26. **Representations of $SL_2(\mathbb{C})$**

26.1. **Harish-Chandra bimodules for $\mathfrak{sl}_2(\mathbb{C})$.** Let us now work out the simplest example, $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. In this case $\mathfrak{h}^* = \mathbb{C}$, $P = \mathbb{Z}$, $\chi_\lambda = \lambda^2$. So by Theorem 25.6, irreducible Harish-Chandra bimodules $\mathbf{L}_\xi$ are parametrized by pairs $\xi = (\mu, \lambda)$ of complex numbers such that $\lambda - \mu$ is an integer, modulo the map $(\mu, \lambda) \mapsto (-\mu, -\lambda)$, and we may (and will) assume that (μ, λ) is a proper representation of ξ, i.e., $\lambda \notin \mathbb{Z}_{<0}$ and if $\lambda = 0$ then $\mu \in \mathbb{Z}_{\le 0}$. Let us describe these bimodules in terms of the principal series bimodules $\mathbf{M}(\lambda, \mu)$.

Proposition 26.1. *(i) The principal series bimodule $\mathbf{M}(\lambda, \mu)$ is irreducible and isomorphic to $\mathbf{M}(-\lambda, -\mu)$ unless λ, μ are nonzero integers of the same sign. Otherwise such bimodules are pairwise non-isomorphic.*

(ii) If λ, μ are both nonzero integers of the same sign then $\mathbf{M}(\lambda, \mu)$ is indecomposable and has a finite dimensional constituent $L^*_{|\lambda|-1} \otimes L_{|\mu|-1}$, which is a submodule if $\lambda > 0$ and quotient if $\lambda < 0$. The other composition factor is $\mathbf{M}(\lambda, -\mu) \cong \mathbf{M}(-\lambda, \mu)$, which is irreducible.

(iii) If $\xi = (\mu, \lambda)$ is a proper representation with $\lambda \notin \mathbb{Z}_{\ge 1}$ then $\mathbf{L}_\xi = \mathbf{M}(\lambda, \mu)$. If $\xi = (\mu, \lambda)$ where $\lambda \in \mathbb{Z}_{\ge 1}$ then $\mathbf{L}_\xi = L^*_{\lambda-1} \otimes L_{\mu-1}$ if $\mu \ge 1$ and $\mathbf{L}_\xi = \mathbf{M}(\lambda, \mu)$ if $\mu \le 0$.

Proof. (i),(ii) Consider first the case when λ and μ are both non-integers. Then the weights $\pm\lambda$ are dominant and $M^\vee_{\pm\mu-1}$ are simple, so by Theorem 25.8 $\mathbf{M}(\lambda, \mu)$ is also simple and isomorphic to $\mathbf{M}(-\lambda, -\mu)$.

Now suppose λ, μ are integers. Recall that $\mathbf{M}(\lambda, \mu)$ decomposes over the diagonal copy of $\mathfrak{g}$ as

$$\mathbf{M}(\lambda, \mu) = \oplus_{j \ge 0} L_{|\lambda - \mu| + 2j}. \tag{18}$$

If $\lambda = 0$ and $\mu \ge 0$, then the equivalence $T_\lambda = T_0$ maps $\mathbf{M}(0, \pm\mu)$ to $M^\vee_{\pm\mu-1}$. So if $\mu = 0$, we have a simple bimodule $\mathbf{M}(0, 0)$. On the other hand, if $\mu > 0$, we have two bimodules $\mathbf{M}(0, -\mu), \mathbf{M}(0, \mu)$ and a natural map

$$a : \mathbf{M}(0, \mu) = \mathrm{Hom}_{\mathrm{fin}}(M_{-1}, M^\vee_{\mu-1}) \to \mathbf{M}(0, -\mu) = \mathrm{Hom}_{\mathrm{fin}}(M_{-1}, M^\vee_{-\mu-1})$$

induced by the surjection $M^\vee_{\mu-1} \to M^\vee_{-\mu-1}$. The kernel of this map is $\mathrm{Ker} a = \mathrm{Hom}_{\mathrm{fin}}(M_{-1}, L_{\mu-1}) = 0$, which implies that a is an isomorphism (as the K-type of the bimodules $\mathbf{M}(0, \mu), \mathbf{M}(0, -\mu)$ is the same by (18)). So we have the simple bimodule $\mathbf{M}(\mu, 0) = \mathbf{M}(-\mu, 0)$. If $\mu = 0, \lambda \neq 0$, the situation is similar, as λ and μ play a symmetric role.

It remains to consider the situation when $\lambda, \mu \in \mathbb{Z} \setminus 0$. So let n, m be positive integers. By Theorem 25.8, the bimodule $\mathbf{M}(n, -m)$ is simple, as it corresponds to the simple module $M^\vee_{-m-1}$. Similarly, $\mathbf{M}(-n, m)$ is simple. Now, we have homomorphisms

$$a : \mathbf{M}(n, m) \to \mathbf{M}(n, -m), b : \mathbf{M}(n, -m) \to \mathbf{M}(-n, -m).$$

Since $\mathbf{M}(n, -m)$ is simple and $a \neq 0$, it is surjective, so in view of (18) we have a short exact sequence

$$0 \to L^*_{n-1} \otimes L_{m-1} \to \mathbf{M}(n, m) \to \mathbf{M}(n, -m) \to 0.$$

Similarly, since $b \neq 0$, it is injective, so in view of (18) we have a short exact sequence

$$0 \to \mathbf{M}(n, -m) \to \mathbf{M}(-m, -n) \to L^*_{n-1} \otimes L_{m-1} \to 0.$$

Moreover, these sequences are not split by Theorem 25.8. This proves (i),(ii).

(iii) follows immediately from (i),(ii). The proposition is proved. □

Example 26.2. One may also describe explicitly the projectives $\mathbf{P}_\xi$. As an example let us do so for $\xi = (-1, 0)$. Consider the tensor product $L_1 \otimes U_0$, which is a projective object. We have $(L_1 \otimes U_0) \otimes_{U_0} M_{-1} = L_1 \otimes M_{-1} = P_{-2}$, the big projective object with composition series $[M_{-2}, \mathbb{C}, M_{-2}]$. Thus $L_1 \otimes U_0 = \mathbf{P}_\xi$. Over the diagonal copy of $\mathfrak{g}$ we have

$$\mathbf{P}_\xi = L_1 \otimes U_0 = L_1 \otimes (L_0 \oplus L_2 \oplus ...) = 2L_1 \oplus 2L_3 \oplus ...$$

Thus we have a short exact sequence

$$0 \to \mathbf{L}_\xi \to \mathbf{P}_\xi \to \mathbf{L}_\xi \to 0, \tag{19}$$

where $\mathbf{L}_\xi = \mathbf{M}(0, -1) = \mathbf{M}(0, 1)$, which is not split.

This shows that the functor $T_\lambda = T_0$ is not exact in this case. Indeed, $T_0(\mathbf{L}_\xi) = M^\vee_0$ ($M^\vee_0 \in \mathcal{O}(0)$ with presentation $P_{-2} \to P_{-2} \to M^\vee_0 \to 0$ and $H_0(M^\vee_0) = \mathbf{M}(0, 1)$), so the image of (19) under T_0 is the sequence

$$0 \to M^\vee_0 \to P_{-2} \to M^\vee_0 \to 0,$$

which is not exact in the leftmost nontrivial term (the cohomology is $\mathbb{C}$). This sequence is, however, exact in the category $\mathcal{O}(0)$, which has just two indecomposable objects $M^\vee_0$ and P_{-2} (so $\mathcal{O}(0)$ is not closed under taking subquotients and the inclusion $\mathcal{O}(0) \hookrightarrow \mathcal{O}$ is not exact).

26.2. **Representations of $SL_2(\mathbb{C})$.** Let us now consider representations of $G = SL_2(\mathbb{C})$. We have $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $K = SU(2)$. We have already classified the irreducible Harish-Chandra (bi)modules and shown that the only ones are finite dimensional modules and principal series modules. Moreover, we realized the principal series module

$\mathbf{M}(\lambda,\mu)$ as the space of K-finite vectors in the space of smooth functions $F : G \to \mathbb{C}$ such that

$$F(gb) = F(g)t(b)^{\lambda-\mu}|t(b)|^{2\mu-2},\ b \in B,$$

where $B \subset G$ is the subgroup of upper triangular matrices. Thus, similarly to the real case, setting $\mu - \lambda = m \in \mathbb{Z}$, we may represent $\mathbf{M}(\lambda,\mu)$ as the space of polynomial tensor fields on $\mathbb{CP}^1 = S^2$ of the form

$$\omega = \phi(u)(du)^{\frac{m}{2}}|du|^{1-\mu},$$

and we have an admissible realization of $\mathbf{M}(\lambda,\mu)$ by the vector space $C^\infty_{\lambda-1,\mu-1}(G/B)$ of smooth tensor fields of the same form. The (right) action of the group G on this space is given by

$$\left(\phi \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(u) = \phi\left(\frac{au+b}{cu+d}\right)(cu+d)^{-m}|cu+d|^{2\mu-2}.$$

We may also upgrade this realization to a Hilbert space realization by completing it with respect to the inner product

$$\|\omega\|^2 = \int_{S^2} |\phi(u)|^2 dA,$$

where dA is the rotation-invariant probability measure on S^2. However, this inner product is not G-invariant, in general; it is only G-invariant if $\mathrm{Re}\mu = \frac{m}{2}$, i.e., $\mu = \frac{m}{2} + s$, $s \in i\mathbb{R}$. This shows that the $(\mathfrak{g}, K)$-modules $\mathbf{M}(-\frac{m}{2}+s, \frac{m}{2}+s)$ are unitary and irreducible for any imaginary s, with the Hilbert space completion being $L^2_{-\frac{m}{2}+s-1,\frac{m}{2}+s-1}(G/B)$ – the **unitary principal series**.

Also the trivial representation is obviously unitary. Are there any other unitary irreducible representations? Clearly, they cannot be finite dimensional. However, the answer is yes. To find them, let us first determine which $\mathbf{M}(\lambda,\mu)$ are Hermitian. It is easy to show that this happens whenever $\lambda^2 = \overline{\mu}^2$, i.e., $\lambda = \pm\overline{\mu}$. If $\lambda = -\overline{\mu}$, we get $2\mathrm{Re}\mu = m$, so $\mu = \frac{m}{2} + s$, $\lambda = -\frac{m}{2} + s$, $s \in i\mathbb{R}$, exactly as above. On the other hand, if $\lambda = \overline{\mu}$ then we get $\mu - \overline{\mu} = m$, which implies that $m = 0$, i.e., $\lambda = \mu \in \mathbb{R}$. In this case by Theorem 25.6 the module $\mathbf{M}(\mu,\mu)$ is irreducible if and only if $\mu \notin \mathbb{Z}$. Thus we see that for $0 < |\mu| < 1$, this module is unitary, as we have a continuous family of simple Hermitian modules $X(c) := \mathbf{M}(\sqrt{c}, \sqrt{c})$ for $c \in (-\infty, 1)$, and these modules are in the unitary principal series for $c \le 0$. This family of unitary modules for $c > 0$ $(0 < |\mu| < 1)$ is called the **complementary series**; it is analogous to the complementary series in the real case.

It remains to consider the intervals $m < |\mu| < m+1$ for $m \in \mathbb{Z}_{\ge 0}$. If $\mathbf{M}(\mu,\mu)$ is unitary for at least one point in such interval, then it is so for the whole interval, and taking the limit $\mu \to m+1$, we see that $L^*_{m+1} \otimes L_{m+1}$, which is a composition factor of $\mathbf{M}(m+1, m+1)$, would have to be unitary, which it is not. This shows that we have no unitary modules in these intervals. Thus we obtain the following result.

Theorem 26.3. *(Gelfand-Naimark) The irreducible unitary representations of $SL_2(\mathbb{C})$ are Hilbert space completions of the following unitary Harish-Chandra modules:*

- *Unitary principal series $\mathbf{M}(-\frac{m}{2}+s, \frac{m}{2}+s)$, $m \in \mathbb{Z}$, $s \in i\mathbb{R}$;*
- *Complementary series $\mathbf{M}(s,s)$, $-1 < s < 1$;*
- *The trivial representation $\mathbb{C}$.*

Here $\mathbf{M}(-\frac{m}{2}+s, \frac{m}{2}+s) \cong \mathbf{M}(\frac{m}{2}-s, -\frac{m}{2}-s)$, $\mathbf{M}(s,s) = \mathbf{M}(-s,-s)$ and there are no other isomorphisms.

Exercise 26.4. Compute the map $M \mapsto M^\vee$ from Exercise 5.17 on the set of irreducible Harish-Chandra modules for $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$.

Exercise 26.5. The following exercise is the complex analog of Exercise 9.6.

(i) Show that for $-1 < s < 0$ the formula

$$(f,g)_s := \int_{\mathbb{C}^2} f(y)\overline{g(z)}|y-z|^{-2s-2} dy d\overline{y} dz d\overline{z}$$

defines a positive definite inner product on the space $C_0(\mathbb{C})$ of continuous functions $f : \mathbb{C} \to \mathbb{C}$ with compact support (*Hint*: pass to Fourier transforms).

(ii) Deduce that if f is a measurable function on $\mathbb{C}$ then

$$0 \le (f,f)_s \le \infty,$$

so measurable functions f with $(f,f)_s < \infty$ modulo those for which $(f,f)_s = 0$ form a Hilbert space $\mathcal{H}_s$ with inner product $(,)_s$, which is the completion of $C_0(\mathbb{C})$ under $(,)_s$.

(iii) Let us view $\mathcal{H}_s$ as the space of tensor fields $f(y)|dy|^{1-s}$, where f is as in (ii). Show that the complementary series unitary representation $\widehat{\mathbf{M}}(s,s)$ of $SL_2(\mathbb{C})$ may be realized in $\mathcal{H}_s$ with G acting naturally on such tensor fields.

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