9. **Representations of $SL_2(\mathbb{R})$**

9.1. **Irreducible $(\mathfrak{g}, K)$-modules for $SL_2(\mathbb{R})$.** Let us now apply the general theory to the simplest example – representations of the group $G = SL_2(\mathbb{R})$ of real 2 by 2 matrices with determinant 1. Note that $SL_2(\mathbb{R}) \cong SU(1,1)$, and in this realization the maximal compact subgroup $SO(2)$ becomes $U(1)$. So we have $\mathrm{Lie}(G) = \mathfrak{g} = \mathfrak{su}(1,1)$, hence $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}_2(\mathbb{C})$ with standard basis e, f, h, so that a maximal compact subgroup K of G consists of elements e^{ith}, $t \in [0, 2\pi)$. Thus a $(\mathfrak{g}, K)$-module is the same thing as a $\mathfrak{g}_{\mathbb{C}}$-module with a weight decomposition and integer weights.

Let us classify irreducible $(\mathfrak{g}, K)$-modules M. To this end, recall that we have the central Casimir element $C \in U(\mathfrak{g}_{\mathbb{C}})$ given by

$$C = fe + \frac{(h+1)^2}{4},$$

and note that by the PBW theorem, $U(\mathfrak{g}_{\mathbb{C}})$ is free as a right module over the commutative subalgebra $\mathbb{C}[h, fe] = \mathbb{C}[h, C]$ with basis $1, f^n, e^n$, $n \geq 1$. Thus if v is a nonzero weight vector of M then M is spanned by $v, f^n v, e^n v$. It follows that weight subspaces of M are 1-dimensional, and $P(M)$ is an arithmetic progression with step 2. Thus we have four cases:

1. $P(M)$ is finite. Then $M = L_m$, the $m+1$-dimensional irreducible representation.

2. $P(M)$ is infinite, bounded above. In this case let v have the maximal weight m. Then $f^n v$, $n \geq 0$ is a basis of M, and we have $hv = mv, ev = 0$. Thus $M = M_m$ is the Verma module with highest weight $m \in \mathbb{Z}$. This module is irreducible iff $m < 0$ (Exercise 8.11). Thus in this case we get modules $M_{-m} = M_{-m}^+$, $m \geq 1$.

3. $P(M)$ is infinite, bounded below. The situation is completely parallel (with f replaced by e) and we obtain lowest weight Verma modules M_m^- for $m \geq 1$. The $(\mathfrak{g}, K)$-modules M_m^-, M_{-m}^+ are called the **discrete series modules** for $m \geq 2$, and **limit of discrete series** for $m = 1$.

4. $P(M)$ is unbounded on both sides. Let c be the scalar by which C acts on M. We have two cases – the even case $P(M) = 2\mathbb{Z}$ and the odd case $P(M) = 2\mathbb{Z}+1$. In both cases we have a basis v_n, $n \in P(M)$ such that

$$hv_n = nv_n,\ fv_n = v_{n-2},\ ev_n = \Lambda_n v_{n+2}, \tag{4}$$

where $\Lambda_n \neq 0$. To compute Λ_n, we write

$$\Lambda_n v_n = fev_n = (C - \tfrac{(h+1)^2}{4})v_n = (c - \tfrac{(n+1)^2}{4})v_n.$$

Thus
$$\Lambda_n = c - \tfrac{(n+1)^2}{4}.$$
Let $c = \frac{s^2}{4}$. Then

$$\Lambda_n = \tfrac{1}{4}(s-1-n)(s+1+n). \tag{5}$$

Thus we can replace v_n by its multiple w_n so that
$$hw_n = nw_n,\ fw_n = \tfrac{1}{2}(s-1+n)w_{n-2},\ ew_n = \tfrac{1}{2}(s-1-n)w_{n+2}.$$
These formulas define $\mathfrak{g}_{\mathbb{C}}$-modules for any $s \in \mathbb{C}$. We will denote these modules by $P_\pm(s)$ (plus for the even case, minus for the odd case). The $(\mathfrak{g}, K)$-modules $P_\pm(s)$ are called the **principal series modules**. We see that $P_+(s)$ is irreducible if $s \notin 2\mathbb{Z}+1$ and $P_-(s)$ is irreducible iff $s \notin 2\mathbb{Z}$, and $P_\pm(s) = P_\pm(-s)$ in this case.

Moreover, when these conditions fail, we have short exact sequences
$$0 \to L_{2m} \to P_+(2m+1) \to M^+_{-2m-2} \oplus M^-_{2m+2} \to 0,\ m \in \mathbb{Z}_{\ge 0},$$
$$0 \to M^+_{-2m-2} \oplus M^-_{2m+2} \to P_+(-2m-1) \to L_{2m} \to 0,\ m \in \mathbb{Z}_{\ge 0},$$
$$0 \to L_{2m+1} \to P_-(2m+2) \to M^+_{-2m-3} \oplus M^-_{2m+3} \to 0,\ m \in \mathbb{Z}_{\ge 0},$$
$$0 \to M^+_{-2m-3} \oplus M^-_{2m+3} \to P_-(-2m-2) \to L_{2m+1} \to 0,\ m \in \mathbb{Z}_{\ge 0},$$
and for $s=0$ we have an isomorphism
$$P_-(0) \cong M^+_{-1} \oplus M^-_1.$$
All these modules except $P_-(0)$ are indecomposable. Thus we see that $P_\pm(s) \ncong P_\pm(-s)$ when it is reducible and $s \neq 0$.

As a result, we get

Proposition 9.1. *The simple $(\mathfrak{g}, K)$-modules (or equivalently, Harish-Chandra modules) are $L_m, m \in \mathbb{Z}_{\ge 0}$, M^-_m, M^+_{-m}, $m \in \mathbb{Z}_{\ge 1}$, and $P_+(s)$, $s \notin 2\mathbb{Z}+1$, $P_-(s)$, $s \notin 2\mathbb{Z}$, with the only isomorphisms $P_\pm(s) \cong P_\pm(-s)$.*

Exercise 9.2. Let $\widetilde{P}_+(s), \widetilde{P}_-(s)$ be the modules defined by (4),(5); so they are isomorphic to $P_+(s), P_-(s)$ when s is not an odd integer, respectively not a nonzero even integer. But we will consider $\widetilde{P}_+(s)$ when $s = 2k+1$ and $\widetilde{P}_-(s)$ when $s = 2k$, $k \neq 0$ (where k is an integer).

(i) Compute the Jordan-Hölder series of $\widetilde{P}_+(s), \widetilde{P}_-(s)$ and show that they are uniserial, i.e., have a unique filtration with irreducible successive quotients.

(ii) Do there exist isomorphisms $\widetilde{P}_+(s) \cong P_+(s)$, $\widetilde{P}_-(s) \cong P_-(s)$?

9.2. **Realizations.** Let us discuss realizations of these representations by admissible representations of G. For L_m there is nothing to discuss, so we'll focus on principal series and discrete series.

The realization of principal series has already been discussed in Example 5.3. Namely, let $B \subset G$ be the subgroup of upper triangular matrices b with diagonal entries $(t(b), t(b)^{-1})$. As before we consider the spaces

$$\mathbb{V}_+(s) = \{F \in C^\infty(G) : F(gb) = F(g)|t(b)|^{s-1}\},$$
$$\mathbb{V}_-(s) = \{F \in C^\infty(G) : F(gb) = F(g)|t(b)|^{s-1}\mathrm{sign}(t(b))\}.$$

These are admissible representations of G acting by left multiplication. Let us compute $\mathbb{V}_\pm(s)^{\mathrm{fin}}$. To this end, note that the group $K = U(1) = S^1$ acts transitively on G/B with stabilizer $\mathbb{Z}/2 = \{\pm 1\}$. Thus, pulling the function F back to K, we can realize $\mathbb{V}_\pm(s)$ as the space $\mathbb{V}_\pm$ of functions $F \in C^\infty(S^1)$ such $F(-z) = \pm F(z)$.

A more geometric way of thinking about this is the following. Given a Lie group G and a closed subgroup B with Lie algebras $\mathfrak{g}, \mathfrak{b}$, every finite-dimensional representation V of B gives rise to a vector bundle $E_V := (G \times V)/B$ over G/B, where the action of B on $G \times V$ is given by $(g, v)b = (gb, b^{-1}v)$. For example, the tangent bundle $T(G/B)$ is obtained from the representation $V = \mathfrak{g}/\mathfrak{b}$. In our example, $\mathfrak{g}/\mathfrak{b}$ is the 1-dimensional representation of B given by $b \mapsto t(b)^{-2}$. Thus sections of the tangent bundle on G/B (i.e., vector fields) can be interpreted as functions F on G such that

$$F(gb) = F(g)t(b)^2.$$

It follows that elements of $\mathbb{V}_+(s)$ can be interpreted as sections of the bundle $\mathrm{K}^{\frac{1-s}{2}}$ where $\mathrm{K} = T^*(G/B)$ is the canonical bundle, which coincides with the cotangent bundle since $\dim(G/B) = 1$ (this bundle is trivial topologically but the action of diffeomorphisms of $G/B = S^1$, in particular, of elements of $SL_2(\mathbb{R})$ on its sections depends on s). In other words, elements of $\mathbb{V}_+(s)$ can be interpreted as "tensor fields of non-integer rank": $\phi(u)(d\ \mathrm{arg} u)^{\frac{1-s}{2}}$, where $u = e^{i\theta}$, θ is the angle coordinate on $G/B = \mathbb{RP}^1$ and ϕ is a smooth function. Similarly, elements of $\mathbb{V}_-(s)$ can be interpreted as expressions $u^{\frac{1}{2}}\phi(u)(d\ \mathrm{arg} u)^{\frac{1-s}{2}}$, i.e., two-valued smooth sections of the same bundle which change sign when one goes around the circle. Thus the Lie algebra action on these modules is by the vector fields

$$h = 2u\partial_u,\ f = \partial_u,\ e = -u^2\partial_u,$$

but they act on elements of $\mathbb{V}_\pm(s)$ not as on functions but as on tensor fields. Thus $\mathbb{V}_\pm(s)^{\mathrm{fin}} \subset \mathbb{V}_\pm(s)$ is the subspace of vectors such that

$\phi \in \mathbb{C}[u, u^{-1}]$. Taking the basis $w_{2k} = u^k (d\ \mathrm{arg} u)^{\frac{1-s}{2}}$ in the even case and $w_{2k+1} = u^{k+\frac{1}{2}} (d\ \mathrm{arg} u)^{\frac{1-s}{2}}$ in the odd case, we have

$$hw_n = nw_n,\ fw_n = \tfrac{1}{2}(s-1+n)w_{n-2},\ ew_n = \tfrac{1}{2}(s-1-n)w_{n+2}.$$

Thus we get that $\mathbb{V}_\pm(s)^{\mathrm{fin}} \cong P_\pm(s)$ for all $s \in \mathbb{C}$.

In particular, at points where $P_\pm(s)$ are reducible, this gives realizations of the discrete series. Namely, consider the modules $\mathbb{V}_+(-r)$ for odd $r \geq 1$ and $\mathbb{V}_-(-r)$ for even $r \geq 1$. The space $\mathbb{V}_+(-r)$ consists of elements $\phi(u)(\frac{du}{iu})^{\frac{1+r}{2}}$ where ϕ is smooth (note that $d\ \mathrm{arg} u = \frac{du}{iu}$). So it has the subrepresentation $\mathbb{V}_+^0(-r)$ of forms that extend holomorphically to the disk $|u| \leq 1$. Thus means that $\phi(u) = \sum_{N \geq 0} a_N u^{N + \frac{1+r}{2}}$, where a_N is a rapidly decaying sequence (faster than any power of N). In other words, $\mathbb{V}_+^0(-r)$ consists of elements $\psi(u)(du)^{\frac{1+r}{2}}$, where ψ is smooth on the disk $|u| \leq 1$ and holomorphic for $|u| < 1$. Thus the eigenvalues of h on $\mathbb{V}_+^0(-r)$ are $1 + r + 2N$, hence $\mathbb{V}_+^0(-r)^{\mathrm{fin}} = M_{r+1}^-$.

Also, $\mathbb{V}_+(-r)$ has a subrepresentation $\mathbb{V}_+^\infty(-r)$ of forms that extend holomorphically to $|u| \geq 1$ (including infinity), which means that $\phi(u) = \sum_{N \geq 0} a_N u^{-N - \frac{1+r}{2}}$. In other words, $\mathbb{V}_+^\infty(-r)$ consists of elements $\psi(u^{-1})(du^{-1})^{\frac{1+r}{2}}$, where ψ is smooth on the disk $|u| \leq 1$ and holomorphic for $|u| < 1$. Thus we get $\mathbb{V}_+^\infty(-r)^{\mathrm{fin}} = M_{-r-1}^+$.

Similarly, for even r we get $\mathbb{V}_-^0(-r)^{\mathrm{fin}} = M_{r+1}^-$, $\mathbb{V}_-^\infty(-r)^{\mathrm{fin}} = M_{-r-1}^+$.

9.3. **Unitary representations.** These Fréchet space realizations can easily be made Hilbert space realizations, by completing with respect to the usual L^2-norm given by

$$\|\phi\|^2 = \frac{1}{2\pi} \int_0^{2\pi} |\phi(e^{i\theta})|^2 d\theta.$$

However, this norm is only preserved by G when s is imaginary. In this case we obtain that the completed representations $\widehat{\mathbb{V}}_\pm(s)$, in particular $\widehat{\mathbb{V}}_-^0(0), \widehat{\mathbb{V}}_-^\infty(0)$, are unitary. It follows that the Harish-Chandra modules $P_\pm(s)$ for $s \in i\mathbb{R}$ and M_1^-, M_{-1}^+ are unitary.

It turns out, however, that there are other irreducible unitary representations. Let us classify them. It suffices to classify irreducible unitary Harish-Chandra modules. Note that the relevant anti-involution on $\mathfrak{g}$ is given by $e^\dagger = -f$, $f^\dagger = -e$, $h^\dagger = h$. Let M be irreducible and $v \in M$ a vector of weight n. Then if $(,)$ is an invariant Hermitian form on M then

$$(ev, ev) = -(fev, v) = ((\tfrac{n+1}{2})^2 - c)(v, v),$$

where c is a Casimir eigenvalue on M. We see that a nonzero invariant Hermitian form exists iff $c = \frac{s^2}{4} \in \mathbb{R}$, and such a form can be chosen positive definite iff $c < (\frac{n+1}{2})^2$ for every $n \in P(M)$. This shows that all discrete series representations are unitary and also determines the unitarity range of s for the principal series representations. Thus we obtain the following theorem.

Theorem 9.3. *(Gelfand-Naimark* [GN]*, Bargmann* [Ba]*). The irreducible unitary representations of $SL_2(\mathbb{R})$ are Hilbert space completions of the following unitary Harish-Chandra modules:*
- *Discrete series and limit of discrete series M_m^-, M_{-m}^+, $m \in \mathbb{Z}_{\geq 1}$;*
- *Unitary principal series $P_\pm(s)$, $s \in i\mathbb{R}$, $s \neq 0$;*
- *The* **complementary series** *$P_+(s)$, $s \in \mathbb{R}$, $0 < |s| < 1$;*
- *The trivial representation $\mathbb{C}$.*

Here $P_\pm(s) \cong P_\pm(-s)$ and there are no other isomorphisms.

Let us discuss explicit Hilbert space realizations of the unitary representations. We have already described such unitary realizations of principal series in $L^2(S^1)$, except the complementary series. For discrete series we only gave realizations for $m = 1$, as M_1^-, M_{-1}^+ are direct summands in $P_-(0)$. However, one can give a realization for any m. To this end, note that $G = SL_2(\mathbb{R})$ acts by fractional linear transformations on the disk $|u| \leq 1$. Moreover, we have the Poincaré (hyperbolic) metric on the disk which is G-invariant. The volume element for this metric looks like

$$\mu = \frac{dud\overline{u}}{(1-|u|^2)^2}.$$

Thus for expressions $\omega = \psi(u)(du)^{\frac{m}{2}}$ where $m \geq 2$ is an integer and $\psi(u)$ is holomorphic for $|u| < 1$ we may define the G-invariant norm

$$\|\omega\|^2 = \int_{|u|<1} \frac{\omega\overline{\omega}}{\mu^{\frac{m}{2}-1}} = \int_{|u|<1} |\psi(u)|^2(1-|u|^2)^{m-2} dud\overline{u}.$$

Hence the Hilbert space completion $\widehat{M_m^-}$ may be realized as the space H_m of holomorphic $\frac{m}{2}$-forms $\omega = \psi(u)(du)^{\frac{m}{2}}$ for $|u| < 1$ for which $\|\omega\|^2 < \infty$ (note that this space is nonzero only if $m \geq 2$).

Likewise, $\widehat{M_{-m}^+}$ can be similarly realized via antiholomorphic forms. Indeed, conjugation by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (of determinant -1) defines an outer automorphism of $SL_2(\mathbb{R})$ which is induced by complex conjugation on the unit disk, and this automorphism exchanges M_m^- with M_{-m}^+.

Exercise 9.4. Let G_ℓ be the ℓ-fold cover of $PSL_2(\mathbb{R})$ (for example, $G_1 = PSL_2(\mathbb{R})$, $G_2 = SL_2(\mathbb{R})$). Classify irreducible admissible representations (up to infinitesimal equivalence) and irreducible unitary representations of G_ℓ for all ℓ.

Hint. The maximal compact subgroup of G_ℓ is K_ℓ, the ℓ-fold cover of $PSO(2)$. Thus irreducible Harish-Chandra modules for G_ℓ are irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules on which the element h acts diagonalizably with eigenvalues in $\frac{2}{\ell}\mathbb{Z}$.

Exercise 9.5. Compute the matrix coefficients of the principal series modules, $\psi_{m,n}(g) = (w_m, gw_n)$, $g \in SL_2(\mathbb{R})$.

Hint. Write g as $g = U_1 D U_2$ where

$$U_k = \exp(i\theta_k h) \in SO(2),\ \theta_k \in \mathbb{R}/2\pi\mathbb{Z}$$

for $k = 1, 2$ and $D = \mathrm{diag}(a, a^{-1})$ is diagonal, and express $\psi_{m,n}(g)$ as $e^{i(n\theta_2 - m\theta_1)}\psi(m, n, a, s)$. Write the function $\psi(m, n, a, s)$ in terms of the Gauss hypergeometric function ${}_2F_1$.

Exercise 9.6. (i) Show that for $-1 < s < 0$ the formula

$$(f, g)_s := \int_{\mathbb{R}^2} f(y)\overline{g(z)}|y - z|^{-s-1} dydz$$

defines a positive definite inner product on the space $C_0(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{C}$ with compact support (*Hint*: pass to Fourier transforms).

(ii) Deduce that if f is a measurable function on $\mathbb{R}$ then

$$0 \le (f, f)_s \le \infty,$$

so measurable functions f with $(f, f)_s < \infty$ modulo those for which $(f, f)_s = 0$ form a Hilbert space $\mathcal{H}_s$ with inner product $(,)_s$, which is the completion of $C_0(\mathbb{R})$ under $(,)_s$.

(iii) Let us view $\mathcal{H}_s$ as the space of tensor fields $f(y)(dy)^{\frac{1-s}{2}}$, where f is as in (ii). Show that the complementary series unitary representation $\widehat{P}_+(s)$ of $SL_2(\mathbb{R})$ may be realized in $\mathcal{H}_s$ with G acting naturally on such tensor fields. (*Hint:* show that the differential form $\frac{dydz}{(y-z)^2}$ is invariant under simultaneous Möbius transformations of y, z by the same matrix).

18.757 Representations of Lie Groups
Fall 2023