18. **Maps of finite type, Duflo-Joseph theorem**

18.1. **Maps of finite type.** Let M, N be $\mathfrak{g}$-modules. Let $\mathrm{Hom}_{\mathrm{fin}}(M,N)$ be the space of linear maps from M to N which generate a finite dimensional $\mathfrak{g}$-module under the adjoint action $a \circ T := [a,T]$. The elements of $\mathrm{Hom}_{\mathrm{fin}}(M,N)$ are called **linear maps of finite type**. For example, a module homomorphism is a map of finite type, as it generates a trivial 1-dimensional $\mathfrak{g}$-module.

Exercise 18.1. Show that any map of finite type has the form $(f \otimes 1) \circ \Phi$, where $f \in V^*$ for some finite dimensional $\mathfrak{g}$-module V and $\Phi : M \to V \otimes N$ is a module homomorphism.

Note that $\mathrm{Hom}_{\mathrm{fin}}(M,N)$ is a $\mathfrak{g}$-bimodule with bimodule structure given by

$$(a,b) \circ T := aT + Tb,$$

$a, b \in \mathfrak{g}$. Moreover, it is clear that if M has central character χ and N has central character θ then $\mathrm{Hom}_{\mathrm{fin}}(M,N)$ has central character (θ, χ).

Proposition 18.2. *If $M, N \in \mathcal{O}$ then $\mathrm{Hom}_{\mathrm{fin}}(M,N)$ is an admissible $\mathfrak{g}$-bimodule.*

Proof. We must show that fo every simple finite dimensional $\mathfrak{g}$-module V, the space

$$\mathrm{Hom}_{\mathfrak{g}}(V, \mathrm{Hom}_{\mathrm{fin}}(M,N)) = \mathrm{Hom}_{\mathfrak{g}}(V, \mathrm{Hom}_{\mathbb{C}}(M,N))$$

is finite dimensional. Let $\mu(M,N,V)$ be its dimension (a nonnegative integer of infinity). Since the functor $(M,N) \mapsto \mathrm{Hom}_{\mathbb{C}}(M,N)$ is exact in both arguments, for any short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

we have

$$\mu(M_2,N,V) = \mu(M_1,N,V) + \mu(M_3,N,V),$$
$$\mu(N,M_2,V) = \mu(N,M_1,V) + \mu(N,M_3,V).$$

Thus, since M, N have finite length, it suffices to establish the result for M, N simple. Then M is a quotient of M_λ and N a submodule of $M_\mu^\vee$ for some λ, μ, so $\mathrm{Hom}_{\mathbb{C}}(M,N) \subset \mathrm{Hom}_{\mathbb{C}}(M_\lambda, M_\mu^\vee)$. But by Exercise 8.13, for any finite dimensional $\mathfrak{g}$-module V,

$$\mathrm{Hom}_{\mathfrak{g}}(V, \mathrm{Hom}_{\mathbb{C}}(M_\lambda, M_\mu^\vee)) \cong \mathrm{Hom}_{\mathfrak{g}}(V \otimes M_\lambda, M_\mu^\vee) \cong$$

$$\mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V^* \otimes M_\mu^\vee) \cong V^*[\lambda - \mu].$$

This implies the statement. ∎

Proposition 18.3. *For $M, N \in \mathcal{O}$ and a finite dimensional $\mathfrak{g}$-module V we have*

$$\mathrm{Hom}_{\mathrm{fin}}(M, V \otimes N) = V \otimes \mathrm{Hom}_{\mathrm{fin}}(M, N).$$

Exercise 18.4. Prove Proposition 18.3.

Proposition 18.5. *Let V be a finite dimensional $\mathfrak{g}$-module. Then for any $\lambda \in \mathfrak{h}^*$, we have*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V \otimes M_\lambda) = \dim V[0].$$

Thus the multiplicity of V in $\mathrm{Hom}_{\mathrm{fin}}(M_\lambda, M_\lambda)$ *equals* $\dim V[0]$.

Proof. By Exercise 8.14, the statement holds if M_λ is irreducible, i.e., generically. Thus $\dim \mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V \otimes M_\lambda) \geq \dim V[0]$, and it remains to prove the opposite inequality. Let M_μ be the simple Verma submodule of M_λ. Given $\Phi : M_\lambda \to V \otimes M_\lambda$, we claim that the restriction of Φ to M_μ must land in $V \otimes M_\mu$. Indeed, otherwise we will have a nonzero (hence injective) homomorphism $M_\mu \to V \otimes (M_\lambda/M_\mu)$, which is impossible by growth considerations.

But by Exercise 8.14, the statement holds if λ is replaced by μ. So if it does not hold for λ then there is a nonzero Φ which kills M_μ. Thus Φ defines a nonzero homomorphism $M_\lambda/M_\mu \to M_\lambda \otimes V$, which is impossible since $M_\lambda \otimes V$ is a free, hence torsion free $U(\mathfrak{n}_-)$-module, while every homogeneous vector in M_λ/M_μ is torsion (as this module does not contain free $U(\mathfrak{n}_-)$-submodules by growth considerations). This establishes the proposition. □

Remark 18.6. Note that Proposition 18.5 does not extend to maps $M_{\lambda+\nu} \to V \otimes M_\lambda$ where $\nu \in P$ is nonzero. Namely, if M_λ is irreducible then we have $\dim \mathrm{Hom}_{\mathfrak{g}}(M_{\lambda+\nu}, V \otimes M_\lambda) = \dim V[\nu]$, so in general $\dim \mathrm{Hom}_{\mathfrak{g}}(M_{\lambda+\nu}, V \otimes M_\lambda) \geq \dim V[\nu]$, and the inequality can, in fact, be strict. The simplest example is $\mathfrak{g} = \mathfrak{sl}_2$, $V = \mathbb{C}$, $\lambda = 0$, $\nu = -2$, in which case the left hand side is 1 and the right hand side is 0.

Also the expectation value map

$$\langle , \rangle : \mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V \otimes M_\lambda) \to V[0]$$

need not be an isomorphism, even though its source and target have the same dimension. The simplest example is $\mathfrak{g} = \mathfrak{sl}_2$, $\lambda = 0$, and V is the adjoint representation. We have

$$\dim \mathrm{Hom}_{\mathfrak{g}}(M_0, V \otimes M_0) = \dim \mathrm{Hom}_{\mathfrak{g}}(M_0, V \otimes M_{-2}) = 1,$$

so the only (up to scaling) nonzero homomorphism $\Phi : M_0 \to V \otimes M_0$ in fact lands in $V \otimes M_{-2} \subset V \otimes M_0$. Thus $\langle \Phi \rangle = 0$.

18.2. **The Duflo-Joseph theorem.**

Proposition 18.7. *The action homomorphism*

$$\phi : U_{\chi_{\lambda+\rho}} \to \mathrm{Hom}_{\mathrm{fin}}(M_\lambda, M_\lambda)$$

is injective.

Proof. Let $M_\mu \subset M_\lambda$ be a simple Verma submodule with highest weight vector v. Let $B_{\mu,\beta} : U(\mathfrak{n}_+)[\beta] \otimes U(\mathfrak{n}_-)[-\beta] \to \mathbb{C}$ be the pairing defined by the equality

$$abv = B_{\mu,\beta}(a, b)v.$$

As M_μ is simple, this pairing is nondegenerate.

Consider the multiplication map

$$\xi : U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+) \to U_{\chi_{\lambda+\rho}}.$$

We claim that the map $\phi \circ \xi$ is injective, hence so are ξ and $\phi|_{\mathrm{Im}\xi}$. Indeed, let $x \in U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+)$ be a nonzero element. We can uniquely write $x = \sum_{\alpha \in Q_+} x_\alpha$, where $x_\alpha \in U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+)[\alpha]$. Let $\beta \in Q_+$ be a minimal element such that $x_\beta = \sum_i b_i \otimes a_i \neq 0$, where $\{a_i\}$ is a basis of $U(\mathfrak{n}_+)[\beta]$. Let $\{a_i^*\}$ be the dual basis of $U(\mathfrak{n}_-)[-\beta]$ with respect to $B_{\mu,\beta}$. Then

$$(\phi \circ \xi)(x)a_j^* v = b_j v.$$

Since b_j are not all zero, there exists j such that $b_j v \neq 0$. It follows that $(\phi \circ \xi)(x) \neq 0$, as claimed.

Thus, denoting the PBW filtration by F_n, we have

$$\dim F_n(U_{\chi_{\lambda+\rho}}/\mathrm{Ker}\phi) \geq \dim F_n(U(\mathfrak{n}_-) \otimes U(\mathfrak{n}_+)) \geq Cn^{\dim \mathfrak{g} - r}$$

for some $C > 0$. On the other hand, assume that $\mathrm{Ker}\phi \neq 0$ and consider the nonzero ideal

$$\mathrm{gr}(\mathrm{Ker}\phi) \subset (S\mathfrak{g})_0 = \mathcal{O}(\mathcal{N}).$$

This ideal contains a principal ideal $\mathcal{O}(\mathcal{N})f$, where $f \in \mathcal{O}(\mathcal{N})$ is a nonzero homogeneous element. Since $\mathcal{O}(\mathcal{N})$ is a domain (Proposition 17.6(iii)), this ideal is a free $\mathcal{O}(\mathcal{N})$-module generated by f.

$$\dim F_n(U_{\chi_{\lambda+\rho}}/\mathrm{Ker}\phi) = \dim \mathrm{gr}_{\leq n}(\mathcal{O}(\mathcal{N})/\mathrm{gr}(\mathrm{Ker}\phi)) =$$

$$\leq \dim \mathrm{gr}_{\leq n}(\mathcal{O}(\mathcal{N})/\mathcal{O}(\mathcal{N})f) \leq C' n^{\dim \mathfrak{g} - r - 1}.$$

for some $C' > 0$. So we get that $Cn^{\dim \mathfrak{g} - r} \leq C' n^{\dim \mathfrak{g} - r - 1}$. This is a contradiction, so $\mathrm{Ker}\phi = 0$ and thus ϕ is injective. □

Corollary 18.8. *(The Duflo-Joseph theorem)* ϕ *is an isomorphism.*

Proof. Consider the restriction ϕ_V of ϕ to the V^*-isotypic component. Thus
$$\phi_V : \mathrm{Hom}_{\mathfrak{g}}(V^*, (U_{\chi_{\lambda+\rho}})_{\mathrm{ad}}) \to \mathrm{Hom}_{\mathfrak{g}}(M_\lambda, V \otimes M_\lambda).$$
By Kostant's theorem, the source of this map has dimension $\dim V[0]$, while by Proposition 18.5, so does the target. Since by Proposition 18.7 ϕ_V is injective, it follows that ϕ_V is an isomorphism for all V, hence so is ϕ. □

Corollary 18.9. *If V is a finite dimensional $\mathfrak{g}$-module then the natural map $V \otimes U_{\chi_{\lambda+\rho}} \to \mathrm{Hom}_{\mathrm{fin}}(M_\lambda, V \otimes M_\lambda)$ is an isomorphism.*

Proof. This follows from Proposition 18.3 and Corollary 18.8. □

18.3. **Central characters of Harish-Chandra bimodules.**

Corollary 18.10. *Let V be a finite dimensional $\mathfrak{g}$-module and $\lambda \in \mathfrak{h}^*$.*

(i) The left central characters occurring in $V \otimes U_{\chi_\lambda}$ are $\chi_{\lambda+\nu}$ where ν runs over weights of V.

(ii) If M is a $\mathfrak{g}$-module with central character χ_λ then the central characters occurring in $V \otimes M$ are among $\chi_{\lambda+\nu}$ where ν runs over weights of V.

(iii) If M is a nonzero Harish-Chandra $\mathfrak{g}$-bimodule with central character (χ_λ, χ_μ) then there is $w \in W$ such that $w\lambda - \mu \in P$.

Proof. (i) This follows from Corollary 18.9.

(ii) follows from (i) and the isomorphism
$$V \otimes M \cong (V \otimes U_{\chi_\lambda}) \otimes_{U_{\chi_\lambda}} M.$$

(iii) This follows from (i) since by Corollary 14.5 any irreducible Harish-Chandra bimodule is a quotient of $V \otimes U_{\chi_\mu}$ for some μ, V. □

Let $HC_{\theta,\chi}(\mathfrak{g})$ be the category of Harish-Chandra $\mathfrak{g}$-bimodules with generalized central character (θ, χ).

Corollary 18.11. *The category of Harish-Chandra $\mathfrak{g}$-bimodules $HC(\mathfrak{g})$ has a decomposition according to generalized central characters:*
$$HC(\mathfrak{g}) = \oplus_{\gamma,\lambda} HC_{\chi_{\lambda+\gamma},\chi_\lambda}(\mathfrak{g}),$$
where $\gamma \in P_+$ and $\lambda \in \mathfrak{h}^*/\mathrm{Stab}(\gamma)$ (here $\mathrm{Stab}(\gamma)$ is the stabilizer of γ in W). In particular, if (θ, χ) cannot be written as $(\chi_{\lambda+\gamma}, \chi_\lambda)$, $\lambda \in \mathfrak{h}^*$, $\gamma \in P_+$, then $HC_{\theta,\chi}(\mathfrak{g}) = 0$.

Proof. This follows from Exercise 15.5 and Corollary 18.10. □

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