28. **D-modules - I**

We would now like to formulate the Beilinson-Bernstein localization theorems. We first review generalities about differential operators and D-modules.

28.1. **Differential operators.** Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. Let X be a smooth affine algebraic variety over $\mathbf{k}$. Let $\mathcal{O}(X)$ be the algebra of regular functions on X. Following Grothendieck, we define inductively the notion of a *differential operator of order (at most) N on X*. Namely, a differential operator of order -1 is zero, and a k-linear operator $L : \mathcal{O}(X) \to \mathcal{O}(X)$ is a differential operator of order $N \geq 0$ if for all $f \in \mathcal{O}(X)$, the operator $[L, f]$ is a differential operator of order $N - 1$.

Let $D_N(X)$ denote the space of differential operators of order N. We have

$$0 = D_{-1}(X) \subset \mathcal{O}(X) = D_0(X) \subset D_1(X) \subset ... \subset D_N(X) \subset ...$$

and $D_i(X)D_j(X) \subset D_{i+j}(X)$, which implies that the nested union $D(X) := \cup_{i\geq 0} D_i(X)$ is a filtered algebra.

Definition 28.1. $D(X)$ is called **the algebra of differential operators** on X.

Exercise 28.2. Prove the following statements.

1. $[D_i(X), D_j(X)] \subset D_{i+j-1}(X)$ for $i, j \geq 0$. In particular, $[,]$ makes $D_1(X)$ a Lie algebra naturally isomorphic to $\mathrm{Vect}(X) \ltimes \mathcal{O}(X)$, where $\mathrm{Vect}(X)$ is the Lie algebra of vector fields on X.

2. Suppose $x_1, ..., x_n \in \mathcal{O}(X)$ are regular functions such that $dx_1, ..., dx_n$ form a basis in each cotangent space to X. Let $\partial_1, ..., \partial_n$ be the corresponding vector fields. For $\mathbf{m} = (m_1, ..., m_n) \in \mathbb{Z}^n_{\geq 0}$, let $|\mathbf{m}| := \sum_{i=1}^n m_i$ and $\partial^{\mathbf{m}} := \partial_1^{m_1} ... \partial_n^{m_n}$. Then $D_N(X)$ is a free finite rank $\mathcal{O}(X)$-module (under left multiplication) with basis $\{\partial^{\mathbf{m}}\}$ with $|\mathbf{m}| \leq N$, and $D(X)$ is a free $\mathcal{O}(X)$-module with basis $\{\partial^{\mathbf{m}}\}$ for all $\mathbf{m}$.

3. One has $\mathrm{gr}\, D(X) = \oplus_{i\geq 0} \Gamma(X, S^i TX) = \mathcal{O}(T^*X)$. In particular, $D(X)$ is left and right Noetherian.

4. $D(X)$ is generated by $\mathcal{O}(X)$ and elements L_v, $v \in \mathrm{Vect}(X)$ (depending linearly on v), with defining relations

$$[f, g] = 0,\ [L_v, f] = v(f),\ L_{fv} = fL_v,\ [L_v, L_w] = L_{[v,w]}, \tag{20}$$

where $f, g \in \mathcal{O}(X), v, w \in \mathrm{Vect}(X)$.

5. If $U \subset X$ is an affine open set then the multiplication map $\mathcal{O}(U) \otimes_{\mathcal{O}(X)} D(X) \to D(U)$ is a filtered isomorphism.

28.2. D**-modules.**

Definition 28.3. A **left** (respectively, **right**) D**-module** on X is a left (respectively, right) $D(X)$-module.

Example 28.4. 1. $\mathcal{O}(X)$ is an obvious example of a left D-module on X. Also, $\Omega(X)$ (the space of top differential forms on X) is naturally a right D-module on X, via $\rho(L) = L^*$ (the adjoint differential operator to L with respect to the "integration pairing" between functions and top forms). More precisely, $f^* = f$ for $f \in \mathcal{O}(X)$, and L_v^* is the action of the vector field $-v$ on top forms (by Lie derivative). Finally, $D(X)$ is both a left and a right D-module on X.

2. Suppose $\mathbf{k} = \mathbb{C}$, and f is a holomorphic function defined on some open set in X (in the usual topology). Then $M(f) := D(X)f$ is a left D-module. We have a natural surjection $D(X) \to M(f)$ whose kernel is the left ideal generated by the linear differential equations satisfied by f. E.g. $M(1) = \mathcal{O}(X) = D(X)/D(X)\mathrm{Vect}(X)$, $M(x^s) = D(\mathbb{C})/D(\mathbb{C})(x\partial - s)$ if $s \notin \mathbb{Z}_{\geq 0}$, $M(e^x) = D(\mathbb{C})/D(\mathbb{C})(\partial - 1)$. Similarly, if ξ is a distribution (e.g., a measure) then $\xi \cdot D(X)$ is a right D-module. For instance, $\delta \cdot D(\mathbb{C}) = D(\mathbb{C})/xD(\mathbb{C})$, where δ is the delta-measure on the line.

Exercise 28.5. Show that $\mathcal{O}(X)$ is a simple $D(X)$-module. Deduce that for any nonzero regular function f on X, $M(f) = \mathcal{O}(X)$.

28.3. D**-modules on non-affine varieties.** Now assume that X is a smooth variety which is not necessarily affine. Recall that a **quasicoherent sheaf** on X is a sheaf M of $\mathcal{O}_X$-modules (in Zariski topology) such that for any affine open sets $U \subset V \subset X$ the restriction map induces an isomorphism of $\mathcal{O}(U)$-modules $\mathcal{O}(U) \otimes_{\mathcal{O}(V)} M(V) \cong M(U)$. Exercise 28.2(5) implies that there exists a canonical quasicoherent sheaf of algebras D_X on X such that $\Gamma(U, D_X) = D(U)$ for any affine open set $U \subset X$. This sheaf is called the **sheaf of differential operators on** X.

Definition 28.6. A **left** (respectively, **right**) D**-module** on X is a quasicoherent sheaf of left (respectively, right) D_X-modules. The categories of left (respectively, right) D-modules on X (with obviously defined morphisms) are denoted by $\mathcal{M}_l(X)$ and $\mathcal{M}_r(X)$.

It is clear that these are abelian categories. We will mostly use the category $\mathcal{M}_l(X)$ and denote it shortly by $\mathcal{M}(X)$.

Note that if X is affine, this definition is equivalent to the previous one (by taking global sections).

As before, the basic examples are $\mathcal{O}_X$ (a left D-module), Ω_X (a right D-module), D_X (both a left and a right D-module).

We see that the notion of a D-module on X is local. For this reason, many questions about D-modules are local and reduce to the case of affine varieties.

28.4. **Connections.** The definition of a D_X-module can be reformulated in terms of connections on an $\mathcal{O}_X$-module. Namely, in differential geometry we have a theory of connections on vector bundles. An algebraic vector bundle on X is the same thing as a coherent, locally free $\mathcal{O}_X$-module. It turns out that the usual definition of a connection, when written algebraically, makes sense for any $\mathcal{O}_X$-module (i.e., quasicoherent sheaf), not necessarily coherent or locally free.

Namely, let X be a smooth variety and Ω_X^i be the $\mathcal{O}_X$-module of differential i-forms on X.

Definition 28.7. A **connection** on an $\mathcal{O}_X$-module M is a $\mathbf{k}$-linear morphism of sheaves $\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega_X^1$ such that

$$\nabla(fm) = f\nabla(m) + m \otimes df$$

for local sections f of $\mathcal{O}_X$ and m of M.

Thus for each $v \in \mathrm{Vect}(X)$ we have the operator of covariant derivative $\nabla_v : M \to M$ given on local sections by $\nabla_v(m) := \nabla(m)(v)$.

Exercise 28.8. Let X be an affine variety. Show that the operator $m \mapsto ([\nabla_v, \nabla_w] - \nabla_{[v,w]})m$ is $\mathcal{O}(X)$-linear in v, w, m.

Given a connection ∇ on M, define the $\mathcal{O}_X$-linear map

$$\nabla^2 : M \to M \otimes_{\mathcal{O}_X} \Omega_X^2$$

given on local sections by

$$\nabla^2(m)(v, w) := ([\nabla_v, \nabla_w] - \nabla_{[v,w]})m.$$

This map is called the **curvature** of ∇. We say that ∇ is **flat** if its curvature vanishes: $\nabla^2 = 0$.

Proposition 28.9. *A left D_X-module is the same thing as an $\mathcal{O}_X$-module with a flat connection.*

Proof. Given an $\mathcal{O}_X$-module M with a flat connection ∇, we extend the $\mathcal{O}_X$-action to a D_X-action by $\rho(L_v) = \nabla_v$. The first three relations of (20) then hold for any connection, while the last relation holds due to flatness of ∇. Conversely, the same formula can be used to define a flat connection ∇ on any D_X-module M. $\square$

Exercise 28.10. Show that if a left D-module M on X is $\mathcal{O}$-coherent (i.e. a coherent sheaf on X) then it is locally free, i.e., is a vector bundle with a flat connection, and vice versa.

28.5. **Direct and inverse images.** Let $\pi : X \to Y$ be a morphism of smooth affine varieties. This morphism gives rise to a homomorphism $\pi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, making $\mathcal{O}(X)$ an $\mathcal{O}(Y)$-module, and a morphism of vector bundles $\pi_* : TX \to \pi^*TY$. This induces a map on global sections $\pi_* : \mathrm{Vect}(X) \to \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} \mathrm{Vect}(Y)$.

Define

$$D_{X\to Y} = \mathcal{O}(X) \otimes_{\mathcal{O}(Y)} D(Y).$$

This is clearly a right $D(Y)$-module. Let us show that it also has a commuting left $D(X)$-action. The left action of $\mathcal{O}(X)$ is obvious, so it remains to construct a flat connection. Given a vector field v on X, let

$$\nabla_v(f \otimes L) = v(f) \otimes L + f\pi_*(v)L,\ f \in \mathcal{O}(X),\ L \in D(Y), \tag{21}$$

where we view $\pi_*(v)$ as an element of $D_{X\to Y}$. This is well defined since for $a \in \mathcal{O}(Y)$ one has $[\pi_*(v), a] = v(a) \otimes 1$.

Exercise 28.11. Show that this defines a flat connection on $D_{X\to Y}$.

Now we define the **inverse image functor** $\pi^* : \mathcal{M}_l(Y) \to \mathcal{M}_l(X)$ by

$$\pi^!(N) = D_{X\to Y} \otimes_{D(Y)} N$$

and the *direct image functor* $\pi_* : \mathcal{M}_r(X) \to \mathcal{M}_r(Y)$ by

$$\pi_*(M) = M \otimes_{D(X)} D_{X\to Y}.$$

Note that at the level of quasicoherent sheaves, π^* is the usual inverse image.

These functors are right exact are compatible with compositions. Also by definition, $D_{X\to Y} = \pi^!(D(Y))$.

Note that $\pi^!(N) = \mathcal{O}(X)\otimes_{\mathcal{O}(Y)} N$ as an $\mathcal{O}(X)$-module (i.e., the usual pullback of $\mathcal{O}$-modules), with the connection defined by the formula similar to (21):

$$\nabla_v(f \otimes m) = v(f) \otimes m + f\nabla_{\pi_*(v)}(m),\ f \in \mathcal{O}(X),\ m \in M.$$

This means that the definition of $\pi^!$ is local both on X and on Y. On the contrary, the definition of π_* is local only on Y but not on X. For example, if Y is a point and $\dim X = d$ then $\pi_*\Omega_X = H^d(X, \mathbf{k})$, the algebraic de Rham cohomology of X of degree d.

Thus we can use the same definition locally to define $\pi^!$ for any morphism of smooth varieties, and π_* for an affine morphism (i.e. such that $\pi^{-1}(U)$ is affine for any affine open set $U \subset Y$), for example, a closed embedding. On the other hand, due to the non-local nature of direct image with respect to X the correct functor π_* for a non-affine morphism is not the derived functor of anything and can be defined only in the derived category.