20. BGG reciprocity and BGG Theorem

20.1. A vanishing lemma for Ext groups.

Lemma 20.1. *Let $X \in \mathcal{O}$ be a free $U(\mathfrak{n}_-)$-module. Then for any $\mu \in \mathfrak{h}^*$ we have*

$$\mathrm{Ext}^i_{\mathcal{O}}(X, M_\mu^\vee) = 0,\ \ i > 0.$$

Proof. Fix a projective resolution $P_\bullet$ of X in $\mathcal{O}$ and consider the complex $\mathrm{Hom}_{\mathfrak{g}}(P_\bullet, M_\mu^\vee)$ which computes the desired Ext groups. Since P_i have a weight decomposition,

$$\mathrm{Hom}_{\mathfrak{g}}(P_\bullet, M_\mu^\vee) = \mathrm{Hom}_{\mathfrak{g}}(P_\bullet, \widehat{M}_\mu^\vee),$$

where $\widehat{M}_\mu^\vee := \prod_{\beta \in \mathfrak{h}^*} \widehat{M}_\mu^\vee[\beta]$ is the completion of $M_\mu^\vee$. We have

$$\widehat{M}_\mu^\vee = \mathrm{Coind}_{\mathfrak{b}_-}^{\mathfrak{g}}(\mathbb{C}_\mu) := \mathrm{Hom}_{\mathfrak{b}_-}(U(\mathfrak{g}), \mathbb{C}_\mu) \cong \mathrm{Hom}_{\mathbb{C}}(U(\mathfrak{n}_+), \mathbb{C}_\mu).$$

Thus, Frobenius reciprocity yields

$$\mathrm{Hom}_{\mathfrak{g}}(P_\bullet, \widehat{M}_\mu^\vee) = \mathrm{Hom}_{\mathfrak{b}_-}(P_\bullet, \mathbb{C}_\mu).$$

By Proposition 16.6(ii), P_i are free $U(\mathfrak{n}_-)$-modules, so the exact sequence of $U(\mathfrak{n}_-)$-modules

$$... \to P_1 \to P_0 \to X \to 0$$

is split. Thus the complex $\mathrm{Hom}_{\mathfrak{b}_-}(P_\bullet, \mathbb{C}_\mu)$ is exact in positive degrees, which implies the statement. □

20.2. Standard filtrations.

A **standard (or Verma) filtration** on $X \in \mathcal{O}$ is a filtration for which successive quotients are Verma modules. X is called **standardly filtered** if it admits a standard filtration. It is clear that every standardly filtered object X is necessarily a free $U(\mathfrak{n}_-)$-module.

Corollary 20.2. *If X is standardly filtered then $\mathrm{Ext}^i_{\mathcal{O}}(X, M_\mu^\vee) = 0$ for all $\mu \in \mathfrak{h}^*$ and $i > 0$.*

Proof. This follows from Lemma 20.1. □

The converse also holds. In fact, we have

Theorem 20.3. *X is standardly filtered if and only if*

$$\mathrm{Ext}^1_{\mathcal{O}}(X, M_\lambda^\vee) = 0$$

for all $\lambda \in \mathfrak{h}^*$.

Proof. Let E be a finite dimensional vector space, and suppose we have a short exact sequence in $\mathcal{O}$:

$$0 \to K \to E \otimes M_\lambda \to Z \to 0$$

with $K[\lambda] = 0$.

Lemma 20.4. *If* $\mathrm{Ext}^1_{\mathcal{O}}(Z, M_\mu^\vee) = 0$ *for all* $\mu \in \mathfrak{h}^*$ *then* $K = 0$ *and* $Z \cong E \otimes M_\lambda$.

Proof. The long exact sequence of cohomology yields

$$... \to \mathrm{Hom}(E \otimes M_\lambda, M_\mu^\vee) \to \mathrm{Hom}(K, M_\mu^\vee) \to \mathrm{Ext}^1_{\mathcal{O}}(Z, M_\mu^\vee) = 0.$$

For $\lambda \neq \mu$, we have $\mathrm{Hom}(M_\lambda, M_\mu^\vee) = 0$, so it follows that $\mathrm{Hom}(K, M_\mu^\vee) = 0$. But we also have $\mathrm{Hom}(K, M_\lambda^\vee) = 0$, as $K[\lambda] = 0$, while every nonzero submodule of $M_\lambda^\vee$ contains L_λ. It follows that $K = 0$. □

Now let us prove the theorem. We only need to prove the "if" direction. We argue by induction in the length of X (with the base case $X = 0$ being trivial). Let λ be a maximal weight in $P(X)$ and $E := X[\lambda]$. Let Z be the submodule of X generated by E; it is a quotient of $E \otimes M_\lambda$ by a submodule K with $K[\lambda] = 0$. We have a short exact sequence

$$0 \to Z \to X \to Y \to 0.$$

Thus from the long exact sequence of cohomology we get an exact sequence

$$... \to \mathrm{Hom}(Z, M_\mu^\vee) \to \mathrm{Ext}^1_{\mathcal{O}}(Y, M_\mu^\vee) \to \mathrm{Ext}^1_{\mathcal{O}}(X, M_\mu^\vee) = 0.$$

It follows that for $\mu \neq \lambda$ we have $\mathrm{Ext}^1_{\mathcal{O}}(Y, M_\mu^\vee) = 0$, as in this case $\mathrm{Hom}(Z, M_\mu^\vee) = 0$ (since Z is a quotient of $E \otimes M_\lambda$). On the other hand, if $\mu = \lambda$ then by the argument in the proof of Lemma 20.1 we have

$$\mathrm{Ext}^1_{\mathcal{O}}(Y, M_\lambda^\vee) = \mathrm{Ext}^1_{\mathcal{C}}(Y, \mathbb{C}_\lambda),$$

where $\mathcal{C}$ is the category of $\mathfrak{h}$-semisimple $\mathfrak{b}_-$-modules. But $\mathrm{Ext}^1_{\mathcal{C}}(Y, \mathbb{C}_\lambda) = 0$, as all weights of Y are not $> \lambda$ and hence any short exact sequence of $\mathfrak{b}_-$-modules

$$0 \to \mathbb{C}_\lambda \to \widetilde{Y} \to Y \to 0$$

canonically splits. By the induction assumption, it follows that Y is standardly filtered, so by Corollary 20.2, $\mathrm{Ext}^i(Y, M_\mu^\vee) = 0$ for all $i \geq 1$, in particular for $i = 1, 2$. Thus the long exact sequence of Ext groups gives

$$\mathrm{Ext}^1(Z, M_\mu^\vee) = \mathrm{Ext}^1(X, M_\mu^\vee) = 0,$$

hence $Z = E \otimes M_\lambda$ by Lemma 20.4. This completes the induction step. □

Corollary 20.5. *(i) Every $X \in \mathcal{O}$ which is a free $U(\mathfrak{n}_-)$-module is standardly filtered. In particular, for any $\lambda \in \mathfrak{h}^*$ and finite dimensional $\mathfrak{g}$-module V, the module $V \otimes M_\lambda$ is standardly filtered.*

(ii) Any projective object $P \in \mathcal{O}$ is standardly filtered.

Proof. (i) Follows from Theorem 20.3 and Lemma 20.1.

(ii) Immediate from Theorem 20.3. □

20.3. **BGG reciprocity.** Denote by $d_{\lambda\mu}$ the multiplicity of L_μ in the Jordan-Hölder series of M_λ. Since characters of L_μ are linearly independent, these numbers are determined from the formula

$$\sum_\mu d_{\lambda\mu}\mathrm{ch}(L_\mu) = \mathrm{ch}(M_\lambda) = \frac{e^\lambda}{\prod_{\alpha\in R_+}(1-e^{-\alpha})}.$$

Thus the knowledge of $d_{\lambda\mu}$ is equivalent to the knowledge of the characters $\mathrm{ch}(L_\lambda)$.

Since by Proposition]20.5(ii) the projective covers P_λ of L_λ are standardly filtered, we may also define the multiplicities $d^*_{\lambda\mu}$ of M_μ in P_λ. These are independent on the choice of the standard filtration and are determined by the formula

$$\mathrm{ch}(P_\lambda) = \sum_\mu d^*_{\lambda\mu}\mathrm{ch}(M_\lambda) = \sum_\mu d^*_{\lambda\mu}\frac{e^\lambda}{\prod_{\alpha\in R_+}(1-e^{-\alpha})}.$$

Theorem 20.6. *(BGG reciprocity) We have $d^*_{\lambda\mu} = d_{\mu\lambda}$.*

Proof. We compute $\dim \mathrm{Hom}(P_\lambda, M_\mu^\vee)$ in two ways. First using the standard filtration of P_λ and Lemma 20.1, we have $\dim \mathrm{Hom}(P_\lambda, M_\mu^\vee) = d^*_{\lambda\mu}$. On the other hand, using that the multiplicity of L_λ in $M_\mu^\vee$ is $d_{\mu\lambda}$, we get $\dim \mathrm{Hom}(P_\lambda, M_\mu^\vee) = d_{\mu\lambda}$. □

Let $c_{\lambda\mu} = \dim \mathrm{Hom}(P_\lambda, P_\mu)$ be the entries of the Cartan matrix C of $\mathcal{O}$. They are equal to the multiplicities of L_λ in P_μ.

Corollary 20.7. *We have*

$$c_{\lambda\mu} = \sum_\nu d_{\nu\lambda}d_{\nu\mu}.$$

In other words, $C = D^T D$ where $D = (d_{\lambda\mu})$.

Note that since D is upper triangular with respect to the partial order $\le$ with ones on the diagonal, it can be uniquely recovered from C by Gauss decomposition. Thus the knowledge of D is equivalent to the knowledge of C.

Example 20.8. Consider the structure of the category $\mathcal{O}_\chi$ for $\mathfrak{g} = \mathfrak{sl}_2$. The only interesting case is $\chi = \chi_{\lambda+1}$ for $\lambda \in \mathbb{Z}_{\geq 0}$. Then the simple objects are $X = L_\lambda$ (finite dimensional) and $Y = M_{-\lambda-2}$. By Proposition 16.4, the projective cover P_X is just the Verma module M_λ, which has composition series $[X, Y]$, starting from the head X. To determine P_Y, consider the tensor product $P := M_{-1} \otimes L_{\lambda+1}$. This is projective with character

$$\mathrm{ch}(P) = \mathrm{ch}(M_\lambda) + \mathrm{ch}(M_{\lambda-2}) + ... + \mathrm{ch}(M_{-\lambda-2}).$$

Thus denoting by Π_λ the projection functor to the generalized central character $\chi_{\lambda+1}$, we get that

$$\mathrm{ch}(\Pi_\lambda(P)) = \mathrm{ch}(M_\lambda) + \mathrm{ch}(M_{-\lambda-2}).$$

Note that $M_{-\lambda-2}$ is not projective since $\mathrm{Ext}^1_{\mathcal{O}}(M_{-\lambda-2}, L_\lambda) \neq 0$ (there is a nontrivial extension $M_\lambda^\vee$). Thus $\Pi_\lambda(P)$ is indecomposable (otherwise one of the summands in the decomposition would have to be $M_{-\lambda-2}$), i.e., $\Pi_\lambda(P) = P_Y$. Since it maps to Y and receives an injection from M_λ, its composition series is $[Y, X, Y]$. This is the **big projective object** of $\mathcal{O}_\chi$. We this get for $\mathcal{O}_\chi$:

$$D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

We can now compute the (basic) algebra A whose module category is equivalent to $\mathcal{O}_\chi$. This is the algebra $A = \mathrm{End}(P_X \oplus P_Y)$, and it has dimension $\sum_{i,j} c_{ij} = 5$. The basis is formed by $1_X, 1_Y$ and morphisms $a : P_X \to P_Y$, $b : P_Y \to P_X$ and $ab : P_Y \to P_Y$. Moreover, we have $ba = 0$. Thus the algebra A is the path algebra of the quiver with two vertices x, y with edges $a : x \to y$ and $b : y \to x$ with the only relation $ba = 0$.

20.4. **The duality functor.** Let $\tau : \mathfrak{g} \to \mathfrak{g}$ be the **Cartan involution** given by $\tau(e_i) = f_i$, $\tau(f_i) = e_i$, $\tau(h_i) = -h_i$. For $X \in \mathcal{O}$ let X^τ be the module X twisted by τ, and $X^\vee = (X^\tau)^*_{\mathrm{fin}}$, the $\mathfrak{h}$-finite part of $(X^\tau)^*$. The following proposition is easy:

Proposition 20.9. *(i) $X^\vee \in \mathcal{O}$ and has the same character and composition series as X.*

(ii) $(M_\lambda)^\vee = M_\lambda^\vee$, $L_\lambda^\vee = L_\lambda$.

(iii) the assignment $X \mapsto X^\vee$ is an involutive equivalence of categories $\mathcal{O} \to \mathcal{O}^{\mathrm{op}}$ which preserves the decomposition into $\mathcal{O}_\chi(S)$.

Corollary 20.10. *$\mathcal{O}$ has enough injectives, namely the injective hull of L_λ is $P_\lambda^\vee$.*

20.5. **The Jantzen filtration.** It turns out that every Verma module M_λ carries a canonical finite filtration by submodules called the **Jantzen filtration**, which plays an important role in studying category $\mathcal{O}$. In fact, this filtration is defined much more generally, as follows.

Let k be a field and V, U be free $k[[t]]$-modules of the same rank $d < \infty$, and let $B \in \mathrm{Hom}(V, W)$ be such that $\det B := \wedge^d B$ is nonzero. Let $V_0 := V/tV$. Define $V_m \subset V_0$ to be the space of all $v_0 \in V_0$ such that there exists a lift $v \in V$ of v_0 for which $Bv \in t^m W$. It is clear that $V_0 \supset V_1 \supset V_2 \supset ...$ with $V_1 = \mathrm{Ker} B(0)$, and $V_m = 0$ for some m. Thus we get a finite descending filtration $\{V_j\}$ of V_0 called the **Jantzen filtration** attached to B.

Exercise 20.11. (i) Show that there exist unique nonnegative integers $n_1 \le ... \le n_d$ such that for some bases $e_1, ..., e_d$ of V and $f_1, ..., f_d$ of W over $k[[t]]$ one has $Be_i = t^{n_i} f_i$, and that $\mathrm{Coker} B \cong \oplus_{i=1}^d (k[t]/t^{n_i})$ as a $k[[t]]$-module. Deduce that the order of vanishing of $\det B$ at $t = 0$ equals $\dim_k \mathrm{Coker} B = \sum_{i=1}^d n_i$.

(ii) Suppose $\dim V_j = d_j$ (so $d_0 = d$). Show that for all $j \in \mathbb{Z}_{\ge 0}$, $n_i = j$ if and only if $d - d_j < i \le d - d_{j+1}$, and deduce the **Jantzen sum formula**: the order of vanishing of $\det B$ at $t = 0$ equals $\sum_{j \ge 1} d_j$.

(iii) Suppose that V, W are modules over some $k[[t]]$-algebra A with $A_0 := A/tA$ (for example, $A = A_0[[t]]$ and V, W are A_0-modules), and B is an A-module homomorphism. Show that the Jantzen filtration of V_0 attached to B is a filtration by A_0-submodules.

The Jantzen filtration on M_λ is now defined using the homomorphism $B : M_{\lambda(t)} \to M_{\lambda(t)}^\vee$ over $A_0 := U(\mathfrak{g})$ corresponding to the Shapovalov form, where $\lambda(t) := \lambda + t\rho$. Namely, we define it separately on each weight subspace. For example, $(M_\lambda)_1 = J_\lambda$ is the maximal proper submodule of M_λ.

Exercise 20.12. (Jantzen sum formula for M_λ) Use the Jantzen sum formula of Exercise 20.11 and the formula for the determinant of the Shapovalov form (Exercise 8.15) to show that

$$\sum_{j \ge 1} \mathrm{ch}((M_\lambda)_j) = \sum_{\alpha \in R_+ : (\lambda+\rho, \alpha^\vee) \in \mathbb{Z}_{\ge 1}} \mathrm{ch}(M_{\lambda - (\lambda+\rho,\alpha^\vee)\alpha}).$$

20.6. **The BGG theorem.** The following is the converse to Theorem 15.11.

Theorem 20.13. *(Bernstein – I. Gelfand – S. Gelfand) If $L_{\mu-\rho}$ occurs in the composition series of $M_{\lambda-\rho}$ (i.e., $d_{\lambda-\rho,\mu-\rho} \neq 0$) then $\mu \preceq \lambda$.*

Proof. It is clear that $\lambda - \mu \in Q_+$. The proof is by induction in the integer $n := (\lambda - \mu, \rho^\vee)$. If $n = 0$, the statement is obvious, so we only need to justify the induction step for $n > 0$. Then $L_{\mu-\rho}$ occurs in $J_{\lambda-\rho} = (M_{\lambda-\rho})_1$, the degree 1 part of the Jantzen filtration of M_λ. Thus by the Jantzen sum formula (Exercise 20.12), $L_{\mu-\rho}$ must occur in $M_{\lambda-\rho-(\lambda,\alpha^\vee)\alpha} = M_{s_\alpha\lambda-\rho}$ for some $\alpha \in R_+$ such that $(\lambda, \alpha^\vee) \in \mathbb{Z}_{\geq 1}$. By the induction assumption, we then have $\mu \preceq s_\alpha\lambda$. But $s_\alpha\lambda \prec \lambda$, so we get $\mu \prec \lambda$. $\square$

Corollary 20.14. *The following conditions on $\mu \leq \lambda$ are equivalent.*

(i) $\mu \preceq \lambda$

(ii) $L_{\mu-\rho}$ *occurs in* $M_{\lambda-\rho}$.

(iii) $\dim \mathrm{Hom}(M_{\mu-\rho}, M_{\lambda-\rho}) \neq 0$.

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