## 14. Harish-Chandra isomorphism, maximal quotients

14.1. **The Harish-Chandra isomorphism.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. By the PBW theorem, we then have a linear isomorphism

$$\mu : U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \to U(\mathfrak{g})$$

given by multiplication. We also have the linear map

$$\beta : U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \to U(\mathfrak{h})$$

given by

$$a_- \otimes h \otimes a_+ \mapsto \varepsilon(a_-)\varepsilon(a_+)h,\ \ a_\pm \in U(\mathfrak{n}_\pm), h \in U(\mathfrak{h}),$$

where $\varepsilon : U(\mathfrak{n}_\pm) \to \mathbb{C}$ is the augmentation homomorphism (the counit). Thus we get a linear map

$$HC := \beta \circ \mu^{-1} : U(\mathfrak{g}) \to U(\mathfrak{h}) = S\mathfrak{h} = \mathbb{C}[\mathfrak{h}^*]$$

called the **Harish-Chandra map**.

**Theorem 14.1.** *(Harish-Chandra) (i) If $b \in U(\mathfrak{g})$ and $c \in Z(\mathfrak{g})$ then $HC(bc) = HC(b)HC(c)$. In particular, the restriction of $HC$ to $Z(\mathfrak{g})$ is an algebra homomorphism.*

*(ii) Define the shifted action of $W$ on $\mathfrak{h}^*$ by $w \bullet x := w(x+\rho)-\rho$ where $\rho$ is the half sum of positive roots (or, equivalently, sum of fundamental weights). Then $HC$ maps $Z(\mathfrak{g})$ into the space of invariants $\mathbb{C}[\mathfrak{h}^*]^{W\bullet}$. That is, for any $b \in Z(\mathfrak{g})$ we have $HC(b)(\lambda) = f_b(\lambda + \rho)$ for some $f_b \in \mathbb{C}[\mathfrak{h}^*]^W$.*

*(iii) If $V$ be a highest weight representation of $\mathfrak{g}$ with highest weight $\lambda$ then*

$$f_b(\lambda + \rho) = (v_\lambda^*, bv_\lambda)$$

*where $v_\lambda$ is a highest weight vector of $V$ and $v_\lambda^*$ the lowest weight vector of $V^*$ such that $(v_\lambda^*, v_\lambda) = 1$. Thus if $b \in Z(\mathfrak{g})$ then $HC(b)(\lambda)$ is the scalar by which $b$ acts on a highest weight module with highest weight $\lambda$.*

*(iv) The map $HC : Z(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]^{W\bullet}$ is a filtered algebra homomorphism and $\mathrm{gr}(HC) = \mathrm{Res}$, the Chevalley restriction homomorphism $(S\mathfrak{g})^{\mathfrak{g}} \to (S\mathfrak{h})^W$.*

*(v) $HC$ is an algebra isomorphism.*

The isomorphism $HC : Z(\mathfrak{g}) \to \mathbb{C}[\mathfrak{h}^*]^{W\bullet}$ is called the **Harish-Chandra isomorphism**.

*Proof.* Let $b = a_-ha_+ \in U(\mathfrak{g})$. We have

$$(v_\lambda^*, bv_\lambda) = (v_\lambda^*, a_-ha_+v_\lambda) = \varepsilon(a_-)\varepsilon(a_+)\lambda(h) = HC(b)(\lambda).$$

Thus

$$HC(bc)(\lambda) = (v_\lambda^*, bcv_\lambda) = (v_\lambda^*, bv_\lambda)(v_\lambda^*, cv_\lambda) = HC(b)(\lambda)HC(c)(\lambda)$$

since $c$ is central; namely, the last factor is just the eigenvalue of $c$ on $V$. This proves (i).

To establish (ii),(iii), it remains to show that for $b \in Z(\mathfrak{g})$, $HC(b)$ is invariant under the shifted action of all $w \in W$. To this end, it suffices to show this for $w = s_i$, a simple reflection. For this purpose, consider the Verma module $M_\lambda$ with $(\lambda + \rho, \alpha_i^\vee) = n \in \mathbb{Z}_{>0}$. Then $f_i^n v_\lambda$ generates a copy of $M_{\lambda - n\alpha_i} = M_{s_i \bullet \lambda}$ inside $M_\lambda$. Thus we get $HC(b)(\lambda) = HC(b)(s_i \bullet \lambda)$. Since this holds on a Zariski dense set, it holds identically, which yields (ii),(iii).

(iv) follows immediately from (iii).

Finally, (v) follows from (iv) and the Chevalley restriction theorem, since any filtered map whose associated graded is an isomorphism is itself an isomorphism. □

**Remark 14.2.** Kostant theorems and the Harish-Chandra isomorphism extend trivially to reductive Lie algebras.

14.2. **Maximal quotients.** Let $\mathfrak{g}$ be a semisimple Lie algebra and $M$ a $\mathfrak{g}$-module on which the center $Z(\mathfrak{g})$ acts by a character

$$\chi : Z(\mathfrak{g}) \to \mathbb{C}$$

(for example, $M$ is irreducible). In view of the Harish-Chandra isomorphism theorem, we have $\chi = \chi_\lambda$, where

$$\chi_\lambda(z) = HC(z)(\lambda)$$

for a unique $\lambda \in \mathfrak{h}^*$ modulo the shifted action of $W$. As mentioned in Subsection 7.2, the element $\chi_\lambda$ is called the **infinitesimal character** or **central character** of $M$.

If $M$ is a $\mathfrak{g}$-bimodule then it carries two actions of $Z(\mathfrak{g})$, by left and by right multiplication. If these actions are by characters, then they are called the **left and right infinitesimal characters** of $M$. The infinitesimal character of $M$ is then the pair $(\theta, \chi)$ where $\theta$ is the left infinitesimal character and $\chi$ the right infinitesimal character of $M$.

For a character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ let

$$U_\chi = U_\chi(\mathfrak{g}) := U(\mathfrak{g})/(z - \chi(z), z \in Z(\mathfrak{g})).$$

This algebra is called the **maximal quotient** of $U(\mathfrak{g})$ with infinitesimal character $\chi$, as every $U(\mathfrak{g})$-module with such infinitesimal character factors through $U_\chi$. Note that $U_\chi$ is a $\mathfrak{g}$-bimodule with infinitesimal character $(\chi, \chi)$ (as it is a $U_\chi$-bimodule).

Theorem 13.5 immediately implies

**Corollary 14.3.** *For any finite-dimensional irreducible* $\mathfrak{g}$*-module* $V$ *we have* $\dim \mathrm{Hom}_{\mathfrak{g}}(V, U_\chi) = \dim V[0]$*, where* $\mathfrak{g}$ *acts on* $U_\chi$ *by the adjoint action. Thus* $U_\chi$ *is a Harish-Chandra* $\mathfrak{g}$*-bimodule.*

**Corollary 14.4.** *If* $V$ *is a finite-dimensional* $\mathfrak{g}$*-bimodule then* $V \otimes U_\chi$ *is a Harish-Chandra* $\mathfrak{g}$*-bimodule.*

*Proof.* This follows from Corollary 14.3 and Exercise 5.12. □

**Corollary 14.5.** *(i) Every irreducible* $\mathfrak{g}$*-bimodule* $M$ *locally finite under the adjoint* $\mathfrak{g}$*-action is a quotient of* $V \otimes U_\chi$ *for some finite-dimensional irreducible* $\mathfrak{g}$*-module* $V$ *with trivial right action of* $\mathfrak{g}$*, where* $\chi$ *is the right infinitesimal character of* $M$*.*

*(ii) Every irreducible* $\mathfrak{g}$*-bimodule locally finite under the adjoint* $\mathfrak{g}$*-action is a Harish-Chandra bimodule.*

*Proof.* (ii) follows from (i) and Corollary 14.4, so it suffices to prove (i). By Dixmier's lemma (Lemma 7.2), $M$ has some infinitesimal character $(\theta, \chi)$. Let $V \subset M$ be an irreducible finite-dimensional subrepresentation under $\mathfrak{g}_{\mathrm{ad}}$. Let us view $V^*$ as a $\mathfrak{g}$-bimodule with action

$$(af)(x) = -f(ax), \;\; fb = 0$$

for $a, b \in \mathfrak{g}, x \in V, f \in V^*$, and consider the tensor product $V^* \otimes M$, which is a $\mathfrak{g}$-bimodule with action

$$a \circ (f \otimes m) := af \otimes m + f \otimes am, \;\; (f \otimes m) \circ b := f \otimes mb.$$

The canonical element $u \in V^* \otimes V \subset V^* \otimes M$ is $\mathfrak{g}_{\mathrm{ad}}$-invariant (i.e., commutes with $\mathfrak{g}$). Thus we have a bimodule homomorphism $\psi : U(\mathfrak{g}) \to V^* \otimes M$ given by $\psi(c) := uc = \sum v_i^* \otimes v_i c$, where $v_i$ is a basis of $V$ and $v_i^*$ the dual basis of $V^*$. Moreover, since the right infinitesimal character of $M$ is $\chi$, this homomorphism descends to $\overline{\psi} : U_\chi \to V^* \otimes M$. This gives rise to a nonzero homomorphism of bimodules $\xi : V \otimes U_\chi \to M$, where the right $\mathfrak{g}$-module structure of $V$ is trivial. Since $M$ is irreducible, $\xi$ is surjective. Thus the result follows from Corollary 14.4. □

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