23. **Projective functors - II**

23.1. **The Grothendieck group of $\mathcal{O}$.** The Grothendieck group $K(\mathcal{O})$ of $\mathcal{O}$ is freely spanned by the classes of simple modules $[L_{\lambda-\rho}]$ or, more conveniently, by the classes of Verma modules $[M_{\lambda-\rho}]$, which we'll denote δ_λ; so it is a basis of $K(\mathcal{O})$. Put an inner product on $K(\mathcal{O})$ by declaring this basis to be orthonormal. Note that if P is projective then

$$([P],[M]) = \dim \mathrm{Hom}(P,M).$$

Indeed, in this case $\dim \mathrm{Hom}(P,M)$ is a linear function of $[M]$, and for $M = L_\mu$ by the BGG reciprocity we have:

$$\dim \mathrm{Hom}(P_\lambda, M_\mu) = d_{\mu\lambda} = d^*_{\lambda\mu} = (\sum_\nu d^*_{\lambda\nu}\delta_{\nu+\rho}, \delta_{\mu+\rho}) = ([P_\lambda],[M_\mu]).$$

Since every projective functor F is exact, it defines an endomorphism $[F]$ of $K(\mathcal{O})$. For example,

$$[F_V]\delta_\lambda = \sum_\beta m_V(\beta)\delta_{\lambda+\beta},$$

where $m_V(\beta)$ is the weight multiplicity of β in V. Clearly $[F_1 \oplus F_2] = [F_1] + [F_2]$ and $[F_1 \circ F_2] = [F_1][F_2]$.

Theorem 23.1. *(i) If F_1, F_2 are projective functors with $[F_1] = [F_2]$ then $F_1 \cong F_2$.*

(ii) If $(F, F^\vee)$ are an adjoint pair of projective functors then $[F]$ is adjoint to $[F^\vee]$ under the inner product on $K(\mathcal{O})$.

(iii) For a projective functor F, its left and right adjoint are isomorphic.

Proof. (i) By Corollary 22.6, to prove (i), it suffices to show that

$$F_1(M_{\lambda-\rho}) \cong F_2(M_{\lambda-\rho})$$

for all dominant λ. These objects are projective, so it is enough to check that they have the same character (or define the same element of $K(\mathcal{O})$). This implies the claim.

(ii) We need to show that $([F]x,y) = (x,[F^\vee]y)$. It suffices to take $x = [P]$ for projective P and $y = [M]$. Then, since $F(P)$ is projective, we have

$$([F][P],[M]) = ([F(P)],[M]) = \dim \mathrm{Hom}(F(P),M) =$$

$$\dim \mathrm{Hom}(P, F^\vee(M)) = ([P],[F^\vee(M)]) = ([P],[F^\vee][M]),$$

as claimed.

(iii) follows from (i),(ii). ∎

23.2. W**-invariance.** We have an action of the Weyl group W on $K(\mathcal{O})$ by $w\delta_\lambda := \delta_{w\lambda}$.

Theorem 23.2. *If F is a projective functor then $[F]$ commutes with W on $K(\mathcal{O})$.*

Proof. We may assume that $F = \Pi_\chi \circ F \circ \Pi_\theta$ for $\chi, \theta \in \mathfrak{h}^*/W$ and F is indecomposable. Let λ be a dominant weight such that $\theta = \chi_\lambda$. Define

$$S = \{\mu \in \lambda + P : \chi_\mu = \chi\}.$$

Let us say that λ **dominates** χ if for every $\mu \in S$ we have $\lambda - \mu \in P_+$.

Lemma 23.3. *When λ dominates χ then*
(i) Theorem 23.2 holds;
(ii) For each $\mu \in S$ there exists an indecomposable projective functor F_μ sending $M_{\lambda-\rho}$ to $P_{\mu-\rho}$.

Proof. (i) For a finite dimensional $\mathfrak{g}$-module V, let $G_V := \Pi_\chi \circ F_V \circ \Pi_\theta$. Since the character of V is W-invariant, $[F_V]$ commutes with W, hence so does $[G_V]$. Thus its suffices to show that $[F]$ is an integer linear combination of $[G_V]$ for various V.

By Proposition 22.7(ii), $F(M_{\lambda-\rho}) = P_{\mu-\rho}$, where $\mu \in S$. Let $\beta := \lambda - \mu$. By our assumption, $\beta \in P_+$. Define $n(\beta) := (\beta, 2\rho^\vee)$, a non-negative integer. We will prove the required statement by induction in $n(\beta)$.

The base of induction is $n(\beta) = 0$, hence $\beta = 0$ and $\mu = \lambda$. So $F(M_{\lambda-\rho}) = P_{\lambda-\rho} = M_{\lambda-\rho}$. This implies that $F = \Pi_\theta$, so $[F]$ is clearly commutes with W.

So it remains to justify the induction step. Let $L := L_\beta^*$, a finite dimensional $\mathfrak{g}$-module. Consider the decomposition of the functor G_L into indecomposables (which we have shown to exist in Proposition 22.7(ii)): $G_L = \oplus_j F_{\nu_j}$, where $\nu_j \in S$ and $F_{\nu_j}(M_{\lambda-\rho}) = P_{\nu_j-\rho}$ (this direct sum may contain repetitions). So $G_L(M_{\lambda-\rho}) = \oplus_j P_{\nu_j-\rho}$. Thus

$$[G_L]\delta_\lambda = \sum_{j,\gamma} d^*_{\nu_j,\gamma}\delta_\gamma = \sum_{j,\gamma} d_{\gamma,\nu_j}\delta_\gamma = \sum_j \delta_{\nu_j} + \sum_{j,\gamma>\nu_j} d_{\gamma,\nu_j}\delta_\gamma.$$

On the other hand,

$$[G_L]\delta_\lambda = [G_L(M_{\lambda-\rho})] = [\Pi_\chi(L \otimes M_{\lambda-\rho})] = [\Pi_\chi]\sum_\eta m_L(\eta)\delta_{\lambda+\eta} =$$

$$[\Pi_\chi]\sum_\eta m_{L_\beta}(\eta)\delta_{\lambda-\eta} = \sum_{\eta:\chi_{\lambda-\eta}=\chi} m_{L_\beta}(\eta)\delta_{\lambda-\eta} = \sum_{\nu:\chi_\nu=\chi} m_{L_\beta}(\beta+\mu-\nu)\delta_\nu =$$

$$\delta_\mu + \sum_{\nu>\mu:\chi_\nu=\chi} m_{L_\beta}(\beta+\mu-\nu)\delta_\nu.$$

These two formulas for $[G_L]\delta_\lambda$ jointly imply that $\nu_j \geq \mu$ for all j, and only one of them equals μ, i.e.,

$$G_L = F_\mu \oplus \bigoplus_{\nu \in S, \nu > \mu} c_{\nu\mu} F_\nu \tag{17}$$

for some constants $c_{\nu\mu} \in \mathbb{Z}_{\geq 0}$. But if $\nu > \mu$ then $n(\lambda - \nu) < n(\lambda - \mu)$, so by the induction assumption $[F_\nu]$ for all $\nu > \mu$ in this sum are linear combinations of $[G_V]$ for various V. Thus so is F_μ. But $F(M_{\lambda-\rho}) = F_\mu(M_{\lambda-\rho})$, so $F \cong F_\mu$ and the induction step follows.

(ii) The functor F_μ from (17) has the desired property. $\square$

Now we are ready to prove the theorem in the general case. So λ no longer needs to dominate χ. However, for sufficiently large integer N, the weight $\lambda + N\rho$ dominates both χ and θ. Let $\theta_N := \chi_{\lambda+N\rho}$. We have shown in Lemma 23.3(ii) that there exists an indecomposable projective functor $G = \Pi_\theta \circ G \circ \Pi_{\theta_N}$ such that $G(M_{\lambda+(N-1)\rho}) = P_{\lambda-\rho} = M_{\lambda-\rho}$. Moreover, by Lemma 23.3(i), W commutes with both $[G]$ and $[F \circ G] = [F][G]$. Thus for $w \in W$,

$$w[F]\delta_\lambda = w[F][G]\delta_{\lambda+N\rho} = [F][G]w\delta_{\lambda+N\rho} = [F]w[G]\delta_{\lambda+N\rho} = [F]w\delta_\lambda = [F]\delta_{w\lambda}.$$

So for $u \in W$,

$$u[F]\delta_{w\lambda} = uw[F]\delta_\lambda = [F]uw\delta_\lambda = [F]u\delta_{w\lambda},$$

i.e.,

$$u[F]\delta_\mu = [F]u\delta_\mu$$

for all $\mu \in \mathfrak{h}^*$, as claimed. $\square$

Lemma 23.4. *Let $\lambda \in \mathfrak{h}^*$ be dominant and $\phi, \psi \in \lambda + P$, $\psi \preceq \phi$. Then $(\lambda - \phi)^2 \leq (\lambda - \psi)^2$, and if $(\lambda - \phi)^2 = (\lambda - \psi)^2$ then $\psi \in W_\lambda \phi$.*

Proof. Consider the subgroup $W_{\lambda+Q} \subset W$. By Proposition 15.12, it is the Weyl group of a root system $R' \subset R$. Let us first prove the result when $\mu <_\alpha \lambda$, $\alpha \in R$, i.e., $\psi = s_\alpha \phi$, $\psi \neq \phi$. Then $\alpha \in R'$ and thus by Proposition 16.1

$$(\lambda, \alpha^\vee) = a \in \mathbb{Z}_{\geq 1},\ (\phi, \alpha^\vee) = -(\psi, \alpha^\vee) = b \in \mathbb{Z}_{\geq 0}.$$

We have $\lambda = \frac{1}{2}a\alpha + \lambda'$, $\phi = \frac{1}{2}b\alpha + \phi'$, $\psi = -\frac{1}{2}b\alpha + \phi'$. where λ', ϕ' are orthogonal to α. Thus

$$(\lambda - \psi)^2 - (\lambda - \phi)^2 = ((\tfrac{a+b}{2})^2 - (\tfrac{a-b}{2})^2)\alpha^2 = ab\alpha^2.$$

So this is ≥ 0, and if it is zero then either $b = 0$, in which case $\phi = \psi$ and there is nothing to prove, or $a = 0$, so $s_\alpha \lambda = \lambda$ and $s_\alpha \in W_\lambda$, as claimed.

Now let us consider the general case. By assumption, there is a chain

$$\psi = \psi_m <_{\alpha^m} \psi_{m-1} ... <_{\alpha^1} \psi_0 = \phi,$$

where $\alpha^1, ..., \alpha^m$ are positive roots of R. Thus, as we've shown,

$$(\lambda - \psi_i)^2 \leq (\lambda - \psi_{i-1})^2$$

for all $i \geq 1$, so $(\lambda - \phi)^2 \leq (\lambda - \psi)^2$. Moreover, if $(\lambda - \phi)^2 = (\lambda - \psi)^2$ then $(\lambda - \psi_{i-1})^2 = (\lambda - \psi_i)^2$ for all $i \geq 1$ so $\psi_{i-1} \in W_\lambda \psi_i$, hence $\psi \in W_\lambda \phi$. □

Remark 23.5. The last statement of Lemma 23.4 fails if the partial order $\preceq$ is replaced with $\leq$. For example, take $R = A_3$ and $\psi = (0, 3, 1, 2)$, $\phi = (1, 2, 3, 0)$, as in Remark 15.10 (so $\psi < \phi$ but $\psi \not\prec \phi$), and let $\lambda := (1, 1, 0, 0)$. Then $(\lambda - \phi)^2 = (\lambda - \psi)^2 = 10$, but $W_\lambda = \langle (12), (34) \rangle$, so $\psi \notin W_\lambda \phi$.

23.3. **Classification of indecomposable projective functors.** Denote by Ξ_0 the set of pairs (λ, μ) of weights in $\mathfrak{h}^*$ such that $\lambda - \mu \in P$, and let $\Xi := \Xi_0 / W$. So in general an element $\xi \in \Xi$ can be represented by more than one pair. Let us say that the pair (μ, λ) representing ξ is **proper** if λ is dominant and μ is a minimal element of $W_\lambda \mu$ with respect to the partial order $\preceq$ (where W_λ is the stabilizer of λ in W). It is clear that any ξ has a proper representative. This representative is not unique in general, but for every dominant λ in the W-orbit of the second coordinate of ξ, there is a unique μ such that (μ, λ) is a proper representation of ξ (indeed, $W_\lambda \mu$ has a unique minimal element).

Theorem 23.6. *For any $\xi \in \Xi$ there exists an indecomposable projective functor F_ξ such that $F_\xi(M_{\nu-\rho}) = 0$ if $\chi_\nu \neq \chi_\lambda$ and $F_\xi(M_{\lambda-\rho}) = P_{\mu-\rho}$ for any proper representation (μ, λ) of ξ. The assignment $\xi \mapsto F_\xi$ is a bijection between Ξ and the set of isomorphism classes of indecomposable projective functors.*

Proof. For a projective functor F let

$$a_F(\mu, \lambda) := (\delta_\mu, [F]\delta_\lambda)$$

be the matrix coefficients of $[F]$. If λ is dominant then $F(M_{\lambda-\rho})$ is projective, so $a_F(\mu, \lambda) \geq 0$ for all $\mu \in \mathfrak{h}^*$. Since by Theorem 23.2 $[F]$ commutes with W, this holds for all $\lambda \in \mathfrak{h}^*$.

Let $S(F) := \{(\mu, \lambda) \in \mathfrak{h}^* \times \mathfrak{h}^* : a_F(\mu, \lambda) > 0\}$. Since $a_F(\mu, \lambda) \geq 0$, if $F = \oplus_i F_i$ then $S(F) = \cup_i S(F_i)$. Also it is clear that $S(F_V) \subset \Xi_0$. It follows that $S(F) \subset \Xi_0$ for any F, so for $(\mu, \lambda) \in S(F)$ we have $\lambda - \mu \in P$.

Let $S_*(F)$ be the set of elements of $S(F)$ for which $(\lambda - \mu)^2$ has maximal value (it is clear that $(\lambda - \mu)^2$ is bounded on $S(F)$, so $S_*(F)$

is nonempty if $F \neq 0$). Since by Theorem 23.2 $[F]$ commutes with W, both $S(F)$ and $S_*(F)$ are W-invariant.

We claim that if F is indecomposable, then $S_*(F)$ is a single W-orbit. More specifically, recall that $F = F \circ \Pi_{\chi_\lambda}$ for some dominant λ and $F(M_{\lambda-\rho}) = P_{\mu-\rho}$ for some μ.

Lemma 23.7. *In this case $S_*(F) = \xi := W(\mu, \lambda)$ and (μ, λ) is a proper representation of ξ.*

Proof. It suffices to check that if $(\phi, \lambda) \in S_*(F)$ then $\phi \in W_\lambda\mu$ and $\mu \preceq \phi$. So let $(\phi, \lambda) \in S_*(F)$. Since F is indecomposable, $\chi_\mu = \chi_\phi$, so there exists $w \in W$ such that $\mu = w\phi$. Moreover, by Theorem 20.13,

$$[P_{\mu-\rho}] = \sum_{\mu \preceq \eta} d^*_{\mu\eta}\delta_\eta,$$

we get that $\mu \preceq \psi$. Thus we may apply Lemma 23.4 with $\psi = \mu$. It follows that $(\lambda - \phi)^2 \leq (\lambda - \mu)^2$. But by the definition of $S_*(F)$, we have $(\lambda - \phi)^2 \geq (\lambda - \mu)^2$. Thus $(\lambda - \phi)^2 = (\lambda - \mu)^2$. Then Lemma 23.4 implies that $\phi \in W_\lambda\mu$, as claimed. □

Thus to every indecomposable projective functor F we have assigned $\xi = S_*(F)/W \in \Xi$. If (μ, λ) is a proper representation of ξ then it follows that $F(M_{\lambda-\rho}) = P_{\mu-\rho}$, so F is completely determined by ξ by Corollary 22.6. It remains to show that any $\xi \in \Xi$ is obtained in this way. To this end, let $\xi = W(\mu, \lambda)$ (a proper representation), and let V be a finite dimensional $\mathfrak{g}$-module with extremal weight $\mu - \lambda$. Then $(\mu - \lambda)^2 \geq \beta^2$ for any weight β of V, so $(\mu, \lambda) \in S_*(F_V)$. This implies that $(\mu, \lambda) \in S_*(F)$ for some indecomposable direct summand F of F_V. Since $S_*(F)/W$ consists of one element, this F must correspond to the element ξ. □

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