29. **The Beilinson-Bernstein Localization Theorem**

29.1. **The Beilinson-Bernstein localization theorem for the zero central character.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra and U_0 be the maximal quotient of $U(\mathfrak{g})$ corresponding to the central character $\chi_\rho = \chi_{-\rho}$ of the trivial representation of $\mathfrak{g}$. Recall that $\mathrm{gr}(U_0) = \mathcal{O}(\mathcal{N})$. Let G be the corresponding simply connected complex group and $\mathcal{F}$ the flag variety of G; thus $\mathcal{F} \cong G/B$ for a Borel subgroup $B \subset G$. Let $D(\mathcal{F})$ be the algebra of global differential operators on $\mathcal{F}$; it is clear that $\mathrm{gr}D(\mathcal{F}) \subset \mathcal{O}(T^*\mathcal{F})$. Also, we have a natural filtration-preserving action map $a : U(\mathfrak{g}) \to D(\mathcal{F})$, induced by the Lie algebra homomorphism $\mathfrak{g} \to \mathrm{Vect}(\mathcal{F})$.

Theorem 29.1. *(Beilinson-Bernstein) (i) The homomorphism $a : U(\mathfrak{g}) \to D(\mathcal{F})$ factors through a homomorphism $a_0 : U_0 \to D(\mathcal{F})$.*

(ii) One has $\mathrm{gr}(a_0) = p^*$ where p is the Springer map $T^*\mathcal{F} \to \mathcal{N}$.

(iii) $\mathrm{gr}D(\mathcal{F}) = \mathcal{O}(T^*\mathcal{F})$ and a_0 is an isomorphism.

Proof. (i) Let $z \in Z(\mathfrak{g})$ be an element acting by zero in the trivial representation of $\mathfrak{g}$. Our job is to show that for any rational function $f \in \mathbb{C}(\mathcal{F})$ we have $a(z)f = 0$. Writing $\mathcal{F}$ as G/B, we may view f as a rational function on G such that $f(gb) = f(g)$, $b \in B$. The function $a(z)f$ on G is the result of action on f of the right-invariant differential operator L_z corresponding to z: $a(z)f = L_z f$. Since z is central, this operator is also left-invariant: $L_z = R_z$. Since z acts by zero on the trivial representation, using the Harish-Chandra isomorphism, we may write z as $\sum_i c_i b_i$, where $b_i \in \mathfrak{b} := \mathrm{Lie}(B)$ and $c_i \in U(\mathfrak{g})$. Thus $R_z = \sum_i R_{c_i} R_{b_i}$. But $R_{b_i} f = 0$ since f is invariant under right translations by B. Thus $R_z f = 0$ and we are done.

(ii) It suffices to check the statement in degrees 0 and 1, where it is straightforward.

(iii) The statement follows from (i), (ii) and the fact that p^* is an isomorphism (Theorem 27.16). □

The isomorphism a_0 gives rise to two functors: the functor of global sections $\Gamma : \mathcal{M}(\mathcal{F}) \to D(\mathcal{F}) - \mathrm{mod} \cong U_0 - \mathrm{mod}$ and the functor of localization $\mathrm{Loc} : U_0 - \mathrm{mod} \cong D(\mathcal{F}) - \mathrm{mod} \to \mathcal{M}(\mathcal{F})$ given by $\mathrm{Loc}(M)(U) := D(U) \otimes_{D(\mathcal{F})} M$ for an affine open set $U \subset \mathcal{F}$. Note that by definition the functor Loc is left adjoint to Γ.

The following theorem is a starting point for the geometric representation theory of semisimple Lie algebras (in particular, for the original proof of the Kazhdan-Lusztig conjecture).

Theorem 29.2. *(Beilinson-Bernstein localization theorem) The functors Γ and* Loc *are mutually inverse equivalences. Thus the category*

$U_0 - \text{mod}$ *is canonically equivalent to the category of D-modules on the flag variety $\mathcal{F}$.*

We will not give a proof of this theorem here.

Theorem 29.2 motivates the following definition.

Definition 29.3. A smooth algebraic variety X is said to be **D-affine** if the global sections functor $\Gamma : \mathcal{M}(X) \to D(X) - \text{mod}$ is an equivalence (hence Loc is its inverse).

It is clear that any affine variety is D-affine. Also we have

Corollary 29.4. *Partial flag varieties of semisimple groups are D-affine.*

29.2. **Twisted differential operators and D-modules.** We would now like to generalize the localization theorem to nonzero central characters. To do so, we have to replace usual differential operators and D-modules by twisted ones.

Let T be an algebraic torus with character lattice $P := \text{Hom}(T, \mathbb{C}^\times)$ and $\widetilde{X}$ be a principal T-bundle over a smooth algebraic variety X (with T acting on the right). In this case, given $\lambda \in P$, we can define the line bundle $\mathcal{L}_\lambda$ on X whose total space is $\widetilde{X} \times_T \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the 1-dimensional representation of T corresponding to λ, and we can consider the sheaf $D_{\lambda,X}$ of differential operators acting on local sections of $\mathcal{L}_\lambda$ (rather than functions).

Moreover, unlike the bundle $\mathcal{L}_\lambda$, the sheaf $D_{\lambda,X}$ makes sense not just for $\lambda \in P$ but more generally for $\lambda \in P \otimes_{\mathbb{Z}} \mathbb{C}$. Namely, assuming for now that $\lambda \in P$, we may think of rational sections of $\mathcal{L}_\lambda$ as rational functions F on $\widetilde{X}$ such that $F(yt) = \lambda(t)^{-1} F(y)$ for $y \in \widetilde{X}$. A differential operator D on $\widetilde{X}$ may be applied to such a function, and if $\xi \in \mathfrak{t} := \text{Lie}(T)$ then the first order differential operator $R_\xi - \lambda(\xi)$ acts by zero: $(R_\xi - \lambda(\xi))F = 0$. Thus given an affine open set $U \subset X$ with preimage $\widetilde{U} \subset \widetilde{X}$, the space

$$D_\lambda(U) := (D(\widetilde{U})/D(\widetilde{U})(R_\xi - \lambda(\xi), \xi \in \mathfrak{t}))^T$$

is naturally an associative algebra (check it!) which acts on rational sections of $\mathcal{L}_\lambda$. Moreover, it is easy to check that $D_\lambda(U) = D_{\lambda,X}(U)$. Now it remains to note that the definition of $D_\lambda(U)$ does not use the integrality of λ, thus makes sense for all $\lambda \in P \otimes_{\mathbb{Z}} \mathbb{C}$.

Thus for any $\lambda \in P \otimes_{\mathbb{Z}} \mathbb{C}$ we obtain a quasicoherent sheaf of algebras $D_{\lambda,X}$ on X which is called the sheaf of **λ-twisted differential operators**. If $\lambda = 0$, this sheaf coincides with the sheaf D_X of usual differential operators, and in general it has very similar properties, for example $\mathrm{gr}(D_{\lambda,X}(U)) = \mathcal{O}(T^*U)$ for any affine open set $U \subset X$. A quasicoherent sheaf on X with the structure of a (left or right) $D_{\lambda,X}$-module is called a (left or right) **λ-twisted D-module** on X. For example, if $\lambda \in P$ then $\mathcal{L}_\lambda$ is a left $D_{\lambda,X}$-module. The category of such modules is denoted by $\mathcal{M}^\lambda(X)$ (of course, it depends on the principal bundle $\widetilde{X}$ but we do not indicate it in the notation). Note that for $\beta \in P$ we have an equivalence $\mathcal{M}^\lambda(X) \cong \mathcal{M}^{\lambda+\beta}(X)$ defined by tensoring with $\mathcal{L}_\beta$.

Example 29.5. Let $\mathcal{L}$ be a line bundle on X and $c \in k$. Let $\widetilde{X}$ be the subset of nonzero vectors in the total space of $\mathcal{L}$. We have a natural action of $T := k^\times$ on $\widetilde{X}$ by dilations, and c defines a character of $\mathrm{Lie}(T)$. Thus we can define the sheaf $D_{c,L,X}$ of twisted differential operators on X, and if $c \in \mathbb{Z}$ then $D_{c,L,X} = D_X(L^{\otimes c})$ is the sheaf of differential operators on $L^{\otimes c}$. For example, if Ω_X is the canonical bundle of X then $D_{1,\Omega,X} = D_X(\Omega)$ is naturally isomorphic to the sheaf of usual differential operators with opposite multiplication, D_X^{op}.

Thus tensoring with Ω defines a canonical equivalence

$$\mathcal{M}_l(X) \cong \mathcal{M}_r(X)$$

(i.e., the sheaf D_X is Morita equivalent, although not in general isomorphic, to D_X^{op}). We may therefore not distinguish between these categories any more, identifying them by this equivalence, and can use left or right D-modules depending on what is more convenient.

29.3. **The localization theorem for non-zero central characters.** We are now ready to generalize the localization theorem to non-zero central characters. Let U_λ be the minimal quotient of $U(\mathfrak{g})$ corresponding to the central character $\chi_{\lambda-\rho}$. Recall that $\mathrm{gr}(U_\lambda) = \mathcal{O}(\mathcal{N})$.

Let $\widetilde{\mathcal{F}} := G/[B,B]$. We have a right action of $T := B/[B,B]$ on this variety by $y \mapsto yt$, defining the structure of a principal T-bundle $\widetilde{\mathcal{F}} \to \mathcal{F}$. Thus for every $\lambda \in P \otimes_{\mathbb{Z}} \mathbb{C} = \mathfrak{h}^*$ we have a sheaf of λ-twisted differential operators $D_{\lambda,\mathcal{F}} = D_\lambda$ on $\mathcal{F}$. For example, if $\lambda \in P$ then D_λ is the sheaf of differential operators acting on sections of the line bundle $\mathcal{L}_\lambda$ appearing in the Borel-Weil theorem (Theorem 27.3). Let $D_\lambda(\mathcal{F})$ be the algebra of global λ-twisted differential operators on $\mathcal{F}$; it is clear that $\mathrm{gr}D_\lambda(\mathcal{F}) \subset \mathcal{O}(T^*\mathcal{F})$. Also, we have a natural filtration-preserving action map $a : U(\mathfrak{g}) \to D_\lambda(\mathcal{F})$.

Theorem 29.6. *(Beilinson-Bernstein) (i) The map*

$$a : U(\mathfrak{g}) \to D_\lambda(\mathcal{F})$$

factors through a map $a_\lambda : U_\lambda \to D_\lambda(\mathcal{F})$.

(ii) One has $\mathrm{gr}(a_\lambda) = p^*$ *where* p *is the Springer map* $T^*\mathcal{F} \to \mathcal{N}$.
(iii) $\mathrm{gr}D_\lambda(\mathcal{F}) = \mathcal{O}(T^*\mathcal{F})$ *and* a_λ *is an isomorphism.*

Proof. The proof is completely parallel to the proof of Theorem 29.1. □

As in the untwisted case, the isomorphism a_λ gives rise to two functors: the functor of global sections

$$\Gamma : \mathcal{M}^\lambda(\mathcal{F}) \to D_\lambda(\mathcal{F}) - \mathrm{mod} \cong U_\lambda - \mathrm{mod}$$

and the functor of localization

$$\mathrm{Loc} : U_\lambda - \mathrm{mod} \cong D(\mathcal{F}) - \mathrm{mod} \to \mathcal{M}_\lambda(\mathcal{F})$$

given by $\mathrm{Loc}(M)(U) := D_\lambda(U) \otimes_{D_\lambda(\mathcal{F})} M$ for an affine open set $U \subset \mathcal{F}$. Moreover, as before, Loc is left adjoint to Γ.

Let us say that $\lambda \in \mathfrak{h}^*$ is **antidominant** if $-\lambda$ is dominant (cf. Subsection 16.1).

Theorem 29.7. *(Beilinson-Bernstein localization theorem) If* λ *is antidominant then the functors* Γ *and* Loc *are mutually inverse equivalences. Thus the category* $U_\lambda - \mathrm{mod}$ *is canonically equivalent to the category of* D_λ*-modules on the flag variety* $\mathcal{F}$.

Remark 29.8. 1. As explained above, for $\beta \in P$ we have an equivalence $\mathcal{M}^\lambda(\mathcal{F}) \cong \mathcal{M}^{\lambda+\beta}(\mathcal{F})$ defined by tensoring with $\mathcal{L}_\beta$. On the other side of the Beilinson-Bernstein equivalence this corresponds to translation functors defined in Subsection 24.1.

2. The first statement of Theorem 29.7 fails if λ is not assumed antidominant. Indeed, if λ is integral but not antidominant then by the Borel-Weil theorem (Theorem 27.3) $\Gamma(\mathcal{F}, \mathcal{L}_\lambda) = 0$, so the functor Γ is not faithful. The second statement of Theorem 29.7 also fails if $\lambda \in P$ and $\lambda - \rho$ is not regular.

For example, for $\mathfrak{g} = \mathfrak{sl}_2$ and $\lambda \in \mathbb{Z}$, the localization theorem holds for $\lambda \le 0$. For $\lambda \ge 2$ the first statement fails but we still have an equivalence $\mathcal{M}^\lambda(\mathcal{F}) \cong U_\lambda - \mathrm{mod}$ (as $U_\lambda \cong U_{-\lambda+2}$), albeit not given by Γ. But for $\lambda = 1$ there is no such equivalence at all; in fact, one can show that the category $U_\lambda - \mathrm{mod}$, unlike $\mathcal{M}^\lambda(\mathcal{F})$, has infinite cohomological dimension.

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