22. **Projective functors - I**

22.1. **Projective functors and projective θ-functors.** Let $\mathrm{Rep}(\mathfrak{g})_f$ be the category of $\mathfrak{g}$-modules in which the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts through its finite dimensional quotient. We have

$$\mathrm{Rep}(\mathfrak{g})_f = \oplus_{\theta\in\mathfrak{h}^*/W} \mathrm{Rep}(\mathfrak{g})_\theta,$$

where $\mathrm{Rep}(\mathfrak{g})_\theta$ is the category of modules with generalized central character θ. Recall that for a finite dimensional $\mathfrak{g}$-module V we have an exact functor $F_V : \mathrm{Rep}(\mathfrak{g}) \to \mathrm{Rep}(\mathfrak{g})$ given by $X \mapsto V \otimes X$ (e.g., $F_{\mathbb{C}} = \mathrm{Id}$), and that if M has central character χ_λ then

$$F_V(M) = (V \otimes U_{\chi_\lambda}) \otimes_{U_{\chi_\lambda}} M.$$

Recall also that the central characters occurring in the left $\mathfrak{g}$-module $V \otimes U_{\chi_\lambda}$ are $\chi_{\lambda+\beta}$ for $\beta \in P(V)$ (Corollary 18.10); thus the central characters occurring in $F_V(M)$ belong to the same set. It follows that

$$F_V(\mathrm{Rep}(\mathfrak{g})_{\chi_\lambda}) \subset \oplus_{\beta\in P(V)} \mathrm{Rep}(\mathfrak{g})_{\chi_{\lambda+\beta}},$$

hence F_V maps $\mathrm{Rep}(\mathfrak{g})_f$ to itself. Finally note that F_{V^*} is both right and left adjoint to F_V.

Definition 22.1. A **projective functor** is an endofunctor of $\mathrm{Rep}(\mathfrak{g})_f$ which is isomorphic to a direct summand in F_V for some V.

Example 22.2. For $\theta \in \mathfrak{h}^*/W$ let $\Pi_\theta : \mathrm{Rep}(\mathfrak{g})_f \to \mathrm{Rep}(\mathfrak{g})_\theta$ be the projection. Then $\mathrm{Id} = \oplus_{\theta\in\mathfrak{h}^*/W}\Pi_\theta$, hence Π_θ is a projective functor.

It is easy to see that projective functors form a category which is closed under taking compositions, direct summands and finite direct sums, and every projective functor admits a left and right adjoint which are also projective functors (we'll see that they are isomorphic). It is also clear that every projective functor F has a decomposition

$$F = \oplus_{\theta,\chi\in\mathfrak{h}^*/W}\Pi_\chi \circ F \circ \Pi_\theta.$$

Finally, projective functors obviously map category $\mathcal{O}$ to itself and by Proposition 16.5(i) send projectives of this category to projectives.

For a central character $\theta : Z(\mathfrak{g}) \to \mathbb{C}$ let $\mathrm{Rep}(\mathfrak{g})^n_\theta \subset \mathrm{Rep}(\mathfrak{g})_\theta$ be the subcategory of modules annihilated by $(\mathrm{Ker}\theta)^n$. In other words, $\mathrm{Rep}(\mathfrak{g})^n_\theta$ is the category of left modules over the algebra

$$U^{(n)}_\theta := U(\mathfrak{g})/(\mathrm{Ker}\theta)^n U(\mathfrak{g}).$$

Every $M \in \mathrm{Rep}(\mathfrak{g})_\theta$ is the nested union of submodules $M_n \subset M$ of elements killed by $(\mathrm{Ker}\theta)^n$, and $M_n \in \mathrm{Rep}(\mathfrak{g})^n_\theta$. Note that $U^{(1)}_\theta = U_\theta$ and $\mathrm{Rep}(\mathfrak{g})^1_\theta$ is the category of modules with central character θ.

For a projective functor F denote by $F(\theta)$ the restriction of F to $\mathrm{Rep}(\mathfrak{g})^1_\theta$.

Definition 22.3. A **projective θ-functor** is a direct summand in $F_V(\theta)$.

For example, if F is a projective functor then $F(\theta)$ is a projective θ-functor.

Theorem 22.4. *Let F_1, F_2 be projective θ-functors for $\theta = \chi_\lambda$. Let*

$$i_\lambda : \mathrm{Hom}(F_1, F_2) \to \mathrm{Hom}(F_1(M_{\lambda-\rho}), F_2(M_{\lambda-\rho})).$$

Then i_λ is an isomorphism.

Proof. It suffices to assume $F_j = F_{V_j}(\theta)$, $j = 1, 2$. Let $V = V_1^* \otimes V_2$. Then $\mathrm{Hom}(F_1, F_2) = \mathrm{Hom}(\mathrm{Id}(\theta), F_V(\theta))$ and

$$\mathrm{Hom}(F_1(M_{\lambda-\rho}), F_2(M_{\lambda-\rho})) = \mathrm{Hom}_{\mathfrak{g}}(M_{\lambda-\rho}, V \otimes M_{\lambda-\rho}).$$

Thus it suffices to show that the natural map

$$i_\lambda : \mathrm{Hom}(\mathrm{Id}(\theta), F_V(\theta)) \to \mathrm{Hom}_{\mathfrak{g}}(M_{\lambda-\rho}, V \otimes M_{\lambda-\rho})$$

is an isomorphism.

Recall that for associative unital algebras A, B, a right exact functor $F : A - \mathrm{mod} \to B - \mathrm{mod}$ has the form $F(X) = F(A) \otimes_A X$, where $F(A)$ is the corresponding (B, A)-bimodule. Thus if F_1, F_2 are two such functors then $\mathrm{Hom}(F_1, F_2) \cong \mathrm{Hom}_{(B,A)-\mathrm{bimod}}(F_1(A), F_2(A))$. Applying this to $A = U_\theta$ and $B = U(\mathfrak{g})$, we get

$$\mathrm{Hom}(\mathrm{Id}(\theta), F_V(\theta)) = \mathrm{Hom}_{(U(\mathfrak{g}),U_\theta)-\mathrm{bimod}}(U_\theta, V \otimes U_\theta) = (V \otimes U_\theta)^{\mathfrak{g}_{\mathrm{ad}}}.$$

Moreover, upon this identification the map i_λ becomes the natural map

$$i_\lambda : (V \otimes U_{\chi_\lambda})^{\mathfrak{g}_{\mathrm{ad}}} \to \mathrm{Hom}(M_{\lambda-\rho}, V \otimes M_{\lambda-\rho})^{\mathfrak{g}_{\mathrm{ad}}}.$$

But this map is an isomorphism by the Duflo-Joseph theorem, as it is obtained by restricting the Duflo-Joseph isomorphism

$$U_{\chi_\lambda} \cong \mathrm{Hom}_{\mathrm{fin}}(M_{\lambda-\rho}, M_{\lambda-\rho})$$

to the multiplicity space of V^*. □

22.2. Lifting projective θ-functors.

Proposition 22.5. *(i) If F_1, F_2 are projective functors then every morphism $\phi : F_1(\theta) \to F_2(\theta)$ lifts to a morphism $\widehat{\phi} : F_1|_{\mathrm{Rep}(\mathfrak{g})_\theta} \to F_2|_{\mathrm{Rep}(\mathfrak{g})_\theta}$.*

(ii) If $F_1 = F_2$ and $\phi^2 = \phi$ then we can choose $\widehat{\phi}$ so that $\widehat{\phi}^2 = \widehat{\phi}$.

(iii) If ϕ is an isomorphism then so is $\widehat{\phi}$.

Proof. (i) It suffices to show that there exist morphisms

$$\phi_n : F_1|_{\mathrm{Rep}(\mathfrak{g})^n_\theta} \to F_2|_{\mathrm{Rep}(\mathfrak{g})^n_\theta}$$

such that ϕ_n restricts to ϕ_{n-1} and $\phi_1 = \phi$; then $\widehat{\phi}$ is the projective limit of ϕ_n. As before, we may assume without loss of generality that $F_1 = \mathrm{Id}$ and $F_2 = F_V$. As explained in the proof of Theorem 22.4, we have

$$\mathrm{Hom}(F_1|_{\mathrm{Rep}(\mathfrak{g})^n_\theta}, F_2|_{\mathrm{Rep}(\mathfrak{g})^n_\theta}) =$$
$$\mathrm{Hom}_{(U(\mathfrak{g}),U_\theta^{(n)})-\mathrm{bimod}}(U_\theta^{(n)}, V \otimes U_\theta^{(n)}) = (V \otimes U_\theta^{(n)})^{\mathfrak{g}_{\mathrm{ad}}}.$$

This implies the statement, as the map $U_\theta^{(n)} \to U_\theta^{(n-1)}$ is onto and $V \otimes U_\theta^{(n)}$ is a semisimple $\mathfrak{g}_{\mathrm{ad}}$-module.

(ii) Let F be a direct summand in F_V. Let $p : F_V \to F_V$ be the projection to F. Let $A := \mathrm{End}(F(\theta)) = p\mathrm{End}(F_V(\theta))p$ and $\phi \in A$. Let $F^n(\theta)$ be the restriction of F to $\mathrm{Rep}(\mathfrak{g})^n_\theta$, so that $F^1(\theta) = F(\theta)$. We have

$$A_n := \mathrm{End}(F^n(\theta)) = p\mathrm{End}(F^n_V(\theta))p = p(\mathrm{EndV} \otimes U_\theta^{(n)})^{\mathfrak{g}_{\mathrm{ad}}} p.$$

So we have a chain of surjective homomorphisms

$$... \to A_n \to A_{n-1} \to ... \to A_1 = A$$

and our job is to show that ϕ admits a chain of lifts

$$... \mapsto \phi_n \mapsto \phi_{n-1} \mapsto ... \mapsto \phi_1 = \phi$$

such that $\phi_n \in A_n$ and $\phi_n^2 = \phi_n$.

To this end, note that the kernel I of the surjection $A_n \to A_{n-1}$ satisfies $I^2 = 0$, so I is a left and right module over $A_n/I = A_{n-1}$. So we can construct the desired chain of lifts by induction in n as follows. Pick any lift e_* of $e_0 := \phi_{n-1}$. Then $e_* - e_*^2 = a \in I$, and $e_0 a = a e_0$. We look for an idempotent e in the form $e = e_* + b$, $b \in I$. The equation $e^2 = e$ is then equivalent to

$$e_0 b + b e_0 - b = a.$$

Set $b = (2e_0 - 1)a$. Then

$$e_0 b + b e_0 - b = 2e_0 a + (1 - 2e_0)a = a,$$

as desired. Now we can set $\phi_n = e$.

(iii) If $\phi : F_1(\theta) \to F_2(\theta)$ is an isomorphism then it has the inverse $\psi : F_2(\theta) \to F_1(\theta)$ such that $\phi \circ \psi = 1$, $\psi \circ \phi = 1$. Let $\widehat{\phi} = (\phi_n)$ be a lift of ϕ. Our job is to show that ϕ_n are isomorphisms for all n, which yields (iii). We prove it by induction in n.

The base is trivial, so we just need to do the induction step from $n-1$ to n. By the induction assumption, ϕ_{n-1} is invertible with $\phi_{n-1}^{-1} = \psi_{n-1}$.

Let ψ_n be a lift of ψ_{n-1} and consider the composition $\psi_n \circ \phi_n$ in the corresponding algebra A_n. Let I be the kernel of the map $A_n \to A_{n-1}$. Then $\psi_n \circ \phi_n = 1 + a$ where $a \in I$. Since $I^2 = 0$, setting $\psi'_n := (1-a) \circ \psi_n$, we get $\psi'_n \circ \phi_n = 1$. Similarly we can construct ψ''_n such that $\phi_n \circ \psi''_n = 1$. Thus $\psi'_n = \psi''_n$ is the inverse of ϕ_n. This completes the induction step. □

Corollary 22.6. *(i) Let F_1, F_2 be projective functors. Then: any isomorphism $F_1(M_{\lambda-\rho}) \cong F_2(M_{\lambda-\rho})$ lifts to an isomorphism*

$$F_1|_{\mathrm{Rep}(\mathfrak{g})_{\chi_\lambda}} \to F_2|_{\mathrm{Rep}(\mathfrak{g})_{\chi_\lambda}};$$

(ii) Let F be a projective functor. Then any decomposition $F(M_{\lambda-\rho}) = \oplus_i M_i$ can be lifted to a decomposition $F = \oplus_i F_i$ where F_i are projective functors and $F_i(M_{\lambda-\rho}) = M_i$;

(iii) Every projective θ-functor is of the form $F(\theta)$ for a projective functor F.

Proof. (i) follows from Proposition 22.5(i),(iii) and Theorem 22.4.

(ii) follows from Proposition 22.5(ii).

To prove (iii), let H be a projective θ-functor, so $H \oplus H' = F_V(\theta)$. Thus $H(M_{\lambda-\rho}) \oplus H'(M_{\lambda-\rho}) = F_V(M_{\lambda-\rho})$. So by (ii) there is a projective functor F with $F(\theta)(M_{\lambda-\rho}) \cong F(M_{\lambda-\rho}) \cong H(M_{\lambda-\rho})$. Thus $H \cong F(\theta)$ by Theorem 22.4. □

22.3. **Decomposition of projective functors.**

Proposition 22.7. *(i) Each projective functor F is a direct sum of indecomposable projective functors. Moreover, for $F \circ \Pi_\theta$ this sum is finite.*

(ii) If $F = F \circ \Pi_{\chi_\lambda}$ for dominant λ is an indecomposable projective functor then $F(M_{\lambda-\rho}) = P_{\mu-\rho}$ for some $\mu \in \mathfrak{h}^*$.

Proof. (i) We have $F = \oplus_{\theta \in \mathfrak{h}^*/W} F \circ \Pi_\theta$, so it suffices to show the statement for $F \circ \Pi_\theta$. Let $\theta = \chi_\lambda$, and consider $F \circ \Pi_\theta(M_{\lambda-\rho}) \in \mathcal{O}$. Let us write this object as a finite direct sum of indecomposables, $\oplus_{i=1}^N M_i$. Then by Corollary 22.6(ii) we get a decomposition $F \circ \Pi_\theta = \oplus_{i=1}^N F_i$, and all F_i are indecomposable.

(ii) Since F is indecomposable and $M_{\lambda-\rho}$ is projective, $F(M_{\lambda-\rho})$ is indecomposable and projective, so the statement follows. □

18.757 Representations of Lie Groups
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