27. **Geometry of complex semisimple Lie groups**

27.1. **The Borel-Weil theorem.** Let G be a simply connected semisimple complex Lie group with Lie algebra $\mathfrak{g}$ and a Borel subgroup B generated by a maximal torus $T \subset G$ and the 1-parameter subgroups $\exp(te_i)$, $i \in \Pi$. Given an integral weight $\lambda \in P$, we can define the corresponding algebraic (in particular, holomorphic) line bundle $\mathcal{L}_\lambda$ on the flag variety G/B. Namely, the total space $T(\mathcal{L}_\lambda)$ of $\mathcal{L}_\lambda$ is $(G \times \mathbb{C})/B$, where B acts by

$$(g, z)b = (gb, \lambda(b)^{-1}z),$$

and the line bundle $\mathcal{L}_\lambda$ is defined by the projection $\pi : T(\mathcal{L}_\lambda) \to G/B$ to the first component. So this bundle is G-equivariant, i.e., G acts on $T(\mathcal{L}_\lambda)$ by left multiplication preserving the projection map π. We also see that smooth sections of $\mathcal{L}_\lambda$ are smooth functions $F : G \to \mathbb{C}$ such that

$$(g, F(g))b = (gb, F(gb)),$$

which yields

$$F(gb) = \lambda(b)^{-1}F(g).$$

It follows that the space of smooth sections $\Gamma_{C^\infty}(G/B, \mathcal{L}_\lambda)$ coincides with the admissible G-module $C^\infty_{-\lambda,0}(G/B)$, realizing the principal series module $\mathbf{M}(-\lambda + 1, 1) = \mathrm{Hom}_{\mathrm{fin}}(M_{-\lambda}, M_0^\vee)$.

Remark 27.1. Recall that $H^2(G/B, \mathbb{Z}) = P$. It is easy to check that the first Chern class $c_1(\mathcal{L}_\lambda)$ equals λ. This motivates the minus sign in the definition of $\mathcal{L}_\lambda$.

Example 27.2. Let $G = SL_2(\mathbb{C})$, so that B is the subgroup of upper triangular matrices with determinant 1 and $G/B = \mathbb{CP}^1$. Then sections of $\mathcal{L}_m$ are functions $F : G \to \mathbb{C}$ such that $F(gb) = t(b)^{-m}F(g)$, where $t(b) = b_{11}$. Thus $\mathcal{L}_m \cong \mathcal{O}(-m)$.

Let us now consider holomorphic sections of $\mathcal{L}_\lambda$. The space V_λ of such sections is a proper subrepresentation of $C^\infty_{-\lambda,0}(G/B)$, namely the subspace where the left copy of $\mathfrak{g}$ (acting by antiholomorphic vector fields) acts trivially. Thus $V_\lambda^{\mathrm{fin}} = \mathrm{Hom}_{\mathrm{fin}}(M_{-\lambda}, \mathbb{C}) \subset \mathrm{Hom}_{\mathrm{fin}}(M_{-\lambda}, M_0^\vee)$, and $V_\lambda = V_\lambda^{\mathrm{fin}}$ since V_λ^{fin} is finite dimensional. It follows that $V_\lambda^{\mathrm{fin}} = 0$ unless $\lambda \in -P_+$, and in the latter case $V_\lambda = L^*_{-\lambda} = L^-_\lambda = L_{w_0\lambda}$, the finite dimensional representation of G with lowest weight λ. Thus we obtain

Theorem 27.3. *(Borel-Weil) Let $\lambda \in P$. If $\lambda \in P_+$ then we have an isomorphism of G-modules*

$$\Gamma(G/B, \mathcal{L}_{-\lambda}) \cong L^*_\lambda.$$

If $\lambda \notin P_+$ *then* $\Gamma(G/B, \mathcal{L}_{-\lambda}) = 0$.

Example 27.4. Let $G = SL_2(\mathbb{C})$. Then Theorem 27.3 says that

$$\Gamma(\mathbb{CP}^1, \mathcal{O}(m)) \cong L_m = \mathbb{C}^{m+1}$$

as representations of G.

More generally, suppose $\lambda \in P$ and $(\lambda, \alpha_i^\vee) = 0$ for a subset $S \subset \Pi$ of the set of simple roots. Then we have a parabolic subgroup $P_S \subset G$ generated by B and also $\exp(tf_i)$ for $i \in S$, and λ extends to a 1-dimensional representation of P_S. Thus we can define the line bundle $\mathcal{L}_{\lambda,S}$ on the partial flag variety G/P_S in the same way as $\mathcal{L}_\lambda$, and we have $\mathcal{L}_\lambda = p_S^*\mathcal{L}_{\lambda,S}$, where $p_S : G/B \to G/P_S$ is the natural projection.

Note that any holomorphic section of $\mathcal{L}_\lambda$ is just a function when restricted to a fiber $F \cong P_S/B$ of the fibration p_S (a compact complex manifold), so by the maximum principle it must be constant. It follows that $\Gamma(G/B, \mathcal{L}_\lambda) = \Gamma(G/P_S, \mathcal{L}_{\lambda,S})$. Thus we get

Corollary 27.5. *Let* $\lambda \in P$ *with* $(\lambda, \alpha_i^\vee) = 0$, $i \in S$. *Then*

$$\Gamma(G/P_S, \mathcal{L}_{-\lambda,S}) \cong L_\lambda^*.$$

if $\lambda \in P_+$, *otherwise* $\Gamma(G/P_S, \mathcal{L}_{-\lambda,S}) = 0$.

Example 27.6. Let $G = SL_n(\mathbb{C}) = SL(V)$, $V = \mathbb{C}^n$, and $P_S \subset G$ be the subgroup of matrices b such that $b_{r1} = 0$ for $r > 1$ (this corresponds to $S = \{2, ..., n-1\}$). Then $G/P_S = \mathbb{CP}^{n-1} = \mathbb{P}(V)$. The condition $(\lambda, \alpha_i^\vee) = 0$, $i \in S$ means that $\lambda = m\omega_1$, and in this case $\mathcal{L}_{m,S} = \mathcal{O}(-m)$. So Corollary 27.5 says that

$$\Gamma(\mathbb{P}(V), \mathcal{O}(m)) = L_{m\omega_{n-1}} = S^mV^*$$

for $m \geq 0$, and zero for $m < 0$. This is also clear from elementary considerations, as by definition $\Gamma(\mathbb{P}(V), \mathcal{O}(m))$ is the space of homogeneous polynomials on V of degree m.

In fact, for $\lambda \in P_+$ we can construct an isomorphism $L_\lambda^* \cong \Gamma(G/B, \mathcal{L}_{-\lambda})$ explicitly as follows. Let v_λ be a highest weight vector of L_λ, $\ell \in L_\lambda^*$, and $F_\ell(g) := (\ell, gv_\lambda)$. Then

$$F_\ell(gb) = \lambda(b)F_\ell(g).$$

Thus the assignment $\ell \to F_\ell$ defines a linear map $L_\lambda^* \to \Gamma(G/B, \mathcal{L}_{-\lambda})$ which is easily seen to be an isomorphism.

This shows that the bundle $\mathcal{L}_{-\lambda}$ is **globally generated**, i.e., for every $x \in G/B$ there exists $s \in \Gamma(G/B, \mathcal{L}_{-\lambda})$ such that $s(x) \neq 0$. In other words, we have a regular map $i_\lambda : G/B \to \mathbb{P}L_\lambda$ defined as follows.

For $x \in G/B$, choose a basis vector u of the fiber of $\mathcal{L}_{-\lambda}$ at x and define $i_\lambda(x) \in L_\lambda$ by the equality

$$s(x) = i_\lambda(x)(s)u$$

for $s \in \Gamma(G/B, \mathcal{L}_{-\lambda}) \cong L_\lambda^*$. Then $i_\lambda(x)$ is well defined (does not depend on the choice of u) up to scaling and is nonzero, so gives rise to a well defined element of the projective space $\mathbb{P}L_\lambda$. Another definition of this map is

$$i_\lambda(x) = x(\mathbb{C}v_\lambda).$$

This shows that i_λ is an embedding when λ is regular, i.e., in this case the line bundle $\mathcal{L}_\lambda$ is **very ample**. On the other hand, if λ is not necessarily regular and S is the set of j such that $(\lambda, \alpha_j^\vee) = 0$ then $i_\lambda : G/P_S \to \mathbb{P}L_\lambda$ is an embedding, so the bundle $\mathcal{L}_{-\lambda,S}$ over the partial flag variety G/P_S is very ample.

Example 27.7. Let $G = SL_n(\mathbb{C})$ and $\lambda = \omega_k$. Then $S = [1, n-1] \backslash k$, so $P_S \subset G$ is the subgroup of matrices with $g_{ij} = 0$, $i > k, j \le k$ and G/P_S is the Grassmannian $\mathrm{Gr}(k, n)$ of k-dimensional subspaces in $\mathbb{C}^n$. In this case $L_\lambda = \wedge^k \mathbb{C}^n$, so i_λ is the Plücker embedding $\mathrm{Gr}(k, n) \hookrightarrow \mathbb{P}(\wedge^k \mathbb{C}^n)$.

27.2. **The Springer resolution.** Recall that a **resolution of singularities** of an irreducible algebraic variety X is a morphism $p : Y \to X$ from a smooth variety Y that is proper (for example, projective[21]) and birational. Hironaka proved in 1960s that any variety over a field of characteristic zero has a resolution of singularities. However, it is not unique and this theorem does not provide a nice explicit construction of a resolution.

A basic example of a singular variety arising in Lie theory is the nilpotent cone $\mathcal{N}$ of a semisimple Lie algebra $\mathfrak{g}$. This variety turns out to admit a very explicit resolution called the **Springer resolution**, which plays an important role in representation theory.

To define the Springer resolution, consider the cotangent bundle $T^*\mathcal{F}$ of the flag variety $\mathcal{F}$ of G. Recall that $\mathcal{F}$ is the variety of Borel subalgebras $\mathfrak{b} \subset \mathfrak{g}$. For $\mathfrak{b} \in \mathcal{F}$, we have an isomorphism $\mathfrak{g}/\mathfrak{b} \cong T_\mathfrak{b}\mathcal{F}$ defined by the action of G. Thus $T^*\mathcal{F}$ can be viewed as the set of pairs $(\mathfrak{b}, x)$, where $x \in (\mathfrak{g}/\mathfrak{b})^*$. Note that $(\mathfrak{g}/\mathfrak{b})^* \cong \mathfrak{b}^\perp$ under the Killing form, and $\mathfrak{b}^\perp = [\mathfrak{b}, \mathfrak{b}]$ is the maximal nilpotent subalgebra of $\mathfrak{b}$. Thus $T^*\mathcal{F}$ is the variety of pairs $(\mathfrak{b}, x)$ where $\mathfrak{b} \in \mathcal{F}$ is a Borel subnalgebra of $\mathfrak{g}$ and $x \in \mathfrak{b}$ a nilpotent element.

[21]Recall that a morphism $f : X \to Y$ is **projective** if $f = \pi \circ \widetilde{f}$ where $\widetilde{f} : X \to Z \times Y$ is a closed embedding for some projective variety Z and $\pi : Z \times Y \to Y$ is the projection to the second component.

Now we can define the **Springer map** $p : T^*\mathcal{F} \to \mathcal{N}$ given by $p(\mathfrak{b}, x) = x$. Note that this map is G-invariant, so its fibers over conjugate elements of $\mathcal{N}$ are isomorphic.

Theorem 27.8. *The Springer map p is birational and projective, so it is a resolution of singularities.*

Proof. To show that p is birational, it suffices to prove that if $e \in \mathcal{N}$ is regular, the Borel subalgebra $\mathfrak{b}$ containing e is unique. To this end, note that $\dim T^*\mathcal{F} = 2\dim\mathcal{F} = \dim\mathcal{N}$ and the map p is surjective (as any nilpotent element is contained in a Borel subalgebra). Thus p is generically finite, i.e., $p^{-1}(e)$ is a finite set, and our job is to show that it consists of one element.

We may fix a decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ and assume that $e = \sum_{i=1}^r e_i$. Then we have $[\rho^\vee, e] = e$, so the group $\{t^{\rho^\vee}, t \neq 0\} \cong \mathbb{C}^*$ acts on $p^{-1}(e)$ (as any Borel subalgebra containing e also contains te). Since $p^{-1}(e)$ is finite, this action must be trivial. Thus $\rho^\vee$ normalizes every $\mathfrak{b} \in p^{-1}(e)$, hence is contained in every such $\mathfrak{b}$. But $\rho^\vee$ is regular, so is contained in a unique Cartan subalgebra, namely $\mathfrak{h}$. Since every semisimple element in a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is contained in a Cartan subalgebra sitting inside $\mathfrak{b}$, it follows that $\mathfrak{h} \subset \mathfrak{b}$ for all $\mathfrak{b} \in p^{-1}(e)$. Thus $[\omega_i^\vee, e] = e_i \in \mathfrak{b}$ for all i. It follows that $\mathfrak{b} = \mathfrak{b}_+ := \mathfrak{h} \oplus \mathfrak{n}_+$, i.e., $|p^{-1}(e)| = 1$, as claimed.

Now let us show that p is projective. Let $\widetilde{p} : T^*\mathcal{F} \to \mathcal{F} \times \mathcal{N}$ be the map defined by $\widetilde{p}(\mathfrak{b}, x) = (\mathfrak{b}, x)$. This is clearly a closed embedding (the image is defined by the equation $x \in \mathfrak{b}$). But $p = \pi \circ \widetilde{p}$ where $\pi : \mathcal{F} \times \mathcal{N} \to \mathcal{N}$ is the projection to the second component. Thus p is projective, as claimed. □

Remark 27.9. The preimage $p^{-1}(e)$ for $e \in \mathcal{N}$ is called the **Springer fiber**. If e is not regular, $p^{-1}(e)$ has positive dimension. It is a projective variety, which is in general singular, reducible and has complicated structure, but it plays an important role in representation theory.

Example 27.10. Let $\mathfrak{g} = \mathfrak{sl}_2$. Then $\mathcal{N}$ is the usual quadratic cone $yz + x^2 = 0$ in $\mathbb{C}^3$, and $T^*\mathcal{F} = T^*\mathbb{C}P^1$ is the blow-up of the vertex in this cone.

27.3. **The symplectic structure on coadjoint orbits.** Recall that a smooth real manifold, complex manifold or algebraic variety X is **symplectic** if it is equipped with a nondegenerate closed 2-form ω. It is clear that in this case X has even dimension.

Theorem 27.11. *(Kirillov-Kostant) Let G be a connected real or complex Lie group or complex algebraic group. Then every G-orbit in $\mathfrak{g}^*$ has a natural symplectic structure.*

Proof. Let O be a G-orbit in $\mathfrak{g}^*$ and $f \in O$. Then $T_f O = \mathfrak{g}/\mathfrak{g}_f$ where $\mathfrak{g}_f$ is the set of $x \in \mathfrak{g}$ such that $f([x, y]) = 0$ for all $y \in \mathfrak{g}$. Define a skew-symmetric bilinear form $\omega_f : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by $\omega_f(y, z) = f([y, z])$. It is clear that $\mathrm{Ker}\omega_f = \mathfrak{g}_f$, so ω_f defines a nondegenerate form on $\mathfrak{g}/\mathfrak{g}_f = T_f O$. This defines a nondegenerate G-invariant differential 2-form ω on O.

It remains to show that ω is closed. Let L_x be the vector field on O defined by the action of $x \in \mathfrak{g}$; thus $L_{[x,y]} = [L_x, L_y]$. It suffices to show that for any $x, y, z \in \mathfrak{g}$ we have $d\omega(L_x, L_y, L_z) = 0$. By Cartan's differentiation formula we have

$$d\omega(L_x, L_y, L_z) = \mathrm{Alt}(L_x \omega(L_y, L_z) - \omega([L_x, L_y], L_z)),$$

where Alt denotes the sum over cyclic permutations of x, y, z. Since ω is G-invariant, this yields

$$d\omega(L_x, L_y, L_z)(f) = \mathrm{Alt}(\omega(L_y, L_{[x,z]}))(f) = f(\mathrm{Alt}([y, [x, z]])),$$

which vanishes by the Jacobi identity. □

Corollary 27.12. *The singular locus of the nilpotent cone $\mathcal{N}$ has codimension ≥ 2.*

Proof. This follows since $\mathcal{N}$ has finitely many orbits (Exercise 17.8) and by Theorem 27.11 they all have even dimension. □

Corollary 27.13. *$\mathcal{N}$ is normal (i.e., the algebra $\mathcal{O}(\mathcal{N})$ is integrally closed in its quotient field).*

Proof. This follows from Corollary 27.12 since $\mathcal{N}$ is a complete intersection and any complete intersection whose singular locus has codimension ≥ 2 is necessarily normal ([H], Chapter II, Prop. 8.23). □

27.4. **The algebra of functions on $T^*\mathcal{F}$.** We will first recall some facts about normal algebraic varieties.

Proposition 27.14. *Let Y be an irreducible normal algebraic variety. Then*

(i) ([Eis]*, Proposition 11.5) The singular locus of Y has codimension ≥ 2.*

(ii) ([Eis]*, Proposition 11.4) If $U \subset Y$ is an open subset and $Y \backslash U$ has codimension ≥ 2 then any regular function f on U extends to a regular function on Y. In particular, any regular function on the smooth locus of Y extends to a regular function on Y.*

(iii) Zariski main theorem ([H]*, Corollary III.11.4). If X is irreducible and $p : X \to Y$ is a proper birational morphism then fibers of p are connected.*

Proposition 27.15. *Let Y be an irreducible normal affine algebraic variety and $p : X \to Y$ be a resolution of singularities. Then the homomorphism $p^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an isomorphism.*

Proof. It is clear that p^* is injective, so we only need to show it is surjective. Let $f \in \mathcal{O}(X)$. Since every fiber of p is proper, and also connected due to normality of Y by Proposition 27.14(iii), f is constant along this fiber. So $f = h \circ p$ for $h : Y \to \mathbb{C}$ a rational function. It remains to show that h is regular. We know that h is regular on the smooth locus of Y (as it is defined at all points of Y). Thus the result follows from the normality of Y and Proposition 27.14(i),(ii). □

Theorem 27.16. *Let $p : T^*\mathcal{F} \to \mathcal{N}$ be the Springer resolution. Then the map $p^* : \mathcal{O}(\mathcal{N}) \to \mathcal{O}(T^*\mathcal{F})$ is an isomorphism of graded algebras.*

Proof. This follows from Proposition 27.15 and the normality of $\mathcal{N}$ (Corollary 27.13). □

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