21. **Multiplicities in category $\mathcal{O}$**

The multiplicities $d_{\lambda\mu}$ are complicated in general, and the (eventually successful) attempt to understand them was one of the main developments that led to creation of geometric representation theory. These multiplicities are given by the **Kazhdan-Lusztig conjecture** (1979) proved by Beilinson-Bernstein and independently by Brylinski-Kashiwara in 1981. By now several proofs of this conjecture are known, but they are complicated and beyond the scope of this course. However, let us give the statement of this result. To simplify the exposition, we do so for $\mathcal{O}_{\chi_\lambda}$ when $\lambda \in P_+$; it turns out that this case captures all the complexity of the situation, and the general case is similar.

21.1. **The Hecke algebra.**

Even to formulate the Kazhdan-Lusztig conjecture, we need to introduce an object which seemingly has nothing to do with our problem - the **Hecke algebra** of W. Namely, recall that W is defined by generators $s_i, i = 1, ..., r$ subject to the braid relations

$$s_i s_j ... = s_j s_i ..., \; i \neq j,$$

where the length of both words is m_{ij} such that $a_{ij}a_{ji} = 4\cos^2 \frac{\pi}{m_{ij}}$(for $a_{ji}a_{ij} = 0, 1, 2, 3$, $m_{ij} = 2, 3, 4, 6$), and also the relations $s_i^2 = 1$. The same relations of course define the group algebra $\mathbb{Z}W$, in which the last relation can be written as the quadratic relation $(s_i + 1)(s_i - 1) = 0$. The **Hecke algebra** $H_q(W)$ of W is defined over $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ by the generators T_i satisfying the same braid relations

$$T_i T_j ... = T_j T_i ..., \; i \neq j,$$

and the deformed quadratic relations

$$(T_i + 1)(T_i - q) = 0.$$

For every $w \in W$ we can define the element $T_w = T_{i_1} ... T_{i_m}$ for every reduced decomposition $w = s_{i_1} .... s_{i_m}$. This is independent on the reduced decomposition since any two of them can be related by using only the braid relations. Moreover, it is easy to see that the elements T_w span $H_q(W)$, since any non-reduced product of T_i can be expressed via shorter products by using the quadratic relations for T_i. Moreover, we have

Proposition 21.1. *$T_w, w \in W$ are linearly independent, so they form a basis of $H_q(W)$. Thus $H_q(W)$ is a free $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module of rank $|W|$.*

Proof. Let V be the free $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$-module with basis $X_w, w \in W$. Define a left action of the free algebra with generators T_i on V by

$$T_i X_w = X_{s_i w}$$

if $\ell(s_i w) = \ell(w) + 1$ and

$$T_i X_w = (q-1)X_w + qX_{s_i w}$$

if $\ell(s_i w) = \ell(w) - 1$. We claim that this action factors through $H_q(W)$. To show this, define a right action of the same free algebra on V by

$$X_w T_i = X_{w s_i}$$

if $\ell(w s_i) = \ell(w) + 1$ and

$$X_w T_i = (q-1)X_w + qX_{w s_i}$$

if $\ell(w s_i) = \ell(w) - 1$. It is easy to check by a direct computation that these two actions commute:

$$(T_i X_w) T_j = T_i (X_w T_j). \tag{16}$$

Also the elements $X_1 T_w$ clearly span V. Thus to prove the relations of $H_q(W)$ for the left action, it suffices to check them on X_1, which is straightforward.

Since $T_w X_1 = X_w$ are linearly independent, it follows that T_w are linearly independent, as claimed. □

Exercise 21.2. Check identity (16).

The quadratic relation for T_i implies that it is invertible in the Hecke algebra, with inverse

$$T_i^{-1} = q^{-1}(T_i + 1 - q).$$

These inverses satisfy the relation $(T_i^{-1} + 1)(T_i^{-1} - q^{-1}) = 0$ (obtained by multiplying the quadratic relation for T_i by $-T_i^{-2} q^{-1}$), and also the braid relations. It follows that the Hecke algebra has an involutive automorphism D that sends $q^{\frac{1}{2}}$ to $q^{-\frac{1}{2}}$ and each T_i to T_i^{-1}. More generally one has $D(T_w) = T_{w^{-1}}^{-1}$.

21.2. **The Bruhat order.** Recall that the partial **Bruhat order** on W is defined as follows: $y \le w$ if a reduced decomposition of y can be obtained from a reduced decomposition of w by crossing out some s_i; thus $y \le w$ implies that $\ell(y) \le \ell(w)$, and if the equality holds then $y = w$. Moreover, if $\ell(w) = \ell(y) + 1$ then $y < w$ iff $y = y_1 y_2$ and $w = y_1 s_i y_2$ for some i, where $\ell(y) = \ell(y_1) + \ell(y_2)$. In this case we say that w **covers** y, and $y \le w$ iff there exists a sequence $y = x_0 < x_1 < ... < x_m = w$ such that x_{j+1} covers x_j for all j (here $m = \ell(w) - \ell(y)$).

Exercise 21.3. Show that if $y \le w$ then for any dominant $\lambda \in P$, $w\lambda \preceq y\lambda$, and the converse holds if λ is regular (i.e., $W_\lambda = 1$).

Example 21.4. For type A_1 the Bruhat order is the covering relation $1 < s$. For type A_2 the covering relations are

$$1 < s_1, s_2 < s_1 s_2, s_2 s_1 < s_1 s_2 s_1 = s_2 s_1 s_2.$$

21.3. **Kazhdan-Lusztig polynomials.**

Theorem 21.5. *There exist unique polynomials $P_{y,w} \in \mathbb{Z}[q]$ such that*
(a) $P_{y,w} = 0$ unless $y \le w$, and $P_{w,w} = 1$;
(b) If $y < w$ then $P_{y,w}$ has degree at most $\frac{\ell(w)-\ell(y)-1}{2}$;
(c) The elements

$$C_w := q^{-\frac{\ell(w)}{2}} \sum_y P_{y,w}(q) T_y \in H_q(W)$$

satisfy $D(C_w) = C_w$.

Proof. Let $y = s_{i_1} ... s_{i_l}$ be a reduced decomposition of y. Then we have

$$T_{y^{-1}}^{-1} = \prod_{j=1}^{l} T_{i_j}^{-1} = q^{-\ell(y)} \prod_{j=1}^{l} (T_{i_j} + 1 - q).$$

Thus there exist unique polynomials $R_{x,y} \in \mathbb{Z}[q]$ such that

$$D(T_y) = T_{y^{-1}}^{-1} = \sum_x q^{-\ell(x)} R_{x,y}(q^{-1}) T_x,$$

with $R_{x,y} = 0$ unless $x = y$ (in which case $R_{x,y}(q) = 1$) or $\ell(x) < \ell(y)$. It is easy to check that $R_{x,y}$ can be computed using the following recursive rules: for a simple reflection s,

$$R_{x,y} = R_{sx,sy}, \;\; sx < x, sy < y;$$

$$R_{x,y} = (q-1) R_{x,sy} + q R_{sx,sy}, sx > x, sy < y.$$

(we have $R_{x,1} = \delta_{x,1}$ and for $y \ne 1$ there is always i such that $s_i y < y$). This implies by induction in $\ell(y)$ that $R_{x,y} = 0$ unless $x \le y$. Indeed, if $x' := sx < x, y' := sy < y$ then $R_{x,y} = R_{x',y'}$, so if this is nonzero then by the induction assumption $x' \le y'$, hence $sx' \le sy'$, i.e., $x \le y$. On the other hand, if $sx > x$, $sy < y$ and $R_{x,y} \ne 0$ then either $R_{x,sy} \ne 0$ or $R_{sx,sy} \ne 0$, hence either $x \le sy$ or $sx \le sy$. But each one of the inequalities $x \le sy$, $sx \le sy$ implies $x \le y$.

We also see by induction that $\deg R_{x,y} \le \ell(y) - \ell(x)$.

Now it is easy to compute that the condition that $D(C_w) = C_w$ is equivalent to the recursion

$$q^{\frac{\ell(w)-\ell(x)}{2}} P_{x,w}(q^{-1}) - q^{\frac{\ell(x)-\ell(w)}{2}} P_{x,w}(q) =$$

$$\sum_{x<y}(-1)^{\ell(x)+\ell(y)}q^{\frac{-\ell(x)+2\ell(y)-\ell(w)}{2}}R_{x,y}(q^{-1})P_{y,w}(q).$$

We can now see that this recursion has a unique solution $P_{x,w}$ with required properties, as the two terms on the left are supposed to be polynomials in $q^{\frac{1}{2}}$ and $q^{-\frac{1}{2}}$ without constant terms. □

The elements C_w form a basis of the Hecke algebra called the **Kazhdan-Lusztig** basis, and the polynomials $P_{y,w}$ are called the **Kazhdan-Lusztig polynomials**.

21.4. **Kazhdan-Lusztig conjecture.** The Kazhdan-Lusztig conjecture (now a theorem) is:

Theorem 21.6. *(i) $P_{y,w}$ has non-negative coefficients.*
(ii) The multiplicity $[M_{y\bullet\lambda} : L_{w\bullet\lambda}]$ equals $P_{y,w}(1)$.

The polynomials $P_{y,w}$ have the property that if $y \le w$ then $P_{y,w}(0) = 1$, so if in addition $\ell(w) - \ell(y) \le 2$ then $P_{y,w}(q) = 1$ (indeed, it has to be a polynomial of degree 0). Also if $w = w_0$ then $P_{y,w} = 1$ for all y.

Example 21.7. For type A_2 ($\mathfrak{g} = \mathfrak{sl}_3$) we have the following decompositions in the Grothendieck group of $\mathcal{O}_{\chi_\lambda}$ (where we abbreviate $s_{i_1}...s_{i_k}\cdot\lambda$ as $i_1...i_k$:

$$M_{121} = L_{121}$$
$$M_{12} = L_{12} + L_{121}$$
$$M_{21} = L_{21} + L_{121}$$
$$M_1 = L_1 + L_{12} + L_{21} + L_{121}$$
$$M_2 = L_2 + L_{12} + L_{21} + L_{121}$$
$$M_\emptyset = L_\emptyset + L_1 + L_2 + L_{12} + L_{21} + L_{121}.$$

Exercise 21.8. Compute the Cartan matrix of the category $\mathcal{O}_{\chi_\lambda}$ for $\mathfrak{g} = \mathfrak{sl}_3$ for regular weights λ.