17. **The nilpotent cone of $\mathfrak{g}$**

17.1. **The nilpotent cone.** Let $(S\mathfrak{g})_0$ be the quotient of $S\mathfrak{g}$ by the ideal generated by the positive degree part of $(S\mathfrak{g})^{\mathfrak{g}}$, i.e. by the free homogeneous generators $p_1, ..., p_r$ of $(S\mathfrak{g})^{\mathfrak{g}}$ (which exist by Kostant's theorem). The scheme

$$\mathcal{N} := \mathrm{Spec}(S\mathfrak{g})_0 \subset \mathfrak{g}^* \cong \mathfrak{g}$$

is called the **nilpotent cone** of $\mathfrak{g}$. It follows from the Kostant theorem that $p_1, ..., p_r$ is a regular sequence, i.e., this scheme is a complete intersection of codimension r in $\mathfrak{g}$ (see Remark 12.11), i.e., of dimension

$$\dim \mathcal{N} = \dim \mathfrak{g} - r = |R| = 2|R_\pm| = 2\dim \mathfrak{n}_\pm,$$

the number of roots of $\mathfrak{g}$.

Let $x \in \mathfrak{g}$ be a nilpotent element. Recall that then x is conjugate to an element $y \in \mathfrak{n}_+$ and $\mathrm{Ad}(t^{2\rho^\vee})y \to 0$ as $t \to 0$, where $\rho^\vee$ is the half-sum of positive coroots of $\mathfrak{g}$. Thus $p_i(x) = p_i(y) = 0$ and hence $x \in \mathcal{N}(\mathbb{C})$. On the other hand, if x is not nilpotent then $\mathrm{ad}(x)$ is not a nilpotent operator, so $\mathrm{Tr}(\mathrm{ad}(x)^N) \neq 0$ for some N, hence $x \notin \mathcal{N}(\mathbb{C})$. It follows that $\mathcal{N}(\mathbb{C})$ is exactly the set of nilpotent elements of $\mathfrak{g}$, hence the term "nilpotent cone".

For example, for $\mathfrak{g} = \mathfrak{sl}_2$ we have $r = 1$ and

$$p_1(A) = -\det A = x^2 + yz$$

for $A := \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \in \mathfrak{g}$, so $\mathcal{N}$ is the usual quadratic cone in $\mathbb{C}^3$ defined by the equation $x^2 + yz = 0$.

17.2. **The principal $\mathfrak{sl}_2$ subalgebra.** The **principal $\mathfrak{sl}_2$ subalgebra** of $\mathfrak{g}$ is the subalgebra spanned by $e := \sum_{i=1}^r e_i$, $f := \sum_i c_i f_i$ and $h := [e, f] = \sum_i c_i h_i = 2\rho^\vee$. Thus c_i are found from the equations $\sum_i c_i a_{ij} = 2$ for all j, where $A = (a_{ij})$ is the Cartan matrix of $\mathfrak{g}$.

Lemma 17.1. *The restriction of the adjoint representation of $\mathfrak{g}$ to its principal $\mathfrak{sl}_2$-subalgebra is isomorphic to $L_{2m_1} \oplus ... \oplus L_{2m_r}$ for appropriate $m_i \in \mathbb{Z}_{>0}$.*

Proof. Consider the corresponding action of the group $SL_2(\mathbb{C})$. The element $-1 \in SL_2(\mathbb{C})$ acts on $\mathfrak{g}$ by $\exp(2\pi i \rho^\vee) = 1$ since $\rho^\vee$ is an integral coweight. Thus only even highest weight $\mathfrak{sl}_2$-modules may occur in the decomposition of $\mathfrak{g}$. Since $\rho^\vee$ is regular, the 0-weight space of this module (the centralizer $Z_{\mathfrak{g}}(\rho^\vee)$) is $\mathfrak{h}$, i.e., has dimension r. Thus $\mathfrak{g}$ has r indecomposable direct summands over the principal $\mathfrak{sl}_2$, as claimed. □

The numbers m_i (arranged in non-decreasing order) are called the **exponents** of $\mathfrak{g}$. We will soon see that $m_i = d_i - 1$, where d_i are the degrees of $\mathfrak{g}$.

17.3. **Regular elements.** Recall that $x \in \mathfrak{g}$ is **regular** if the dimension of its centralizer is $r = \mathrm{rank}\mathfrak{g}$ (the smallest it can be). Thus regular elements form an open set $\mathfrak{g}_{\mathrm{reg}} \subset \mathfrak{g}$.

Lemma 17.2. *The element $e = \sum_{i=1}^r e_i$ is regular.*

Proof. By Lemma 17.1, the centralizer $Z_{\mathfrak{g}}(e)$ is spanned by the highest vectors of the representations $L_{2m_1}, ..., L_{2m_r}$, hence has dimension r. □

Corollary 17.3. *Let B_+ be the Borel subgroup of G with Lie algebra $\mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$. Then $\mathrm{Ad}(B_+)e$ is the set of elements $\sum_{\alpha \in R_+} c_\alpha e_\alpha$ with $c_\alpha \in \mathbb{C}$ and $c_{\alpha_i} \neq 0$ for all i.*

Proof. Since by Lemma 17.2 $\dim Z_{\mathfrak{g}}(e) = r$, we have

$$\dim[e, \mathfrak{n}_+] \geq |R_+| - r = \dim[\mathfrak{n}_+, \mathfrak{n}_+].$$

Since $[e, \mathfrak{n}_+] \subset [\mathfrak{n}_+, \mathfrak{n}_+]$, we get that $[e, \mathfrak{n}_+] = [\mathfrak{n}_+, \mathfrak{n}_+]$. It follows that if $N_+ = \exp(\mathfrak{n}_+)$ then $\mathrm{Ad}(N_+)e = e + [\mathfrak{n}_+, \mathfrak{n}_+]$ is the set of expressions $\sum_{\alpha \in R_+} c_\alpha e_\alpha$ with $c_{\alpha_i} = 1$ for all i. The statement follows by adding the action of the maximal torus $H = \exp(\mathfrak{h})$, which allows to set c_{α_i} to arbitrary nonzero values. □

17.4. **Properties of the nilpotent cone.**

Proposition 17.4. *The nilpotent cone is reduced.*

Proposition 17.4 is proved in the following exercise.

Exercise 17.5. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra.

(i) Let R_0 be the graded algebra in Theorem 12.2. Show that the top degree of this algebra is $D := \sum_{i=1}^r (d_i - 1)$ and $R_0[D] = \mathbb{C}\Delta$, where $\Delta := \prod_{\alpha \in R_+} \alpha$. Deduce that $\sum_{i=1}^r (d_i - 1) = |R_+|$, , the number of positive roots.

(ii) Let $\mathfrak{g} = \oplus_{i=1}^r L_{2m_i}$ be the decomposition of $\mathfrak{g}$ as a module over the principal $\mathfrak{sl}_2$-subalgebra (e, f, h) given by Lemma 17.1, i.e., m_i are the exponents of $\mathfrak{g}$. Show that $m_1 = 1$ and $\sum_{i=1}^r m_i = |R_+|$. Moreover, show that if $\mu_{\mathfrak{g}}$ is the partition $(m_r, ..., m_1)$ then the conjugate partition $\mu_{\mathfrak{g}}^\dagger$ is $(n_1, ..., n_{\mathrm{h}-1})$, where n_i is the number of positive roots α of height i (i.e., $(\rho^\vee, \alpha) = i$) and $\mathrm{h} := m_r + 1$. Conclude that $\mathrm{h} = (\rho^\vee, \theta) + 1$ where θ is the maximal root, i.e., the **Coxeter number** of $\mathfrak{g}$.

(iii)(a) Let b_i be the lowest weight vectors of L_{2m_i}, and

$$\mathfrak{z}_f := \oplus_{i=1}^r \mathbb{C} b_i \subset \mathfrak{g}$$

be the centralizer of f. Show that $\mathfrak{g} = \mathfrak{z}_f \oplus T_e O_e$, where $O_e = \mathrm{Ad}(G)e$ is the orbit of e. Thus the affine space $e + \mathfrak{z}_f$ is transversal to O_e at e. This affine space is called the **Kostant slice**.

(iii)(b) Consider the $\mathbb{C}^\times$-action on $\mathfrak{g}$ given by

$$t \circ x = t^{\frac{1}{2}\mathrm{ad}(h)-1} x.$$

Show that this action preserves the decomposition of (ii), and the linear coordinates b_i^* on $\mathfrak{z}_f$ have homogeneity degrees m_i+1 under this action.

(iv) Let $(S\mathfrak{g}^*)^{\mathfrak{g}} = \mathbb{C}[p_1, ..., p_r]$, $\deg p_i = d_i$, and let $\widetilde{p}_i(y) := p_i(e+y)$, $y \in \mathfrak{z}_f$. Show that $\widetilde{p}_i$ are polynomials of b_j^* homogeneous under the $\mathbb{C}^\times$-action of (iii) of degrees d_i. Deduce from this and the identity $\sum_i (d_i - 1) = \sum_i m_i$ proved in (i),(ii) that

$$d_i - 1 = m_i$$

and thus $\widetilde{p}_i = b_i^*$ (under appropriate choice of basis). Conclude that the differentials dp_i are linearly independent at $e \in \mathfrak{g}$.

(v) Work out (i)-(iv) explicitly for $\mathfrak{g} = \mathfrak{sl}_n$.

(vi) Prove Proposition 17.4. **Hint:** View $\mathcal{O}(\mathcal{N})$ as an algebra over $\mathcal{R} := S\mathfrak{n}_+ \otimes S\mathfrak{n}_-$. Use the arguments of Subsection 13.1 to show that it is a free $\mathcal{R}$-module of rank $|W|$. Show that the specialization of $\mathcal{O}(\mathcal{N})$ at a generic point $z \in \mathfrak{n}_+^* \times \mathfrak{n}_-^*$ is a semisimple algebra of dimension $|W|$ (use (iv)). Now take $f \in \mathcal{O}(\mathcal{N})$ such that $f^k = 0$ for some k, and deduce that the specialization of f at z is zero. Conclude that $f = 0$.

Proposition 17.6. *(i) The orbit $O_e := \mathrm{Ad}(G)e$ is open and dense in $\mathcal{N}$.*

(ii) All regular nilpotent elements in $\mathfrak{g}$ are conjugate to e.

(iii) $\mathcal{N}$ is an irreducible affine variety. Thus $(S\mathfrak{g})_0$ is an integral domain.

Proof. (i) This follows from Corollary 17.3 and the fact that every nilpotent element in $\mathfrak{g}$ can be conjugated into $\mathfrak{n}_+$.

(ii) The orbit O_x of every regular nilpotent element x has the same dimension as O_e, so the statement follows from (i). Indeed, since O_e is open and dense, $\mathcal{N} \setminus O_e$ has smaller dimension than $\mathcal{N}$, hence can't contain O_x.

(iii) follows from (i) and Proposition 17.4, since O_e is smooth and connected (being an orbit of a connected group), hence irreducible. $\square$

Corollary 17.7. *U_χ is an integral domain for all χ.*

Proof. This follows from Proposition 17.6(iii) since $\mathrm{gr}(U_\chi) = (S\mathfrak{g})_0$. $\square$

Exercise 17.8. Let e be a nilpotent element in a semisimple complex Lie algebra $\mathfrak{g}$, and $\mathfrak{g}^e$ be the centralizer of e. Let $(,)$ be the Killing form of $\mathfrak{g}$.

(i) Show that $(e, \mathfrak{g}^e) = 0$ (prove that for any $x \in \mathfrak{g}^e$, the operator $\mathrm{ad}_e \mathrm{ad}_x$ is nilpotent).

(ii) Show that there exists $h \in \mathfrak{g}$ such that $[h, e] = 2e$ (use that $\mathrm{Im}(\mathrm{ad}_e) = \mathfrak{g}^{\mathrm{e}\perp}$ to deduce that $e \in \mathrm{Im}(\mathrm{ad}_e)$).

(iii) Show that in (ii), h can be chosen semisimple (consider the Jordan decomposition $h = s + n$). From now on we choose h in such a way.

(iv) Show that $\mathbb{C}h \oplus \mathfrak{g}^e$ is a Lie subalgebra of $\mathfrak{g}$.

(v) Assume that $\mathfrak{g}^e$ is nilpotent. Show that there is a basis of $\mathfrak{g}$ in which the operator ad_x is upper triangular for all $x \in \mathbb{C}h \oplus \mathfrak{g}^e$ (use Lie's theorem). Deduce that $(h, x) = 0$ for all $x \in \mathfrak{g}^e$.

(vi) Show that if $\mathfrak{g}^e$ is nilpotent then there are $h, f \in \mathfrak{g}$ such that $[h, e] = 2e$, $[e, f] = h$ and $[h, f] = -2f$. In other words, there is a homomorphism of Lie algebras $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ such that $\phi(E) = e$, $\phi(H) = h$, $\phi(F) = f$. Show that h is semisimple and f is nilpotent.

(vii) (Jacobson-Morozov theorem, part I) Show that the conclusion of (vi) holds for any e (without assuming that $\mathfrak{g}^e$ is nilpotent). (**Hint:** use induction in $\dim \mathfrak{g}$. If $\mathfrak{g}^e$ is not nilpotent, use Jordan decomposition to find a nonzero semisimple element $x \in \mathfrak{g}^e$ and consider the Lie algebra $\mathfrak{g}^x$. Show that $\mathfrak{g}' := [\mathfrak{g}^x, \mathfrak{g}^x]$ is semisimple and $e \in \mathfrak{g}'$).

(viii) Show that for given e, h, the homomorphism ϕ in (vi,vii) is unique (i.e., f is uniquely determined by e, h).

(ix) (Jacobson-Morozov theorem, part II) Show that for a fixed e, $\exp(\mathfrak{g}^e)$ (the Lie subgroup corresponding to $\mathfrak{g}^e$) is a closed Lie subgroup of the adjoint group G_{ad} corresponding to $\mathfrak{g}$, and the element h (hence also f) can be chosen uniquely up to conjugation by $\exp(\mathfrak{g}_e)$. (**Hint**: Let h' be another choice of h, and consider the element $h' - h \in \mathfrak{g}^e$.)

(x) Explain why the Jacobson-Morozov theorem extends to reductive Lie algebras (where by a nilpotent element we mean one that is nilpotent in any finite-dimensional representation). Give an elementary proof of this theorem for $\mathfrak{g} = \mathfrak{gl}_n$ using only linear algebra.

(xi) Show that there are finitely many conjugacy classes of nilpotent elements in $\mathfrak{g}$, i.e., the nilpotent cone $\mathcal{N}$ has finitely many G_{ad}-orbits. (**Hint:** Consider the variety X of homomorphisms $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$ and show that it is a disjoint union of finitely many closed G_{ad}-orbits. To this end, show that the tangent space to X at each $x \in X$ coincides with the tangent space of the orbit Gx at the same point, using that $\mathrm{Ext}^1_{\mathfrak{sl}_2}(\mathbb{C}, \mathfrak{g}) = 0$).

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