = 0$ thanks to the relations satisfied by $v$, so $\widetilde{\eta}$ descends to a map $\eta : U(\mathfrak{g})/I_\lambda = M_\lambda \to V$. Moreover, if $V$ is generated by $v$ then this map is surjective, as desired.

(ii) This follows from (i) since a quotient of any representation with a weight decomposition must itself have a weight decomposition.

(iii) Suppose $v, w$ are two highest weight generators of $V$ of weights $\lambda, \mu$. If $\lambda = \mu$ then they are proportional since $\dim V[\lambda] \le \dim M_\lambda[\lambda] = 1$, as $V$ is a quotient of $M_\lambda$. On the other hand, if $\lambda \neq \mu$, then we can assume without loss of generality that $\lambda - \mu \notin Q_+$ (otherwise switch $\lambda, \mu$). Then $\mu \notin \lambda - Q_+$, hence $\mu \notin P(V)$, a contradiction. □

8.3. **Irreducible highest weight $\mathfrak{g}$-modules.**

**Proposition 8.9.** *For every $\lambda \in \mathfrak{h}^*$, the Verma module $M_\lambda$ has a unique irreducible quotient $L_\lambda$. Moreover, $L_\lambda$ is a quotient of every highest weight $\mathfrak{g}$-module $V$ with highest weight $\lambda$.*

*Proof.* Let $Y \subset M_\lambda$ be a proper submodule. Then $Y$ has a weight decomposition, and cannot contain a nonzero multiple of $v_\lambda$ (as otherwise $Y = M_\lambda$), so $P(Y) \subset (\lambda - Q_+) \setminus \{\lambda\}$. Now let $J_\lambda$ be the sum of all proper submodules $Y \subset M_\lambda$. Then $P(J_\lambda) \subset (\lambda - Q_+) \setminus \{\lambda\}$, so $J_\lambda$ is also a proper submodule of $M_\lambda$ (the maximal one). Thus, $L_\lambda := M_\lambda / J_\lambda$ is an irreducible highest weight module with highest weight $\lambda$. Moreover, if $V$ is any nonzero quotient of $M_\lambda$ then the kernel $K$ of the map $M_\lambda \to V$ is a proper submodule, hence contained in $J_\lambda$. Thus the surjective map $M_\lambda \to L_\lambda$ descends to a surjective map $V \to L_\lambda$. The kernel of this map is a proper submodule of $V$, hence zero if $V$ is irreducible. Thus in the latter case $V \cong L_\lambda$. □

**Corollary 8.10.** *Irreducible highest weight $\mathfrak{g}$-modules are classified by their highest weight $\lambda \in \mathfrak{h}^*$, via the bijection $\lambda \mapsto L_\lambda$.*

**Exercise 8.11.** Let $\mathfrak{g} = \mathfrak{sl}_2$ with standard generators $e, f, h$ and identify $\mathfrak{h}^* \cong \mathbb{C}$ via $\lambda \mapsto \lambda(h)$. Show that $M_\lambda$ is irreducible if $\lambda \notin \mathbb{Z}_{\ge 0}$, while for $\lambda$ a nonnegative integer we have $J_\lambda = M_{-\lambda-2}$, so $L_\lambda$ is the $\lambda + 1$-dimensional irreducible representation of $\mathfrak{sl}_2$.

It is known from the theory of finite-dimensional representations of $\mathfrak{g}$ that its irreducible finite-dimensional representations are $L_\lambda$ with $\lambda \in P_+$. Thus we have

**Proposition 8.12.** *$L_\lambda$ is finite-dimensional if and only if $\lambda \in P_+$.*

Note that the "only if" direction of this proposition follows immediately from Exercise 8.11.

8.4. **Exercises.**

**Exercise 8.13.** Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra, and $V$ a finite-dimensional representation of $\mathfrak{g}$. Let $\lambda, \mu \in \mathfrak{h}^*$ be weights for $\mathfrak{g}$, and $X, Y$ be representations of $\mathfrak{g}$ with $P(X) \subset \lambda - Q_+$, $P(Y) \subset \mu - Q_+$, and $X[\lambda] = \mathbb{C}v_\lambda$, $Y[\mu] = \mathbb{C}v_\mu$ for nonzero vectors

$v_\lambda, v_\mu$. Given a linear map $\Phi : X \to V \otimes Y$, let the **expectation value** of $\Phi$ be defined by

$$\langle \Phi \rangle := (\mathrm{Id} \otimes v_\mu^*, \Phi v_\lambda) \in V$$

where $v_\mu^* \in Y[\mu]^*$ is such that $(v_\mu^*, v_\mu) = 1$. In other words, we have

$$\Phi v_\lambda = \langle \Phi \rangle \otimes v_\mu + \text{lower terms}$$

where the lower terms have lower weight than $\mu$ in the second component.

(i) Show that if $\Phi$ is a homomorphism then $\langle \Phi \rangle$ has weight $\lambda - \mu$.

(ii) Let $M_\lambda$ be the Verma module with highest weight $\lambda \in \mathfrak{h}^*$, and $\overline{M}_{-\mu}$ be the **lowest weight** Verma module with lowest weight $-\mu$, i.e., generated by a vector $v_{-\mu}$ with defining relations $hv_{-\mu} = -\mu(h)v_{-\mu}$ for $h \in \mathfrak{h}$ and $f_i v_{-\mu} = 0$. Show that the map $\Phi \mapsto \langle \Phi \rangle$ defines an isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V \otimes \overline{M}^*_{-\mu}) \cong V[\lambda - \mu]$$

where $*$ denotes the restricted dual (the direct sum of duals of all weight subspaces).

(iii) Let $\lambda \in P_+$ and $V[\nu]_\lambda$ be the subspace of vectors $v \in V[\nu]$ of weight $\nu$ which satisfy the equalities $f_i^{(\lambda,\alpha_i^\vee)+1} v = 0$ for all $i$. Show that a map $\Phi \in \mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V \otimes \overline{M}^*_{-\mu})$ factors through $L_\lambda$ iff $\langle \Phi \rangle \in V[\lambda - \mu]_\lambda$, i.e., $f_i^{(\lambda,\alpha_i^\vee)+1} \langle \Phi \rangle = 0$ (for this, use that $e_j f_i^{(\lambda,\alpha_i^\vee)+1} v_\lambda = 0$, and that the kernel of $M_\lambda \to L_\lambda$ is generated by the vectors $f_i^{(\lambda,\alpha_i^\vee)+1} v_\lambda$). Deduce that the map $\Phi \mapsto \langle \Phi \rangle$ defines an isomorphism $\mathrm{Hom}_{\mathfrak{g}}(L_\lambda, V \otimes \overline{M}^*_{-\mu}) \cong V[\lambda - \mu]_\lambda$.

(iv) Now let both $\lambda, \mu$ be in $P_+$. Show that every homomorphism $L_\lambda \to V \otimes \overline{M}^*_{-\mu}$ in fact lands in $V \otimes L_\mu \subset V \otimes \overline{M}^*_{-\mu}$. Deduce that the map $\Phi \mapsto \langle \Phi \rangle$ defines an isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(L_\lambda, V \otimes L_\mu) \cong V[\lambda - \mu]_\lambda.$$

(v) Let $V = \mathbb{C}^n$ be the vector representation of $SL_n(\mathbb{C})$. Determine the weight subspaces of $S^m V$, and compute the decomposition of $S^m V \otimes L_\mu$ into irreducibles for all $\mu \in P_+$ (use (iv)).

(vi) For any $\mathfrak{g}$, compute the decomposition of $\mathfrak{g} \otimes L_\mu$, $\mu \in P_+$, where $\mathfrak{g}$ is the adjoint representation of $\mathfrak{g}$ (again use (iv)).

In both (v) and (vi) you should express the answer in terms of the numbers $k_i$ such that $\mu = \sum_i k_i \omega_i$ and the Cartan matrix entries of $\mathfrak{g}$.

**Exercise 8.14.** (D. N. Verma) (i) Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a finite-dimensional simple complex Lie algebra, and $\lambda, \mu \in \mathfrak{h}^*$. Show that every nonzero homomorphism $M_\mu \to M_\lambda$ is injective. (Use that $U(\mathfrak{n}_-)$

has no zero divisors). Deduce that if $M_\lambda$ is reducible then there exists $\lambda' \in \lambda - Q_+$, $\lambda' \neq \lambda$ with $M_{\lambda'} \subset M_\lambda$.

(ii) Show that for every $\lambda \in \mathfrak{h}^*$ there is $\lambda' \in \lambda - Q_+$ with $M_{\lambda'} \subset M_\lambda$ and $M_{\lambda'}$ irreducible. (Assume the contrary and construct an infinite sequence of proper inclusions

$$...M_{\lambda_2} \subset M_{\lambda_1} \subset M_\lambda.$$

Then derive a contradiction by looking at the eigenvalues of the quadratic Casimir $C \in U(\mathfrak{g})$).

(iii) Show that if $M_\mu$ is irreducible then $\dim \mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) \leq 1$. (Look at the growth of the dimensions of weight subspaces).

(iv) Show that $\dim \mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda) \leq 1$ for any $\lambda, \mu \in \mathfrak{h}^*$. (Look at the restriction of a homomorphism $M_\mu \to M_\lambda$ to $M_{\mu'} \subset M_\mu$ which is irreducible).

**Exercise 8.15.** (i) Keep the notation of Exercise 8.14. Let $\lambda \in \mathfrak{h}^*$ be such that $(\lambda, \alpha_i^\vee) = n - 1$ for a positive integer $n$ and simple root $\alpha_i$. Show that there is an inclusion $M_{\lambda - n\alpha_i} \hookrightarrow M_\lambda$.

(ii) Let $\rho$ be the sum of fundamental weights of $\mathfrak{g}$ and $W$ be the Weyl group of $\mathfrak{g}$. For $w \in W$, $\lambda \in \mathfrak{h}^*$ let $w \bullet \lambda := w(\lambda + \rho) - \rho$ (the **shifted action** of $W$). Deduce from (i) that if $\lambda \in P_+$ then for every $w \in W$, there is an inclusion $\iota_w : M_{w \bullet \lambda} \hookrightarrow M_\lambda$, and that if $w = w_1 w_2$ with $\ell(w) = \ell(w_1) + \ell(w_2)$ (where $\ell(w)$ is the length of $w$) then $\iota_w$ factors through $\iota_{w_2}$. In particular, we have an inclusion $M_{w \bullet \lambda} \hookrightarrow M_{w_2 \bullet \lambda}$.

(iii) Show that $M_\lambda$ is irreducible unless $(\lambda + \rho, \alpha^\vee) = 1$ for some $\alpha \in Q_+ \setminus 0$, where $\alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}$ (look at the eigenvalues of the quadratic Casimir).

(iv) For $\beta \in Q_+$ define the **Kostant partition function** $K(\beta)$ to be the number of unordered representations of $\beta$ as a sum of positive roots of $\mathfrak{g}$ (thus $K(\beta) = \dim U(\mathfrak{n}_+)[\beta]$). Also define the **Shapovalov pairing**

$$B_\beta(\lambda) : U(\mathfrak{n}_+)[\beta] \times U(\mathfrak{n}_-)[-\beta] \to \mathbb{C}$$

by the formula

$$xyv_\lambda = B_\beta(\lambda)(x, y)v_\lambda,$$

where $x \in U(\mathfrak{n}_+)[\beta], y \in U(\mathfrak{n}_-)[-\beta]$, and $v_\lambda$ is the highest weight vector of $M_\lambda$. Let

$$D_\beta(\lambda) := \det B_\beta(\lambda),$$

the determinant of the matrix of $B_\beta(\lambda)$ in some bases of $U(\mathfrak{n}_+)[\beta], U(\mathfrak{n}_-)[-\beta]$. This is a (non-homogeneous) polynomial in $\lambda$ well defined up to scaling.

Show that the leading term of $D_\beta$ is

$$D^0_\beta(\lambda) = \text{const} \cdot \prod_{\alpha \in R_+} (\lambda, \alpha^\vee)^{\sum_{n \geq 1} K(\beta - n\alpha)}.$$

(Hint: show that the leading term comes from the product of the diagonal entries of the matrix of the Shapovalov pairing in the PBW bases).

(v) Show that

$$D_\beta(\lambda) = \text{const} \cdot \prod_{\alpha \in Q_+ \setminus 0} ((\lambda + \rho, \alpha^\vee) - 1)^{m_\alpha}$$

for some nonnegative integers $m_\alpha = m_\alpha(\beta)$. Then use (iv) to show that moreover $m_\alpha = 0$ unless $\alpha$ is a multiple of a positive root.

(vi) Let $V, U$ be finite-dimensional vector spaces over a field $k$ of dimension $n$ and $B(t) : V \times U \to k[[t]]$ be a bilinear form. Denote by $V_0 \subset V, U_0 \subset U$ the left and right kernels of $B(0)$. Suppose that $B'(0)$ is a perfect pairing $V_0 \times U_0 \to k$. Show that the vanishing order of $\det B(t)$ at $t = 0$ (computed with respect to any bases of $V, U$) equals $\dim V_0 = \dim U_0$. (*Hint:* Pick a basis $e_1, ..., e_m$ of $V_0$, complete it to a basis $e_1, ..., e_n$ of $V$. Choose vectors $f_{m+1}, ..., f_n \in U$ such that $B(0)(e_i, f_j) = \delta_{ij}$ for $m < i, j \leq n$. Let $f_1, ..., f_m$ be the basis $U_0$ dual to $e_1, ..., e_m$ with respect to $B'(0)$. Show that $\{f_i\}$ is a basis of $U$ and the determinant of $B(t)$ in the bases $\{e_i\}$, $\{f_i\}$ equals $t^m + O(t^{m+1})$.)

(vii) Show that if $\lambda$ is generic on the hyperplane $(\lambda + \rho, \alpha^\vee) = n$ for $n \in \mathbb{Z}_{>0}$ and $\alpha \in R_+$ and $m_{n\alpha}(\beta) > 0$ then $M_\lambda$ contains an irreducible submodule $M_{\lambda - n\alpha}$ and the quotient $M_\lambda / M_{\lambda - n\alpha}$ is irreducible. (Use Casimir eigenvalues to show that the only irreducible modules which could occur in the composition series of $M_\lambda$ are $L_\lambda$ and $L_{\lambda - n\alpha}$ and apply Exercise 8.14).

(viii) Let $\lambda$ be as in (vii) and let $B(\beta, t) := B_\beta(\lambda + t\alpha)$. Show that $B(\beta, t)$ satisfies the assumption of (vi) for all $\beta$.

**Hint:** Use that $\oplus_\beta \text{Ker} B(\beta, 0)$ is naturally identified with $M_{\lambda - n\alpha}$ and $B'(\beta, 0)$ restricts on it to a multiple of its Shapovalov form, and show that one has $B'_{n\alpha}(0)(v_{\lambda - n\alpha}, v_{\lambda - n\alpha}) \neq 0$. For the latter, assume the contrary and show that there exists a homogeneous lift $u$ of $v_{\lambda - n\alpha}$ modulo $t^2$ such that $B_{n\alpha}(t)(u, w) = 0$ modulo $t^2$ for all $w$ of weight $\lambda + (t - n)\alpha$. Deduce that $e_i u$ vanishes modulo $t^2$ for all $i$. Conclude that

$$Cu = ((\lambda + (t - n)\alpha + \rho)^2 - \rho^2)u + O(t^2)$$

and derive a contradiction with

$$Cu = ((\lambda + t\alpha + \rho)^2 - \rho^2)u.$$

(ix) Deduce that $m_{n\alpha}(\beta) = K(\beta - n\alpha)$; in particular, in general $m_{n\alpha}(\beta) \le K(\beta - n\alpha)$.

(x) Prove the **Shapovalov determinant formula**:

$$D_\beta(\lambda) = \prod_{\alpha \in R_+} \prod_{n \ge 1} ((\lambda + \rho, \alpha^\vee) - n)^{K(\beta - n\alpha)}$$

up to scaling.

(xi) Determine all $\lambda \in \mathfrak{h}^*$ for which $M_\lambda$ is irreducible.

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