10. **Chevalley restriction theorem and Chevalley-Shephard-Todd theorem**

10.1. **Chevalley restriction theorem.** Let $\mathfrak{g}$ be a semisimple complex Lie algebra with Cartan subalgebra $\mathfrak{h}$, and let W be the corresponding Weyl group. Given $F \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$, let $\mathrm{Res}(F)$ be its restriction to $\mathfrak{h}$.

Theorem 10.1. *(Chevalley restriction theorem) (i)* $\mathrm{Res}(F) \in \mathbb{C}[\mathfrak{h}]^W$.

(ii) The map $\mathrm{Res} : \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W$ *is a graded algebra isomorpohism.*

Proof. (i) Let G be the adjoint complex Lie group corresponding to $\mathfrak{g}$. Then $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} = \mathbb{C}[\mathfrak{g}]^G$, so F is G-invariant. Thus, denoting by H the maximal torus in G with $\mathrm{Lie}H = \mathfrak{h}$, we see that the normalizer $N(H)$ preserves $\mathrm{Res}(F)$. Since H acts trivially on $\mathfrak{h}$, we get that $W = N(H)/H$ preserves $\mathrm{Res}(F)$, as desired.

(ii) It is clear that Res is a graded algebra homomorphism, so we just need to show that it is bijective. The injectivity of this map follows immediately from the fact that $\mathrm{Res}(F)$ determines the values of F on the subset of semisimple elements $\mathfrak{g}_s \subset \mathfrak{g}$, and this subset is dense in $\mathfrak{g}$.

It remains to prove the surjectivity of Res. Consider the functions

$$F_{\lambda,n}(x) := \mathrm{Tr}_{L_\lambda}(x^n) = \chi_\lambda(x^n),\ x \in \mathfrak{g}$$

in $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$, where χ_λ is the character of L_λ. We'll show that the functions $\mathrm{Res}(F_{\lambda,n})$ for various λ span $\mathbb{C}[\mathfrak{h}]^W[n] = (S^n\mathfrak{h}^*)^W$ for each n, which implies that Res is surjective.

To this end, for every dominant integral weight $\lambda \in P_+$ let m_λ be the orbit sum

$$m_\lambda := \sum_{\mu \in W\lambda} e^\mu \in \mathbb{C}[P]^W.$$

We have

$$\chi_\lambda = \sum_{\mu \le \lambda} N_{\lambda\mu} m_\mu,$$

where $\mu \le \lambda$ means that $\lambda - \mu$ is a (possibly empty) sum of positive roots, and $N_{\lambda\mu}$ is the matrix of weight multiplicities (in particular, $N_{\lambda\lambda} = 1$). This matrix is triangular with ones on the diagonal, so we can invert it and get

$$m_\lambda = \sum_{\mu \le \lambda} \widetilde{N}_{\lambda\mu} \chi_\mu \qquad (6)$$

for some integers $\widetilde{N}_{\lambda\mu}$. Now, for $h \in \mathfrak{h}$, let

$$M_{\lambda,n}(h) := \sum_{\mu \in W\lambda} \mu(h)^n = \frac{|W\lambda|}{|W|} \sum_{w \in W} \lambda(wh)^n.$$

(note that $\mu(x)^n = \mu(x^n)$). By (6) we have

$$M_{\lambda,n}(h) = \sum_{\mu\le\lambda} \widetilde{N}_{\lambda\mu} F_{\mu,n}(h).$$

Thus it suffices to show that $M_{\lambda,n}(h)$ for various λ span $(S^n\mathfrak{h}^*)^W[n]$ for each n. Since averaging over W is a surjection $S^n\mathfrak{h}^* \to (S^n\mathfrak{h}^*)^W$, it suffices to show that the functions λ^n for $\lambda \in P_+$ span $S^n\mathfrak{h}^*$.

Denote the span of these functions by Y. Since P_+ is Zariski dense in $\mathfrak{h}^*$, we find that $\lambda^n \in Y$ for all $\lambda \in \mathfrak{h}^*$. Thus $Y \subset S^n\mathfrak{h}^*$ is a subrepresentation of $GL(\mathfrak{h}^*)$. But $S^n\mathfrak{h}^*$ is an irreducible representation of $GL(\mathfrak{h}^*)$, hence $Y = S^n\mathfrak{h}^*$. This completes the proof of (ii). □

Remark 10.2. 1. Since the Killing form allows us to identify $\mathfrak{g} \cong \mathfrak{g}^*$ and $\mathfrak{h} \cong \mathfrak{h}^*$, the Chevalley restriction theorem is equivalent to the statement that the restriction map $\mathrm{Res} : \mathbb{C}[\mathfrak{g}^*]^{\mathfrak{g}} = (S\mathfrak{g})^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}^*]^W = (S\mathfrak{h})^W$ is a graded algebra isomorphism.

2. The Chevalley restriction theorem trivially generalizes to reductive Lie algebras.

Example 10.3. Let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Then by the fundamental theorem on symmetric functions, $\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[x_1, ..., x_n]^{S_n} = \mathbb{C}[e_1, ..., e_n]$ where

$$e_i(x_1, ..., x_n) = \sum_{k_1<...<k_i} x_{k_1}...x_{k_i}$$

are elementary symmetric functions. The Chevalley restriction theorem thus says that restriction defines an isomorphism between the algebra $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ of conjugation-invariant polynomials of a single matrix A and $\mathbb{C}[e_1, ..., e_n]$. Namely, let $a_i := \mathrm{Tr}(\wedge^i A)$ be the coefficients of the characteristic polynomial of A (up to sign). Then $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} = \mathbb{C}[a_1, ..., a_n]$ and $a_i|_{\mathfrak{h}} = e_i(x_1, ..., x_n)$. Another set of generators are $b_i := \mathrm{Tr}(A^i)$, $1 \le i \le n$; we have $b_i|_{\mathfrak{h}} = p_i(x_1, ..., x_n)$, where

$$p_i(x_1, ..., x_n) := \sum_{k=1}^{n} x_k^i$$

are the power sums, another set of generators of the algebra of symmetric functions. Yet another generating set is $c_i := \mathrm{Tr}(S^i A)$ which restrict to complete symmetric functions

$$h_i(x_1, ..., x_n) = \sum_{k_1\le...\le k_i} x_{k_1}...x_{k_i}.$$

Thus

$$a_i(A) = e_i(x_1, ..., x_n),\ b_i(A) = p_i(x_1, ..., x_n),\ c_i(A) = h_i(x_1, ..., x_n),$$

where $x_1, ..., x_n$ are the eigenvalues of A. Note that $a_1(A) = b_1(A) = c_1(A) = \mathrm{Tr}(A)$ and $a_n(A) = \det A$.

For $\mathfrak{g} = \mathfrak{sl}_n$ (type A_{n-1}), the story is the same, except that $e_1 = p_1 = h_1 = 0$ and $a_1 = b_1 = c_1 = 0$, so they should be removed.

Example 10.4. Similarly, for $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ (types B_n and C_n) we have

$$\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[x_1, ..., x_n]^{S_n \ltimes (\mathbb{Z}/2)^n} =$$

$$\mathbb{C}[x_1^2, ..., x_n^2]^{S_n} = \mathbb{C}[e_2, e_4, ..., e_{2n}] = \mathbb{C}[p_2, p_4, ..., p_{2n}] = C[h_2, h_4, ..., h_{2n}],$$

where e_k, p_k, h_k are symmetric functions of $2n$ variables evaluated at the point $(x_1, ..., x_n, -x_n, ..., -x_1)$, and $e_{2i} = a_{2i}|_{\mathfrak{h}}$, $p_{2i} = b_{2i}|_{\mathfrak{h}}$, $h_{2i} = c_{2i}|_{\mathfrak{h}}$ (note that the odd-indexed symmetric functions evaluate to 0). This is so because the eigenvalues of A are $x_1, ..., x_n, -x_n, ..., -x_1$, and also 0 in the orthogonal case.

The case $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$ (type D_n) is a bit trickier. In this case the Weyl group is $W = S_n \ltimes (\mathbb{Z}/2)^n_+$, where $(\mathbb{Z}/2)^n_+$ is the group of binary n-dimensional vectors with zero sum of coordinates. Thus it is easy to check that

$$\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[e_2, ..., e_{2n-2}, \sqrt{e_{2n}}].$$

where $e_j = e_j(x_1, ..., x_n, -x_n, ..., -x_1)$. The polynomial $\sqrt{e_{2n}} = \mathrm{i}^n x_1 ... x_n$ is the restriction of the **Pfaffian** $\mathrm{Pf}(A) = \sqrt{\det A}$. Thus

$$\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} = \mathbb{C}[a_2(A), ..., a_{2n-2}(A), \mathrm{Pf}(A)].$$

The generators of $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ for exceptional $\mathfrak{g}$ are less explicit, however.

10.2. **Chevalley-Shephard-Todd theorem, part I.** In Examples 10.3, 10.4 we observe that the algebras $\mathbb{C}[\mathfrak{h}]^W$ of Weyl group invariant polynomials for classical groups are free (polynomial) algebras. This is not true for a general finite group: e.g. if $G = \mathbb{Z}/2$ acting on $\mathbb{C}^2$ by $(x, y) \mapsto (-x, -y)$ then the ring of invariants $\mathbb{C}[x, y]^{\mathbb{Z}/2}$ is $\mathbb{C}[a, b, c]$ where $a = x^2, b = xy, c = y^2$, and it is not free – it has a relation $ac = b^2$ (and the set of generators is minimal). It turns out, however, that this is true for all Weyl groups and more generally complex reflection groups.

Definition 10.5. A diagonalizable automorphism $g : V \to V$ of a finite-dimensional complex vector space V is called a **complex reflection** if $\mathrm{rank}(g-1) = 1$; in other words, in some basis $g = \mathrm{diag}(\lambda, 1, ..., 1)$ where $\lambda \neq 0, 1$. A **complex reflection group** is a finite subgroup $G \subset GL(V)$ generated by complex reflections.

For example, the Weyl group $W \subset GL(\mathfrak{h})$ of a semisimple Lie algebra $\mathfrak{g}$ and, more generally, a finite Coxeter group is a complex reflection group, but there are others, e.g. $S_n \ltimes (\mathbb{Z}/m)^n$ acting on $\mathbb{C}^n$ for $m > 2$, or, more generally, the subgroup $G(m, d, n)$ in this group consisting of elements for which the sum of $\mathbb{Z}/m$-coordinates lies in $d \cdot \mathbb{Z}/m$ for some divisor d of m.

It is easy to see that any complex reflection group is uniquely a product of irreducible ones, and irreducible complex reflection groups were classified by Shephard and Todd in 1954. Besides symmetric groups S_n acting on $\mathbb{C}^{n-1}$ and $G(m, d, n)$ acting on $\mathbb{C}^n$ (which includes dihedral groups), there are 34 exceptional groups, which include 19 subgroups of GL_2, 6 exceptional Coxeter groups of rank ≥ 3 ($H_3, H_4, F_4, E_6, E_7, E_8$), and 9 other groups.

Theorem 10.6. *(Chevalley-Shephard-Todd theorem, part I,* [Che], [ST]*) Let V be a finite-dimensional complex vector space and $G \subset GL(V)$ be a finite subgroup. Then $\mathbb{C}[V]^G$ is a polynomial algebra if and only if G is a complex reflection group.*

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Fall 2023